Available online at http://scik.org
J. Math. Comput. Sci. 7 (2017), No. 4, 739-754
ISSN: 1927-5307

COMMON FIXED POINT THEOREMS IN COMPLEX VALUED GENERALIZED METRIC SPACES

ZHONGNA SUN*, MEIMEI SONG

Department of Mathematics, Tianjin University of Technology, Tianjin 300384, P.R. China

Abstract. In this paper, using (CLR) property and the Common $(E.A)$ property common fixed point results for weakly compatible mappings, satisfying integral type contractive condition in complex valued generalized metric spaces are investigated.

Keywords: complex valued generalized metric spaces; weakly compatible mapping; $(E.A)$ property; (CLR) property.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

In 2011, Azam et al. [1] introduced the notion of complex valued metric space which is a generalization of the classical metric space and proved a unique common fixed point theorems for two self-mappings satisfying a rational type inequality. Though complex valued metric spaces form a special class of cone metric space, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many results of analysis

*Corresponding author

E-mail address: 153106301@stud.tjut.edu.cn

Received November 7, 2016

cannot be generalized to cone metric spaces. However, in complex valued metric spaces, we can study improvements of a host of results of analysis involving division. Bhatt et al. [7] initiated the concept of weakly compatible maps to study common fixed point theorem for weakly compatible maps in complex valued metric spaces.

In 2013, Abbas et al. [2] introduced the notion of complex valued generalized metric space. They replaced the triangular inequality in the complex valued metric by the rectangular inequality involving four points and extended the concept of complex valued metric spaces. They proved fixed point theorems involving the rational type contractive conditions for weakly contractive mappings in these spaces. The fixed point theorems in complex valued generalized metric space were obtained by many mathematicians (e.g. [2,3,8]).

In 2002, Branciari [9] proved fixed point theorems for two self-maps under contractive condition of integral type in metric space. In 2013, Manro et al. [10] extended and generalized the theorem of Branciari [9] for a pair of weakly compatible mappings satisfying a general contractive condition of integral type in complex valued metric space.

The aim of this paper is to prove common fixed point theorems for integral type contractive condition in complex valued generalized metric spaces.

2. Preliminaries

The following definitions and lemmas will be used in the sequel. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. We define a partial order $\precsim$ on $\mathbb{C}$ as follows:$z_1 \precsim z_2$, iff $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$.

As a result, we can say that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(I) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$;

(II) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$;

(III) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$;

(IV) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.

In particular, we can write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (I)-(III) is satisfied. Also we will write $z_1 \prec z_2$ if only (III) holds.

We can also prove these results:

(1) $z_1 \precsim z_2$ and $0 \leq r \in R \Longrightarrow rz_1 \precsim z_2$;

(2) $0 \precsim z_1 \precnsim z_2 \Longrightarrow |z_1| < |z_2|$;

(3) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Longrightarrow z_1 \prec z_3$.

Definition 2.1. [1] Let X be a nonempty set. If the mapping $d : X \times X \to \mathbb{C}$ satisfies the following axioms:

(i) $0 \precsim d(x,y)$, for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$, for all $x,y \in X$;

(iii) $d(x,y) \precsim d(x,z) + d(z,y)$. for all $x,y,z \in X$.

Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space.

Example 2.1. Let $X = \mathbb{C}$, define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(x,y) = 2i|x-y|,$$

where $x,y \in X$. Then (X,d) is a complex valued metric space.

Definition 2.2. [2] Let X be a nonempty set. If the mapping $d : X \times X \to \mathbb{C}$ satisfies the following axioms:

(i) $0 \precsim d(x,y)$, for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$, for all $x,y \in X$;

(iii) $d(x,y) \precsim d(x,u) + d(u,v) + d(v,y)$, for all $x,y \in X$ and all distinct $u,v \in X$, each one is different from x and y.

Then d is called a complex valued generalized metric on X and (X,d) is called a complex valued generalized metric space.

Example 2.2. Let $X = \{-2,-1,1,2\}$ and define the complex valued generalized metric on $d : X \times X \to \mathbb{C}$ as

d(-2,2)= d(2,-2)= d(-1,2)= d(2,-1)= d(1,2)= d(2,1)=0.4i

d(-2,-1)= d(-1,-2)= d(-1,1)= d(1,-1)=0.2i

d(-2,1)= d(1,-2)=0.5i

d(-2,-2)= d(-1,-1)= d(1,1)= d(2,2)=0

We can prove that (X,d) is a complex valued generalized metric space but it is not a complex valued metric space since it does not satisfy triangular inequality, $d(-2,1)=0.5i > d(-2,-1)+d(-1,1)=0.2i+0.2i=0.4i$.

Let X be a complex valued generalized metric space and $A\subseteq X$. A point $x\in X$ is called an interior point of a set A whenever there exists $0\prec r\in\mathbb{C}$ such that $B(x,r)=\{y\in X: d(x,y)\prec r\}\subseteq A$. A subset A in X is called open whenever each point of A is an interior point of A. The family $F=\{B(x,r): x\in X, 0\prec r\}$ is a sub-basis for a Hausdorff topology τ on X.

A point $x\in X$ is called a limit point of A whenever for every $0\prec r\in\mathbb{C}$, $B(x,r)\bigcap(A\setminus x)\neq\emptyset$. A subset $B\subseteq X$ is called closed whenever each limit point of B belongs to B.

Definition 2.3. [2] Let $\{x_n\}$ be a sequence in complex valued generalized metric space (X,d) and $x\in X$. Then

(1) x is called the limit of $\{x_n\}$ if for every $\omega\in\mathbb{C}$ with $0\prec\omega$ there is $n_0\in N$ such that $d(x_n,x)\prec\omega$ for all $n>n_0$ and we write $\lim_{n\to\infty}x_n=x$.

(2) $\{x_n\}$ is called a Cauchy sequence if for every $\omega\in\mathbb{C}$ with $0\prec\omega$ there is $n_0\in N$ such that $d(x_n,x_m)\prec\omega$ for all $n,m>n_0$.

(3) (X,d) is said to be a complete complex valued generalized metric space if every Cauchy sequence is convergent in (X,d).

Lemma 2.1. [2] Let (X,d) be a complex valued generalized metric space. Then a sequence $\{x_n\}$ in X converges to x if and only if $|d(x_n,x)|\to 0$ as $n\to\infty$.

Lemma 2.2. [2] Let (X,d) be a complex valued generalized metric space. Then a sequence $\{x_n\}$ in X is a Cauchy sequence if and only if $|d(x_n,x_m)|\to 0$ as $n,m\to\infty$.

Lemma 2.3. $[11]d(x_n,y)\to d(x,y)$ and $d(x,x_n)\to d(x,a)$ whenever $\{x_n\}$ is a sequence in X with $x_n\to x$.

Definition 2.4. [3] Let f and g be two self-mapping on a nonempty set X. Then

(I) $x\in X$ is called to be fixed point of f if $fx=x$.

(II) $x\in X$ is called to be a coincidence point of f and g if $fx=gx$.

(III) $x\in X$ is called to be a common fixed point of f and g if $fx=gx=x$.

Definition 2.5. [3] Let (X,d) be a complex valued generalized metric space. Then two self-mapping f and g are said to be weakly compatible if they commute at their coincidence points,i.e., $x \in X$ with $fx = gx$ implies that $fgx = gfx$.

Definition 2.6. [4]Let (X,d) be a complex valued generalized metric space. Then two self-mapping f and g are said to satisfy the common limit in the range of g property (CLR_g) if there exists a sequence $\{x_n\}$ in X, $x_n \neq x_m$ if $n \neq m$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$$

for some $x \in X$.

Definition 2.7. [5] Let (X,d) be a complex valued generalized metric space. Then two pairs of self maps (S,f) and (T,g) are said to satisfy the Common (E,A) property if there exists two sequence $\{x_n\}$ and $\{y_n\}$ in X $x_n \neq x_m$ and $y_n \neq y_m$ if $n \neq m$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} gy_n = z$$

for some $z \in X$.

3. Main results

In this section, we will prove the existence of fixed points for mappings f and g defined on a complex valued generalized metric space (X,d) satisfying a contractive condition of integral type.

Let $\Phi = \{\phi : \phi : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0,\infty)$, non-negative, non-decreasing and such that for each $\varepsilon \succ 0, \int_0^{\varepsilon} \phi(t)dt \succ 0\}$.

For any $z_1, z_2 \in \mathbb{C}_+$, define

$$[z_1,z_2] = \{r(s) \in \mathbb{C} : r(s) = z_1 + s(z_2 - z_1)\ for\ some\ s \in [0,1]\}.$$
$$(z_1,z_2] = \{r(s) \in \mathbb{C} : r(s) = z_1 + s(z_2 - z_1)\ for\ some\ s \in (0,1]\}.$$

Define $\zeta : [z_1,z_2] \to \mathbb{C}$ as follow:

$$\zeta(x+iy) = \phi_1(x) + i\phi_2(y),$$

where $x+iy \in [z_1,z_2]$ and $\phi_1, \phi_2 \in \Phi$.

We denote the set of all complex integrable function $\zeta : [z_1,z_2] \to \mathbb{C}$ by $\Gamma^1([z_1,z_2],\mathbb{C})$.

Lemma 3.1. [6] Suppose $\zeta \in \Gamma^1([z_1,z_2],\mathbb{C})$ and $\{z_n\}$ be a sequence in $\mathbb{C}_+$, then $\lim_{n\to\infty}\int_0^{z_n}\zeta(s)ds = (0,0)$ if and only if $z_n \to (0,0)$, as $n\to\infty$.

Definition 3.1. [6] A complex valued function $\varphi : R^n \to \mathbb{C}$ is measurable if both its real and imaginary parts are measurable.

In 2016, Sarwar Muhammad defined the lebesgue integral of f to be

$\int_E f = \int_E Re(f) + i\int_E Im(f) = (\int_E Re(f), \int_E Im(f))$,

provided that $Re(f)$ and $Im(f)$ are Lebesgue integrables and defined all complex valued lebesgue integrable functions by $\Gamma^1(E,\mathbb{C})$, where $E\subset R^n$ be a measurable set.

We define $\Psi = \{\varphi : R^n \to \mathbb{C}$ as a complex valued Lebesgue-integrable mapping(i.e.,$\varphi \in \Gamma^1(E,\mathbb{C})$), which is summable and non-vanishing on each measurable subset of R^n, such that for each $\varepsilon \succ 0, \int_0^{\varepsilon}\varphi(t)dt \succ 0\}$

Theorem 3.1. *Let f and g be two self-mapping of a complex valued generalized metric space (X,d) which satisfy the following conditions:*

(I) the pairs (f,g) is weakly compatible,

(II) the pairs (f,g) satisfy (CLR_g) property,

(III) $\forall x,y\in X$, there exists

$$u(x,y)\in M(x,y)$$

such that

$$\int_0^{d(fx,fy)}\varphi(t)dt \prec \alpha\int_0^{u(x,y)}\varphi(t)dt \tag{3.1}$$

where α nonnegative real such that $\alpha<1$ and $\varphi\in\Psi$,

$M(x,y) = \{d(gx,gy), d(gy,fy), d(gx,fy), d(gy,fx)\}$, then f and g have a unique common fixed point in X.

Proof. Since the mappings f and g satisfy the (CLR_g) property, then there exists sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx,$$

for some $x\in X$. We now show that $fx = gx$. Suppose not, i.e., $fx\neq gx$.

From (3.1)

$$\int_0^{d(fx_n,fx)}\varphi(t)dt \prec \alpha\int_0^{u(x_n,x)}\varphi(t)dt \tag{3.2}$$

where

$$u(x_n,x) \in M(x_n,x)$$

$$M(x_n,x) = \{d(gx_n,gx), d(gx_n,fx_n), d(gx,fx), d(gx_n,fx), d(gx,fx_n)\}$$

Two cases arises:

1) If

$$u(x_n,x) = d(gx_n,gx), d(gx_n,fx_n) \; or \; d(gx,fx_n)$$

Taking limit as $n \to \infty$, then (3.2) and lem (3.8) implies

$$\int_0^{d(gx,fx)} \varphi(t)dt \prec \alpha \int_0^{d(gx,gx)} \varphi(t)dt = 0$$

Which is a contradiction.

2) If

$$u(x_n,x) = d(gx,fx) \; or \; d(gx_n,fx)$$

Taking limit as $n \to \infty$, then (3.2) and lem (3.8) implies

$$\int_0^{d(gx,fx)} \varphi(t)dt \prec \alpha \int_0^{d(gx,fx)} \varphi(t)dt$$

Since $\alpha < 1$, which is a contradiction.

Hence, from all cases, $fx = gx$. Now let $z = fx = gx$. Since (f,g) is weakly compatible, $fgx = gfx$, i.e. $fz = gz$.

Next we show that $fz = z$. Suppose it is not, then

$$\int_0^{d(fz,z)} \varphi(t)dt = \int_0^{d(fz,fx)} \varphi(t)dt \prec \alpha \int_0^{u(z,x)} \varphi(t)dt \tag{3.3}$$

where

$$u(z,x) \in M(z,x)$$

$$\begin{aligned} M(z,x) &= \{d(gz,gx), d(gz,fz), d(gx,fx), d(gz,fx), d(gx,fz)\} \\ &= \{d(fz,z), 0, 0, d(fz,z), d(z,fz)\} \\ &= \{d(fz,z), 0\} \end{aligned}$$

Two cases arises:

1) If $u(z,x)=d(fz,z)$, then by (3.3), we have

$$\int_0^{d(fz,z)}\varphi(t)dt \prec \alpha\int_0^{d(fz,z)}\varphi(t)dt$$

Since $\alpha<1$, which is a contradiction.

2) If $u(z,x)=0$, then by (3.3), we have

$$\int_0^{d(fz,z)}\varphi(t)dt \prec 0$$

which is a contradiction.

Thus, $fz=gz=z$.

Hence, z is a common fixed point of f and g.

To show that the fixed point is unique, suppose z' is another fixed point of f and g, i.e. $fz'=gz'=z'$.

From (3.3), we have

$$\int_0^{d(z,z')}\varphi(t)dt=\int_0^{d(fz,fz')}\varphi(t)dt \prec \alpha\int_0^{u(z,z')}\varphi(t)dt$$

where

$$u(z,z')\in M(z,z')$$

$$\begin{aligned}M(z,z')&=\{d(gz,gz'),d(gz,fz),d(gz',fz'),d(gz,fz'),d(gz',fz)\}\\&=\{d(z,z'),0,0,d(z,z'),d(z',z)\}\\&=\{d(z,z'),0\}\end{aligned}$$

Two possible cases arises:

1) If $u(z,z')=d(z,z')$, then by (3.3), we have

$$\int_0^{d(z,z')}\varphi(t)dt \prec \alpha\int_0^{d(z,z')}\varphi(t)dt$$

Since $\alpha<1$, which is a contradiction.

2) If $u(z,z')=0$, then by (3.46), we have

$$\int_0^{d(z,z')}\varphi(t)dt \prec 0$$

which is a contradiction.

Hence, $z = z'$ i.e., f and g have a unique common fixed point in X. This completes the proof.

Theorem 3.2. *Let S, T, f and g be four self-mapping of a complete complex valued generalized metric space (X,d) which satisfy the following conditions:*

(I) $S(X) \subseteq g(X)$ and $T(X) \subseteq f(X)$,

(II) the pairs (S, f) and (T, g) are weakly compatible,

(III) the subspace $f(X)$ or $g(X)$ is closed,

(IV) $\forall x, y \in X$,

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \precsim (a+c)\int_0^{d(fx,gy)} \varphi(t)dt + b\int_0^{d(fx,Sx)} \varphi(t)dt + (b+c)\int_0^{d(gy,Ty)} \varphi(t)dt \quad (3.4)$$

where a, b and c are nonnegative reals such that $a+2b+2c<1$ and $\varphi \in \Psi$, then S, T, f and g have a unique common fixed point in X.

Proof. We construct a sequence $\{y_n\}$ in X such that,

$$y_{2n} = Sx_{2n} = gx_{2n+1}$$

and

$$y_{2n+1} = Tx_{2n+1} = fx_{2n+2}, n \geq 0 \quad (3.5)$$

where $\{x_{2n}\}$ is another sequence in X. Using (3.4), we have,

$$\begin{aligned}\int_0^{d(y_{2n},y_{2n+1})} \varphi(t)dt &= \int_0^{d(Sx_{2n},Tx_{2n+1})} \varphi(t)dt \precsim (a+c)\int_0^{d(fx_{2n},gx_{2n+1})} \varphi(t)dt \\ &\quad + b\int_0^{d(fx_{2n},Sx_{2n})} \varphi(t)dt + (b+c)\int_0^{d(gx_{2n+1},Tx_{2n+1})} \varphi(t)dt \\ &= (a+c)\int_0^{d(y_{2n-1},y_{2n})} \varphi(t)dt + b\int_0^{d(y_{2n-1},y_{2n})} \varphi(t)dt \\ &\quad + (b+c)\int_0^{d(y_{2n},y_{2n+1})} \varphi(t)dt\end{aligned} \quad (3.6)$$

Hence,

$$\int_0^{d(y_{2n},y_{2n+1})} \varphi(t)dt \precsim \frac{a+b+c}{1-b-c}\int_0^{d(y_{2n-1},y_{2n})} \varphi(t)dt$$

Therefore,

$$\int_0^{d(y_{2n},y_{2n+1})} \varphi(t)dt \precsim \delta\int_0^{d(y_{2n-1},y_{2n})} \varphi(t)dt$$

where $\delta = \frac{a+b+c}{1-b-c} < 1$, since $a+2b+2c < 1$. Proceeding in a similar way we have,

$$\begin{aligned} \int_0^{d(y_{2n},y_{2n+1})} \varphi(t)dt &\precsim \delta \int_0^{d(y_{2n-1},y_{2n})} \varphi(t)dt \\ &\precsim \delta^2 \int_0^{d(y_{2n-2},y_{2n-1})} \varphi(t)dt \\ &\precsim \cdots \leq \delta^{2n} \int_0^{d(y_0,y_1)} \varphi(t)dt \end{aligned} \tag{3.7}$$

Finally we conclude that

$$\int_0^{d(y_n,y_{n+1})} \varphi(t)dt \precsim \delta^n \int_0^{d(y_0,y_1)} \varphi(t)dt$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \int_0^{d(y_n,y_{n+1})} \varphi(t)dt = 0 \tag{3.8}$$

Hence from lem(3.3), we get $d(y_n, y_{n+1}) \to 0$, as $n \to \infty$.

We now show that $\{y_n\}$ is Cauchy. Suppose that it is not. Then there exists an $\varepsilon > 0$ and subsequences $\{2m\}$ and $\{2n\}$ such that $2n > 2m > n_0$ with

$$d(y_{2m}, y_{2n}) \succsim \varepsilon, d(y_{2m}, y_{2n-1}) \prec \varepsilon \tag{3.9}$$

Now from rectangular inequality,

$$d(y_{2m-1}, y_{2n-1}) \precsim d(y_{2m-1}, y_{2m}) + d(y_{2m}, y_{2n-2}) + d(y_{2n-2}, y_{2n-1}) \tag{3.10}$$

Therefore, from (3.7), (3.9) and (3.10), we have,

$$\begin{aligned} \int_0^{\varepsilon} \varphi(t)dt \precsim \int_0^{d(y_{2m},y_{2n})} \varphi(t)dt &\precsim \delta \int_0^{d(y_{2m-1},y_{2n-1})} \varphi(t)dt \\ &\precsim \delta \int_0^{d(y_{2m-1},y_{2m})} \varphi(t)dt + \delta \int_0^{d(y_{2m},y_{2n-2})} \varphi(t)dt \\ &+ \delta \int_0^{d(y_{2n-2},y_{2n-1})} \varphi(t)dt \prec \delta \int_0^{\varepsilon} \varphi(t)dt \\ &+ \delta \int_0^{d(y_{2m-1},y_{2m})} \varphi(t)dt + \delta \int_0^{d(y_{2n-2},y_{2n-1})} \varphi(t)dt \end{aligned}$$

Taking the limit as $n, m \to \infty$, from (3.8) we have,

$$\int_0^{\varepsilon} \varphi(t)dt \prec \delta \int_0^{\varepsilon} \varphi(t)dt$$

which is a contraction. Therefore $\{y_n\}$ is Cauchy sequence in X. Since X is complete, there exists point z in X such that,

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} fx_{2n+2} = z$$

Assuming $f(X)$ is closed, $z \in f(X)$ and $z = fu$ for some $u \in X$. We claim that $Su = fu = z$. Using the rectangular inequality[Definition 2.2(iii)] we get,

$$d(Su,z) \precsim d(Su,Tx_{2n+1}) + d(Tx_{2n+1},gx_{2n+1}) + d(gx_{2n+1},z) \tag{3.11}$$

Therefore from (3.11),

$$\begin{aligned}\int_0^{d(Su,z)} \varphi(t)dt &\precsim \int_0^{d(Su,Tx_{2n+1})} \varphi(t)dt + \int_0^{d(Tx_{2n+1},gx_{2n+1})} \varphi(t)dt + \int_0^{d(gx_{2n+1},z)} \varphi(t)dt \\ &\precsim (a+c)\int_0^{d(fu,gx_{2n+1})} \varphi(t)dt + (b+c)\int_0^{d(gx_{2n+1},Tx_{2n+1})} \varphi(t)dt \\ &+ b\int_0^{d(fu,Su)} \varphi(t)dt + \int_0^{d(Tx_{2n+1},gx_{2n+1})} \varphi(t)dt + \int_0^{d(gx_{2n+1},z)} \varphi(t)dt\end{aligned}$$

As $n \to \infty$, we get,

$$\begin{aligned}\int_0^{d(Su,z)} \varphi(t)dt &\precsim (a+c)\int_0^{d(z,z)} \varphi(t)dt + (b+c)\int_0^{d(z,z)} \varphi(t)dt \\ &+ b\int_0^{d(z,Su)} \varphi(t)dt + \int_0^{d(z,z)} \varphi(t)dt + \int_0^{d(z,z)} \varphi(t)dt \\ &= b\int_0^{d(z,Su)} \varphi(t)dt\end{aligned}$$

As $b < 1$, $\int_0^{d(z,Su)} \varphi(t)dt = 0$ implies that $Su = z$ i.e. $fu = Su = z$ and u is a coincidence point of f and S. Since $S(X) \subseteq g(X)$, $Su = gv$ for some $v \in X$. Hence $fu = Su = gv = z$. We claim that $Tv = z$. By the inequality (3.4),

$$\begin{aligned}\int_0^{d(z,Tv)} \varphi(t)dt = \int_0^{d(Su,Tv)} \varphi(t)dt &\precsim (a+c)\int_0^{d(fu,gv)} \varphi(t)dt + b\int_0^{d(fu,Su)} \varphi(t)dt \\ &+ (b+c)\int_0^{d(gv,Tv)} \varphi(t)dt = (a+b+c)\int_0^{d(z,z)} \varphi(t)dt \\ &+ (b+c)\int_0^{d(z,Tv)} \varphi(t)dt = (b+c)\int_0^{d(z,Tv)} \varphi(t)dt\end{aligned}$$

Since $b+c < 1, \int_0^{d(z,Tv)} \varphi(t)dt = 0$ implies that $Tv = z$, hence $Su = fu = Tv = gv = z$. As S and f are weakly compatible, $Sfu = fSu$ i.e. $Sz = fz$. We now prove that $Sz = z$, suppose not,

$Sz \neq z$, then by (3.4),

$$\int_0^{d(Sz,z)} \varphi(t)dt = \int_0^{d(Sz,Tv)} \varphi(t)dt \precsim (a+c)\int_0^{d(fz,gv)} \varphi(t)dt + b\int_0^{d(fz,Sz)} \varphi(t)dt$$
$$+(b+c)\int_0^{d(gv,Tv)} \varphi(t)dt = (a+2c)\int_0^{d(Sz,z)} \varphi(t)dt + b\int_0^{d(fz,Sz)} \varphi(t)dt$$
$$+(b+c)\int_0^{d(z,z)} \varphi(t)dt = (a+2c)\int_0^{d(Sz,z)} \varphi(t)dt$$

Since $a+2c<1$, $\int_0^{d(Sz,z)} \varphi(t)dt$ implies that $Sz=z$ i.e. $Sz=fz=z$. Thus z is a fixed point of S and f. Also since T and g are weakly compatible, $Tgv=gTv$ i.e. $Tz=gz$. We now prove that $Tz=z$, suppose not, $Tz\neq z$, then by (3.4) again,

$$\int_0^{d(z,Tz)} \varphi(t)dt = \int_0^{d(Sz,Tz)} \varphi(t)dt \precsim (a+c)\int_0^{d(fz,gz)} \varphi(t)dt + b\int_0^{d(fz,Sz)} \varphi(t)dt$$
$$+(b+c)\int_0^{d(gz,Tz)} \varphi(t)dt = (a+c)\int_0^{d(Tz,z)} \varphi(t)dt + b\int_0^{d(z,z)} \varphi(t)dt$$
$$+b\int_0^{d(gz,Tz)} \varphi(t)dt = (a+c)\int_0^{d(Tz,z)} \varphi(t)dt$$

Since $a+c<1$, $\int_0^{d(Tz,z)} \varphi(t)dt$ implies that $Tz=z$ and $Sz=fz=Tz=gz=z$, z is a common fixed point of S, T, f and g. To show that the fixed point is unique, suppose that there is another point $w\in X$ such that $Sw=Tw=fw=gw=w$.

From (3.4), we have,

$$\int_0^{d(w,z)} \varphi(t)dt = \int_0^{d(Sw,Tz)} \varphi(t)dt \precsim (a+c)\int_0^{d(fw,gz)} \varphi(t)dt + b\int_0^{d(fw,Sw)} \varphi(t)dt$$
$$+(b+c)\int_0^{d(gz,Tz)} \varphi(t)dt = (a+c)\int_0^{d(w,z)} \varphi(t)dt + b\int_0^{d(w,w)} \varphi(t)dt$$
$$+(b+c)\int_0^{d(z,z)} \varphi(t)dt = (a+c)\int_0^{d(w,z)} \varphi(t)dt$$

Therefore $\int_0^{d(w,z)} \varphi(t)dt = 0$ as $a+c<1$ and $w=z$ which proves the uniqueness of the fixed point. Similar argument holds if $g(X)$ is assumed to be closed. Hence S, T, f and g have a unique common fixed point in X. This completes the proof.

By putting $c=0$, in Theorem 3.2 above, we get the following corollary.

Corollary 3.3. *Let S, T, f and g be four self-mapping of a complete complex valued generalized metric space (X,d) which satisfy the following conditions:*

(I) $S(X)\subseteq g(X)$ and $T(X)\subseteq f(X)$,

(II) the pairs (S,f) and (T,g) are weakly compatible,

(III) the subspace $f(X)$ or $g(X)$ is closed,

(IV) $\forall x,y \in X$,

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \precsim a\int_0^{d(fx,gy)} \varphi(t)dt + b(\int_0^{d(fx,Sx)} \varphi(t)dt + \int_0^{d(gy,Ty)} \varphi(t)dt)$$

where a and b are nonnegative reals such that $a+2b<1$ and $\varphi \in \Psi$, then S, T, f and g have a unique common fixed point in X.

If we put $T = S$ and $g = f$, $b = c = 0$ in Theorem 3.2 above, we get the following corollary.

Corollary 3.4. *Let S and f be two mappings of a complete complex valued generalized metric space (X,d) which satisfy the following conditions:*

(I) $S(X) \subseteq f(X)$,

(II) the pair (S,f) is weakly compatible,

(III) the subspace $f(X)$ is closed,

(IV) $\forall x,y \in X$,

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \precsim a\int_0^{d(fx,gy)} \varphi(t)dt$$

where a is nonnegative real such that $a<1$ and $\varphi \in \Psi$, then S and f have a unique common fixed point in X.

Theorem 3.5. *Let S,T,f and g be four self-mapping of a complex valued generalized metric space (X,d) which satisfy the following condition:*

(I) the pairs (S,f) and (T,g) are weakly compatible;

(II) the pairs (S,f) and (T,g) satisfy the Common property (E,A);

(III) the subspaces $f(X)$ and $g(X)$ are closed;

(IV) $\forall x,y \in X$,

$$\begin{aligned}\int_0^{d(Sx,Ty)} \varphi(t)dt \precsim a\int_0^{d(fx,gy)} \varphi(t)dt + b(\int_0^{d(fx,Sx)} \varphi(t)dt + \int_0^{d(gy,Ty)} \varphi(t)dt)\\ + c(\int_0^{d(fx,Ty)} \varphi(t)dt + \int_0^{d(gy,Sx)} \varphi(t)dt)\end{aligned} \tag{3.12}$$

where a, b and c are nonnegative reals such that $a+2b+2c<1$ and $\varphi \in \Psi$, then S, T, f and g have a unique common fixed point in X.

Proof. Since the pairs (S,f) and (T,g) satisfy the common property (E,A), there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} gy_n = z$$

for some $z \in X$. Since $f(X)$ is closed and $z \in f(X)$, there exists $u \in X$ such that $z = fu$. We claim that $Su = fu = z$. From (3.12) we have,

$$\begin{aligned}\int_0^{d(Su,Ty_n)} \varphi(t)dt \precsim a\int_0^{d(fu,gy_n)} \varphi(t)dt + b(\int_0^{d(fu,Su)} \varphi(t)dt + \int_0^{d(gy_n,Ty_n)} \varphi(t)dt) \\ + c(\int_0^{d(fu,Ty_n)} \varphi(t)dt + \int_0^{d(gy_n,Su)} \varphi(t)dt)\end{aligned}$$

As $n \to \infty$,

$$\int_0^{d(Su,z)} \varphi(t)dt \precsim b\int_0^{d(fu,Su)} \varphi(t)dt + c\int_0^{d(z,Su)} \varphi(t)dt$$

As $b + c < 1$, $d(Su,z) = 0$ and $Su = z$ i.e. $Su = fu = z$. Since S and f are weakly compatible, $Sfu = fSu$ and $Sz = fz$ i.e. z is a coincidence point of S and f. Also since $g(X)$ is closed and $z \in g(X)$, $z = gv$ for some $v \in X$. Hence $Su = fu = gv = z$. We prove that $Tv = gv = z$. Consider from (3.12),

$$\begin{aligned}\int_0^{d(z,Tv)} \varphi(t)dt = \int_0^{d(Su,Tv)} \varphi(t)dt &\precsim a\int_0^{d(fu,gv)} \varphi(t)dt \\ &+ b(\int_0^{d(fu,Su)} \varphi(t)dt + \int_0^{d(gv,Tv)} \varphi(t)dt) \\ &+ c(\int_0^{d(fu,Tv)} \varphi(t)dt + \int_0^{d(gv,Su)} \varphi(t)dt \\ &= (b+c)\int_0^{d(z,Tv)} \varphi(t)dt\end{aligned}$$

Hence $d(z,Tv) = 0$ as $b + c < 1$. Therefore $Tv = gv = z$. Since T and g are weakly compatible, we have $Tgv = gTv$ i.e. $Tz = gz$ and z is a coincidence point of T and g. To show that z is a fixed point, we claim that $Sz = z$. By (3.12),

$$\begin{aligned}\int_0^{d(Sz,z)} \varphi(t)dt = \int_0^{d(Sz,Tv)} \varphi(t)dt &\precsim a\int_0^{d(fz,gv)} \varphi(t)dt \\ &+ b(\int_0^{d(fz,Sz)} \varphi(t)dt + \int_0^{d(gv,Tv)} \varphi(t)dt) \\ &+ c(\int_0^{d(fz,Tv)} \varphi(t)dt + \int_0^{d(gv,Sz)} \varphi(t)dt \\ &= (a+2c)\int_0^{d(z,Sz)} \varphi(t)dt\end{aligned}$$

Since $a+2c<1$, $d(Sz,z)=0$ and $Sz=z$. Hence $Sz=fz=z$. Similarly we can show that $Tz=gz=z$. Thus we have $Sz=fz=Tz=gz=z$ and z is a common fixed point of S, T, f and g in X.

Uniqueness. let z' be another fixed point of S,T, f and g, i.e. $Sz'=Tz'=fz'=gz'=z'$. From (3.12), we have

$$\begin{aligned}\int_0^{d(z,z')}\varphi(t)dt = \int_0^{d(Sz,Tz')}\varphi(t)dt &\precsim a\int_0^{d(fz,gz')}\varphi(t)dt \\ &+b(\int_0^{d(fz,Sz)}\varphi(t)dt+\int_0^{d(gz',Tz')}\varphi(t)dt) \\ &+c(\int_0^{d(fz,Tz')}\varphi(t)dt+\int_0^{d(gz',Sz)}\varphi(t)dt \\ &=(a+2c)\int_0^{d(z,z')}\varphi(t)dt\end{aligned}$$

thus

$$(1-a-2c)\int_0^{d(z,z')}\varphi(t)dt \precsim 0$$

Since $1-a-2c>0$, which is a contradiction, unless $z=z'$. Hence z is a unique common fixed point of S, T, f and g in X. This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. Azam,B. Fisher and M. Khan, Common fixed point theorems in complex valued metric spaces, Numer. Funct.Anal.Optim. 3 (2011), 243-253.

[2] M. Abbas, V. Raji, T. Nazir, S. Radenovi, Common fixed point of mappings satisfying rational inequalities in ordered complex valued generalized metric spaces. Afr. Mat. 26 (1-2) (2013), 17-30.

[3] Sunanda R. Patil, J. N. Salunke, Common fixed point theorems in complex valued rectangular metric spaces, South Asian J. Math. 6 (1) (2016), 10-23.

[4] Wutiphol Sintunavarat and Poom Kumam, Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, J. Appl. Math. 2011 (2011), Article ID 637958.

[5] Y. Liu, J. Wu and Z. Li, Common fixed points of single-valued and multivalued maps, Int. J. Math. Math. Sci, 19 (2005), 3045-3055.

[6] M. Sarwar, M. Zada, N. Mehmood, Common fixed point results for weakly compatible mappings under contractive conditions of integral type in complex valued metric spaces, Trans. A. Razmadze Math. Inst. 170 (2016), 91-106.

[7] S. Bhatt. S. Chaukiyal, R.C. Dimri, A common fixed point theorem for weakly compatible maps in complex valued metric spaces, Int. J. Math. Sci. Appl. 1 (3) (2011) 1385-1389.

[8] D. Singh, O. P.Chauhan, N. Singh, V. Joshi, Complex valued rectangular metric spaces and common fixed point theorems, Bull. Math. Anal. Appl. 7 (2) (2015), Pages 1-13.

[9] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci., 29(9) (2002), 531-536.

[10] S. Manro, S.B. Jeong. S.M. Kang, Fixed point theorems for mappings satisfying a general contractive condition of integral type, Int. J. Math. Anal. (Ruse) 7 (57-60) (2013), 2811-2819.

[11] A. Branciari , A fixed point theorem of Banach-Caccippoli type on a class of generalized metric spaces, Publ. Math. Debrecen, 57 (1-2) (2000), 31-37.