Available online at http://scik.org
J. Math. Comput. Sci. 8 (2018), No. 4, 506-522
https://doi.org/10.28919/jmcs/3035
ISSN: 1927-5307

COUPLED FIXED POINT THEOREM AND T-STABILITY FOR NONLINEAR CONTRACTIVE MAPPINGS IN CONE METRIC SPACES OVER BANACH ALGEBRAS

XIAOYE YANG[1], QING YUAN[2,*]

[1]Department of Mathematics, Ren'ai College, Tianjin University, Tianjin 301636, P. R. China

[2]Department of Mathematics, Linyi University, Shandong, P.R. China

Abstract. In this paper, we establish the existence of coupled coincidence point and prove coincidence point theorem for nonlinear contractive mappings in cone metric space over Banach algebras. Our results generalize some known results in cone metric space. Moreover, we verify the T-stability of iteration sequence.

Keywords: Banach algebra; cone metric space; T-stability.

2010 AMS Subject Classification: 47H09, 47H10.

1. Introduction

Cone metric spaces were introduced as a generalization of normal metric spaces by Huang and Zhang in [1]. They presented the notion of convergence of sequences in cone metric spaces and proved some fixed point theorems. Then after, many authors established the equivalence between some fixed point results in metric and in cone metric spaces see [4-6]. But some

*Corresponding author
E-mail address: yuanqing@lyu.edu.cn
Received May 2, 2017

authors appealed to the equivalence of some metric and cone metric fixed point results (see[6-9]) Recently, Liu and Xu [2] introduced the concept of cone metric apace over Banach algebras, replacing Banach spaces by Banach algebras as the underlying spaces of cone metric spaces. They abstain some fixed point theorems of generalized *Lipschitz* mappings. Moreover they give an example to illustrate that are more useful than the standard results in cone metric spaces.

Bhashkar and *Lashmikantham* in[4] introduced the concept of coupled fixed point of a mappings $F: X\times X \to X$ and investigated some fixed point theorems in partially ordered sets.Sabetghadam et al. in[6] introduced this concept in cone metric spaces.Then after, Lakshmikantham and Ciric in[12] proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in partially ordered complete metric space. Further, M.Abbas and M.Ali Khan[5] introduce the concept of a w-compatible mappings to obtain couple coincidence point and couple point of coincidence for nonlinear contractive mappings in cone metric space with a cone having non-empty interior.

In this paper, we establish the existence of coupled coincidence point and prove coincidence point theorem for nonlinear contractive mappings in cone metric space over Banach algebras.Our results generalize some known results in cone metric space. Moreover, we verify the T-stability of iteration sequence.Our results greatly extend the main work of [4-13].

2. **Preliminaries**

In this section, we give some necessary preliminaries on the Caputo derivative, which will be used in the sequel.

Definition 2.1. *(see[1]) Let $\mathcal{A}$ always be a Banach algebra.That is, $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties, for all $x,y,z \in \mathcal{A}, \alpha \in \mathcal{R}$:*

1. $(xy)z = x(yz)$;

2. $x(y+z) = xy + xz$;

3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;

4. $||x|| \leq ||x||||y||$.

Definition 2.2. *(see[1]) Nonempty closed convex subset* K *of* $\mathcal{A}$ *is called a cone, if for all* $\lambda,\mu \geq 0$

1. $(\theta, e) \subset \mathcal{K}$*,*

2. $\mathcal{K}^2 = \mathcal{K}\mathcal{K} \subset K$*,*

3. $\mathcal{K} \cap (-\mathcal{K}) = \theta$*,*

4. $\lambda\mathcal{K} + \mu\mathcal{K} \subset \mathcal{K}$

On this basic, we define a partial ordering $\preceq$ *with respect to* $\mathcal{K}$ *by* $x \preceq y$ *if and only if* $y - x \in \mathcal{K}$*.We shall write* $x \prec y$ *to indicate that* $x \preceq y$ *but* $x \neq y$*, while* $x \ll y$ *will indicate that* $y - x \in int\mathcal{K}$*, where* $int\mathcal{K}$ *stands for the interior of* $\mathcal{K}$*.A cone* $\mathcal{K}$ *is called normal if there is a number* $M > 0$ *such that for all* $x, y \in \mathcal{A}$*,* $\theta \preceq x \preceq y$ *implies* $\|x\| \leq \|y\|$*. The least positive number satisfying above is called the normal constant of* $\mathcal{K}$*. In the following we always suppose that* $\mathcal{A}$ *is a Banach algebra with a unit e,* K *is a solid cone in* $\mathcal{A}$*, and* $\preceq$ *is a partial ordering with respect to* K.

Definition 2.3. *(see[1])Let X be a non-empty set and* $\mathcal{A}$ *a Banach algebra.Suppose that the mappings* $d : X \times X \longrightarrow \mathcal{A}$ *satisfies: 1.* $\theta \prec d(x,y)$ *for all* $x, y \in X$ *with* $d(x,y) = \theta$ *if and only if* $x = y$*; 2.* $d(x,y) = d(y,x)$ *for all* $x, y \in X$*; 3.* $d(x,y) \preceq d(x,z) + d(z,y)$ *for all* $x,y,z \in X$*. Then d is called a cone metric on X, and* (X,d) *is called a cone metric space over Banach algebra.*

Definition 2.4. *(see[17])Let* (X,d) *be a cone metric space,* $x \in X$ *and* x_n *is a sequence in X.*

1. x_n *converges to x whenever for every* $c \gg \theta$ *there is a natural number N such that* $d(x_n, x) \gg c$ *for all* $n \geq N$*.we denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \longrightarrow x (n \longrightarrow \propto)$*;*

2. x_n *is a Cauchy sequence whenever for every* $c \gg \theta$ *there is a natural number N such that* $d(x_n, x_m) \gg c$ *for all* $n, m \geq N$*;*

3. (X,d) *is a complete cone metric space if every Cauchy sequence is convergent.*

Definition 2.5. *(see[5])An element* $(x,y) \in X \times X$ *is called a coupled fixed point of mappings* $F : X \times X \to X$ *if* $x = F(x,y)$ *and* $y = F(y,x)$

Definition 2.6. *(see[5])An element* $(x,y) \in X \times X$ *is called*

(1)a coupled coincidence point of mappings $F : X \times X \to X$ *and* $g : X \to X$ *if* $g(x) = F(x,y)$ *and* $g(y) = F(y,x)$*, and* (gx, gy) *is called coupled point of coincidence;*

(2)a common coupled fixed point of mappings $F : X \times X \to X$ *and* $g : X \to X$ *if* $x = g(x) = F(x,y)$ *and* $y = g(y) = F(y,x)$*.*

Proposition 2.7. *(see[18])Let $\mathcal{A}$ be a Banach algebra with a unite e, and $x \in \mathcal{A}$.If the spectral radius $\rho(x)$ of x is less than 1, i.e.*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = inf_{n\geq 1}\|x^n\|^{\frac{1}{n}} < 1.$$

then $e-x$ is invertible. Actually,

$$(e-x)^{-1} = \sum_{i=0}^{n} x^i.$$

Lemma 2.8. *(see[19]) Let $u,v,w \in \mathcal{A}$, if $u \ll v$ and $v \ll w$, then $u \ll w$.*

Lemma 2.9. *(see[19])Let $\mathcal{A}$ be a Banach algebra and a_n is a sequence in $\mathcal{A}$.If $a_n \longrightarrow \theta (n \longrightarrow \propto)$, then for any $c \gg \theta$, there exists N such that for all $n \rangle N$, one has $a_n \leq c$.*

Lemma 2.10. *(see[18])$\mathcal{A}$ be a Banach algebra with a unit e, $x \in \mathcal{A}$, then the limit $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ exist and the spectral radius $\rho(x)$ satisfies:*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = inf_{n\geq 1}\|x^n\|^{\frac{1}{n}} < 1.$$

If $\rho(x) < |\lambda|$, then $\lambda e - x$ is invertible in $\mathcal{A}$, moreover,

$$(\lambda e - x)^{-1} = \sum_{i=0}^{\infty} \frac{x^i}{\lambda^{i+1}}.$$

Lemma 2.11. *(see[18])$\mathcal{A}$ be a Banach algebra with a unit e, $a,b \in \mathcal{A}$.If a commutes with b, then*

$\rho(a+b) \leq \rho(a) + \rho(b)$; $\rho(ab) \leq \rho(a)\rho(b)$.

Lemma 2.12. *$\mathcal{A}$ be a Banach algebra with a unit e, x_n is a sequence in $\mathcal{A}$. If there exist x in $\mathcal{A}$ have $\lim_{n\to\infty} x^n = x$, where x_n commutes with x, for any $n > 0$, then*

$$\lim_{n\to\infty} \rho(x^n) = \rho(x).$$

Proof: by lemma 2.11, we have

$$\rho(x_n) - \rho(x) = \rho(x_n - x + x) - \rho(x) \leq \rho(x_n - x) + \rho(x) - \rho(x) = \rho(x_n - x).$$

$$\|\rho(x_n) - \rho(x)\| \leq \rho(x_n - x) \leq \|x_n - x\|.$$

Because x_n converges to x when $x \to \infty$, so

$$\|\rho(x_n) - \rho(x)\| \to 0 (n \to \infty).$$

that is

$$\rho(x_n) \to \rho(x) (n \to \infty).$$

Lemma 2.13. *$\mathcal{A}$ be a Banach algebra and $x \in \mathcal{A}$. If $\rho(x) \leq 1$, then $\lim_{n\to\infty} \|x^n\| = 0$.*

Proof: Since $\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = inf_{n\geq 1}\|x^n\|^{\frac{1}{n}} < 1$, there exist $a > 0$, such that $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} < a \leq 1$. When n is enough big, we have $\|x^n\|^{\frac{1}{n}} \leq a$, then $\|x^n\| \leq a^n$. because $a < 1$, so $a^n \to 0(n \to \infty)$, then $\lim_{n\to\infty} \|x^n\| = 0$.

3. Main Results

Theorem 3.1. *Let (X,Y) be a cone metric space over Banach algebra $\mathcal{A}$ and $\mathcal{K}$ be a solid cone in $\mathcal{A}$.Suppose that the mappings $F : X \to X$ and $g : X \to X$ satisfies the following contractive condition:*

$$d(F(x,y),F(u,v)) \leq k_1 d(gx,gu) + k_2 d(F(x,y),gx) + k_3 d(gu,gv)$$
$$+k_4 d(F(u,v),gu) + k_5 d(F(u,v),gu) + k_6 d(F(u,v),gx)$$

for all $x,y,u,v \in X$, where $k_i \in K (i = 1,\cdots,6)$ are generalized Lipschitz constants with $\rho(k_1) + \rho(k_3) + \rho(k_2 + k_4 + k_5 + k_6) < 1$, if k_1, k_3 commutes with $k_2 + k_4 + k_5 + k_6$, then there exists two sequence gx_n, gy_n in X such that they are two Cauchy sequence.Moreover, if $d(gx_n, gx_m) + d(gy_n, gy_m)$ converges to some non-zero element in $\mathcal{A}$, for any two different Cauchy sequence gx_n, gy_n, then $\mathcal{A}$ is a non-normal cone.

Proof: Let x_0, y_0 be any two arbitrary in X, set $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$,

$g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$, then we have

$$\begin{aligned}
d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
&\le k_1 d(gx_{n-1}, gx_n) + k_2 d(F(x_{n-1}, y_{n-1}), gx_{n-1}) + k_3 d(gy_{n-1}, gy_n) \\
&+ k_4 d(F(x_n, y_n), gx_n) + k_5 d(F(x_{n-1}, y_{n-1}), gx_n) + k_6 d(F(x_n, y_n), gx_{n-1}) \\
&= k_1 d(gx_{n-1}, gx_n) + k_2 d(gx_n, gx_{n-1}) + k_3 d(gy_{n-1}, gy_n) \\
&+ k_4 d(gx_{n+1}, gx_n) + k_5 d(gx_n, gx_n) + k_6 d(gx_{n+1}, gx_{n-1}) \\
&\le k_1 d(gx_{n-1}, gx_n) + k_2 d(gx_n, gx_{n-1}) + k_3 d(gy_{n-1}, gy_n) \\
&+ k_4 d(gx_{n+1}, gx_n) + k_6 d(gx_{n+1}, gx_n) + k_6 d(gx_n, gx_{n-1}) \\
&= (k_1 + k_2 + k_6) d(gx_{n-1}, gx_n) + k_3 d(gy_{n-1}, gy_n) + (k_4 + k_6) d(gx_n, gx_{n+1}).
\end{aligned}$$

From which it follows

$$(1 - k_4 - k_6) d(gx_n, gx_{n+1}) \le (k_1 + k_2 + k_6) d(gx_{n-1}, gx_n) + k_3 d(gy_{n-1}, gy_n). \tag{3.1}$$

Similarly

$$(1 - k_4 - k_6) d(gy_n, gy_{n+1}) \le (k_1 + k_2 + k_6) d(gy_{n-1}, gy_n) + k_3 d(gx_{n-1}, gx_n). \tag{3.2}$$

We also have

$$\begin{aligned}
d(gx_{n+1}, gx_n) &= d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\
&\le k_1 d(gx_n, gx_{n-1}) + k_2 d(F(x_n, y_n), gx_{n-1}) + k_3 d(gy_n, gy_{n-1}) \\
&+ k_4 d(F(x_{n-1}, y_{n-1}), gx_{n-1}) + k_5 d(F(x_n, y_n), gx_{n-1}) + k_6 d(F(x_{n-1}, y_{n-1}), gx_n) \\
&= k_1 d(gx_n, gx_{n-1}) + k_2 d(gx_{n+1}, gx_n) + k_3 d(gy_n, gy_{n-1}) \\
&+ k_4 d(gx_n, gx_{n-1}) + k_5 d(gx_{n+1}, gx_{n-1}) + k_6 d(gx_n, gx_n) \\
&\le k_1 d(gx_n, gx_{n-1}) + k_2 d(gx_{n+1}, gx_n) + k_3 d(gy_n, gy_{n-1}) \\
&+ k_4 d(gx_n, gx_{n-1}) + k_5 d(gx_{n+1}, gx_n) + k_5 d(gx_n, gx_{n-1}) \\
&= (k_1 + k_4 + k_5) d(gx_n, gx_{n-1}) + k_3 d(gy_n, gy_{n-1}) + (k_2 + k_6) d(gx_{n+1}, gx_n).
\end{aligned}$$

that is

$$(1-k_2-k_5)d(gx_{n+1},gx_n) \le (k_1+k_4+k_5)d(gx_{n-1},gx_n)+k_3d(gy_n,gy_{n-1}). \tag{3.3}$$

Similarly

$$(1-k_2-k_5)d(gy_{n+1},gy_n) \le (k_1+k_4+k_5)d(gy_{n-1},gy_n)+k_3d(gx_n,gx_{n-1}). \tag{3.4}$$

Let $\delta_n = d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})$*, now, from (3.1) and (3.2), we obtain*

$$(1-k_4-k_6)\delta_n \le (k_1+k_2+k_3+k_6)\delta_{n-1}. \tag{3.5}$$

Respectively (3.3) and (3.4)

$$(1-k_2-k_5)\delta_n \le (k_1+k_3+k_4+k_5)\delta_{n-1}. \tag{3.6}$$

So we have

$$(2-k_2-k_4-k_5-k_6)\delta_n \le (2k_1+2k_3+k_2+k_4+k_5+k_6)\delta_{n-1}. \tag{3.7}$$

In (3.7) put $k = k_2+k_4+k_5+k_6$*, then*

$$(2e-k)\delta_n \le (2k_1+2k_3+k)\delta_{n-1}. \tag{3.8}$$

Since $\rho(k) \le \rho(k_1)+\rho(k_3)+\rho(k) < 1 < 2$*, then by Lemma2.10, it follows that* $(2e-k)$ *is invertible.*

Furthermore

$$(2e-k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}.$$

By multiplying in both side of (3.8) by $(2e-k)^{-1}$*, we arrive at*

$$\delta_n \le (2e-k)^{-1}(2k_1+2k_3+k)\delta_{n-1}.$$

Denote $h = (2e-k)^{-1}(2k_1+2k_3+k)$*, then by (3.7)we get*

$$\delta_n \le h\delta_{n-1} \le h^2\delta_{n-2} \le \cdots \le h^n\delta_0.$$

by lemma2.10, we conclude that

$$\rho(\sum_{i=0}^{n} \frac{k^i}{2^{i+1}}) \le \sum_{i=0}^{n} \rho(\frac{k^i}{2^{i+1}}) \le \sum_{i=0}^{n} \frac{[\rho(k)]^i}{2^{i+1}}.$$

which implies by lemma 2.12 that

$$\rho(\sum_{i=0}^{\infty}\frac{k^i}{2^{i+1}}) \leq \sum_{i=0}^{\infty}\frac{[\rho(k)]^i}{2^{i+1}}.$$

Since k_1 commutes with k, it follows that

$$\begin{aligned}(2e-k)^{-1}(2k_1+2k_3+k) &= (\sum_{i=0}^{\infty}\frac{k^i}{2^{i+1}})(2k_1+2k_3+k)\\ &= 2(\sum_{i=0}^{\infty}\frac{k^i}{2^{i+1}})k_1+2(\sum_{i=0}^{\infty}\frac{k^i}{2^{i+1}})k_3+\sum_{i=0}^{\infty}\frac{k^{i+1}}{2^{i+1}}\\ &= (2k_1+2k_3+k)(\sum_{i=0}^{\infty}\frac{k^i}{2^{i+1}})\\ &= (2k_1+2k_3+k)(2e-k)^{-1}.\end{aligned}$$

that is to say, $(2e-k)^{-1}$ commutes with $(2k_1+2k_3+k)$, then by lemma 2.11, we gain

$$\begin{aligned}\rho(h) &= \rho((2e-k)^{-1}(2k_1+2k_3+k))\\ &\leq \rho(\sum_{i=0}^{\infty}\frac{k^i}{2^{i+1}})[2\rho(k_1)+2\rho(k_3)+\rho(k)]\\ &\leq \sum_{i=0}^{\infty}\frac{[\rho(k)]^i}{2^{i+1}}[2\rho(k_1)+2\rho(k_3)+\rho(k)]\\ &= \frac{1}{2-\rho(k)}[2\rho(k_1)+2\rho(k_3)+\rho(k)] < 1.\end{aligned}$$

Which establishes that $e-h$ is invertible and $\|h^n\| \to 0 (n\to\infty)$. We have

$$d(gx_m,gx_n) \leq d(gx_m,gx_{m-1})+d(gx_{m-1},gx_{m-2})+\cdots+d(gx_{n+1},gx_n). \tag{3.9}$$

and

$$d(gy_m,gy_n) \leq d(gy_m,gy_{m-1})+d(gy_{m-1},gy_{m-2})+\cdots+d(gy_{n+1},gy_n). \tag{3.10}$$

Therefore

$$\begin{aligned} d(gx_m, gx_n) + d(gy_m, gy_n) &\leq \delta_{m-1} + \delta_{m-2} + \cdots + \delta_n \\ &\leq (h^{m-1} + h^{m-2} + \cdots + h^n)\delta_0 \\ &= (h^{m-n-1} + h^{m-n-2} + \cdots + h + e)h^n\delta_0 \\ &= (\sum_{i=0}^{\infty} h^i)h^n\delta_0 = (e-h)^{-1}h^n\delta_0. \end{aligned}$$

Owing to

$$\|(e-h)^{-1}h^n\delta_0\| \leq \|(e-h)^{-1}\|\|h^n\|\|\delta_0\|(n \to \infty).$$

We have $(e-h)^{-1}h^n\delta_0 \to 0, (n \to \infty)$*, so by using lemma 2.8, 2.9*

$d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})$ *is a Cauchy sequence. Since* $d(gx_m, gx_n) \leq d(gx_m, gx_n) + d(gy_m, gy_n)$ *and* $d(gy_m, gy_n) \leq d(gx_m, gx_n) + d(gy_m, gy_n)$*, then again by* (p_4)*,* gx_n *and* gy_n *are Cauchy sequences in* $g(X)$*.*

Since gx_n *and* gy_n *are Cauchy sequences, there is* N *such that* $d(gx_n, gx_m) \ll C$ *and* $d(gy_n, gy_m) \ll C$*, for all* $n, m > N$*, it is clear that*

$$d(gx_n, gy_n) \leq d(gx_n, gx_m) + d(gx_m, gy_m) + d(gy_m, gy_n) \leq d(gx_m, gy_m) + 2C.$$

$$d(gx_m, gy_m) \leq d(gx_m, gx_n) + d(gx_n, gy_n) + d(gy_n, gy_m) \leq d(gx_n, gy_n) + 2C.$$

$$d(gx_m, gy_m) + 2C - d(gx_n, gy_n) \leq d(gx_n, gy_n) + 2C + 2C - d(gx_n, gy_n) = 4C.$$

by virtue of the normality of $\mathcal{K}$*, then we have*

$$\|d(gx_m, gy_m) + 2C - d(gx_n, gy_n)\| \leq 4M\|C\|.$$

Hence, it ensures us that

$$\begin{aligned} \|d(gx_m, gy_m) - d(gx_n, gy_n)\| &\leq \|(gx_m, gy_m) + 2C - d(gx_n, gy_n)\| \cdot \|2C\| \\ &\leq (4M+2)\|C\| \leq \epsilon. \end{aligned}$$

Which implies that $d(gx_n, gy_n)$*is Cauchy sequence and hence convergent. Next, set* $\lim_{n\to\infty} d(gx_n, gy_n) = a$*, it is evident that* $0 \leq a$*. Finally, we claim that* $a = 0$*. Actually, if there exists* $n_0 \in N$ *such that*

$x_{n_0} = y_{n_0}$, the claim is clear. Whithout loss of generality, we supposed that $gx_n \neq gy_n$, for all $n \in N$. Notice that

$$\begin{aligned} d(gx_{n+1}, gy_{n+1}) &= d(F(x_n, y_n), F(y_n, x_n)) \\ &\leq k_1 d(gx_n, gy_n) + k_2 d(F(x_n, y_n), gx_n) + k_3 d(gy_n, gx_n) \\ &+ k_4 d(F(y_n, x_n), gy_n) + k_5 d(F(x_n, y_n), gy_n) + k_6 d(F(y_n, x_n), gx_n) \\ &= (k_1 + k_3) d(gx_n, gy_n) + k_2 d(gx_{n+1}, gx_n) + k_4 d(gy_{n+1}, gy_n) \\ &+ k_5 d(gx_{n+1}, gy_{n+1}) + k_5 d(gy_{n+1}, gy_n) + k_6 d(gy_{n+1}, gx_{n+1}) + k_6 d(gx_{n+1}, gx_n) \\ &\leq (k_1 + k_3) d(gx_n, gy_n) + (k_2 + k_6) d(gx_{n+1}, gx_n) + (k_4 + k_5) d(gy_{n+1}, gy_n) \\ &+ (k_5 + k_6) d(gx_{n+1}, gy_{n+1}). \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain that

$$a \leq (k_1 + k_3 + k_5 + k_6) a.$$

set $\lambda = k_1 + k_3 + k_5 + k_6$, then it follows that

$$a \leq \lambda a \leq \lambda^2 a \leq \cdots \leq \lambda^n a.$$

Because $\lambda \leq k_1 + k_3 + k$ lead to $\lambda^n \leq (k_1 + k_3 + k)^n$

moreover, by lemma 2.13, $\rho(k_1 + k_3 + k) \leq \rho(k_1) + \rho(k_3) + \rho(k) < 1$ lead to $(k_1 + k_3 + k)^n \to 0 (n \to \infty)$. We claim that for each $C \gg \theta$, there exists $n_0(C)$ such that $\lambda^n \ll C$, such that for all $n > n_0(C)$.Consequently, $a = \theta$, so we obtain a contradiction. The proof is completed.

Theorem 3.2. *Let (X, Y) be a cone metric space over Banach algebra $\mathcal{A}$ and $\mathcal{K}$ be a solid cone in $\mathcal{A}$.Suppose that the mappings $F : X \to X$ and $g : X \to X$ satisfies the following contractive condition:*

$$\begin{aligned} d(F(x,y), F(u,v)) \leq k_1 d(gx, gu) + k_2 d(F(x,y), gx) + k_3 d(gu, gv) \\ + k_4 d(F(u,v), gu) + k_5 d(F(u,v), gu) + k_6 d(F(u,v), gx) \end{aligned}$$

for all $x, y, u, v \in X$, where $k_i \in K (i = 1, \cdots, 6)$ are generalized Lipschitz constants with $\rho(k_1) + \rho(k_3) + \rho(k_2 + k_4 + k_5 + k_6) < 1$, if k_1, k_3 commutes with $k_2 + k_4 + k_5 + k_6$, then F and g have *a couple coincidence point in X.Moreover, for arbitrary $x,y \in X$, iterative sequence $F^n(x,y)$ converges to the fixed point.Further, F has a property P.*

Proof: by using Theorem 3.1, we known gx_n and gy_n are two Cauchy sequences in $g(X)$, so there exists x and y in X such that $gx_n \to gx$ and $gy_n \to gy$.Now, we prove that $F(x,y) = gx$ and $F(y,x) = gy$.For that we have

$$\begin{aligned}
d(F(x,y),gx) &\leq d(F(x,y),gx_{n+1}) + d(gx_{n+1},gx) = d(F(x,y),F(x_n,y_n)) + d(gx_{n+1},gx) \\
&\leq k_1 d(gx,gx_n) + k_2 d(F(x,y),gx) + k_3 d(gy,gy_n) + k_4 d(F(x_n,y_n),gx_n) \\
&+ k_5 d(F(x,y),gx_n) + k_6 d(F(x_n,y_n),gx) + d(gx_{n+1},gx) \\
&= k_1 d(gx,gx_n) + k_2 d(F(x,y),gx) + k_3 d(gy,gy_n) + k_4 d(gx_{n+1},gx_n) \\
&+ k_5 d(F(x,y),gx_n) + k_6 d(gx_{n+1},gx) + d(gx_{n+1},gx) \\
&\leq k_1 d(gx,gx_n) + k_2 d(F(x,y),gx) + k_3 d(gy,gy_n) + k_4 d(gx_{n+1},gx) + k_4 d(gx,gx_n) \\
&+ k_5 d(F(x,y),gx) + k_5 d(gx,gx_n) + k_6 d(gx_{n+1},gx) + d(gx_{n+1},gx).
\end{aligned}$$

which implies that

$$\begin{aligned}
(3.11) \quad (e-k_2-k_5)d(F(x,y),gx) &\leq (k_1+k_4+k_5)d(gx_n,gx) + (e+k_4+k_6)d(gx_{n+1},gx) \\
&+ k_3 d(gy_n,gy).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
d(F(x,y),gx) &\leq d(gx_{n+1},F(x,y)) + d(gx_{n+1},gx) = d(F(x_n,y_n),F(x,y)) + d(gx_{n+1},gx) \\
&\leq k_1 d(gx_n,gx) + k_2 d(F(x_n,y_n),gx_n) + k_3 d(gy_n,gy) + k_4 d(F(x,y),gx) \\
&+ k_5 d(F(x_n,y_n),gx) + k_6 d(F(x,y),gx_n) + d(gx_{n+1},gx) \\
&= k_1 d(gx_n,gx) + k_2 d(gx_{n+1},gx_n) + k_3 d(gy_n,gy) + k_4 d(F(x,y),gx) \\
&+ k_5 d(gx_{n+1},gx) + k_6 d(F(x,y),gx_n) + d(gx_{n+1},gx) \\
&\leq k_1 d(gx_n,gx) + k_2 d(gx_{n+1},gx) + k_2 d(gx,gx_n) + k_3 d(gy_n,gy) + k_4 d(F(x,y),gx) \\
&+ k_5 d(gx_{n+1},gx) + k_6 d(F(x,y),gx) + k_6 d(gx,gx_n) + d(gx_{n+1},gx).
\end{aligned}$$

which implies that

$$(3.12)\quad (e-k_4-k_6)d(F(x,y),gx) \leq (k_1+k_2+k_6)d(gx_n,gx)+(e+k_2+k_5)d(gx_{n+1},gx) + k_3 d(gy_n,gy).$$

Combining (3.11) and (3.12) yields that

$$(3.13)\quad (2e-k_2-k_4-k_5-k_6)d(F(x,y),gx) \leq (2k_1+k_2+k_4+k_5+k_6)d(gx_n,gx) + (2e+k_2+k_4++k_5+k_6)d(gx_{n+1},gx)+2k_3 d(gy_n,gy).$$

Put $k = k_2+k_4++k_5+k_6$*, then*

$$(2e-k)d(F(x,y),gx) \leq (2k_1+k)d(gx_n,gx)+(2e+k)d(gx_{n+1},gx)+2k_3 d(gy_n,gy).$$

Consequently, we obtain that

$$d(F(x,y),gx) \leq (2e-k)^{-1}(2k_1+k)d(gx_n,gx)+(2e-k)^{-1}(2e+k)d(gx_{n+1},gx) + (2e-k)^{-1}2k_3 d(gy_n,gy).$$

In view of $gx_n \to gx, gy_n \to gy(n\to\infty)$*, then for each* $C \gg \theta$*, there exists* $N_m,(m=1,2,3)$ *such that for all* $n > N_m$ *have* $d(gx_n,gx) \ll \frac{(2e-k)C}{3(2k_1+k)}, d(gx_{n+1},gx) \ll \frac{(2e-k)C}{3(2e+k)}$ *and* $d(gx_{n+1},gx) \ll \frac{(2e-k)C}{3(2k_3)}$*, for all* $n > minN_1,N_2,N_3$*, thus*

$$d(F(x,y),gx) \leq \frac{C}{3}+\frac{C}{3}+\frac{C}{3} = C.$$

It follows that $d(F(x,y),gx)=\theta$*, and hence* $F(x,y)=gx$*.Similarly,* $F(y,x)=gy$*.Hence* (x,y) *is coupled coincidence point of the mappings* F *and* g*. In the following we shall show the couple coincidence point is unique.Suppose that* $(x,y),(x^*,y^*)\in X\times X$ *with* $g(x)=F(x,y), g(y)=F(y,x)$ *and* $g(x^*)=F(x^*,y^*), g(y^*)=F(y^*,x^*)$*, then*

$$\begin{aligned} d(gx,gx^*) &= d(F(x,y),F(x^*,y^*)) \\ &\leq k_1 d(gx,gx^*)+k_2 d(F(x,y),gx)+k_3 d(gy,gy^*) \\ &+ k_4 d(F(x^*,y^*),gx^*)+k_5 d(F(x,y),gx^*)+k_6 d(F(x^*,y^*),gx) \\ &= (k_1+k_5+k_6)d(gx,gx^*)+k_3 d(gy,gy^*). \end{aligned}$$

similarly

$$d(gy,gy^*) \leq (k_1+k_5+k_6)d(gy,gy^*)+k_3d(gx,gx^*).$$

Thus

$$d(gx,gx^*)+d(gy,gy^*) \leq (k_1+k_3+k_5+k_6)[d(gx,gx^*)+d(gy,gy^*)].$$

Denote $h = k_1+k_3+k_5+k_6$, *then*

$$(3.14)\quad d(gx,gx^*)+d(gy,gy^*) \leq h[d(gx,gx^*)+d(gy,gy^*)] \leq \cdots \leq h^n d(gx,gx^*)+d(gy,gy^*).$$

By the proof of Theorem 3.1, we claim that, for each $c \gg \theta$, *there exists* N *such that* $h^n \ll c$ *for all* $n > N$. *Consequently,* $d(gx,gx^*)+d(gy,gy^*) = \theta$, *that is* $x = x^*, y = y^*$.

Theorem 3.3. *Let* (X,Y) *be a cone metric space over Banach algebra* $\mathcal{A}$ *and* $\mathcal{K}$ *be a solid cone in* $\mathcal{A}$*.Suppose that the mappings* $F: X \to X$ *and* $g: X \to X$ *satisfies the following contractive condition:*

$$d(F(x,y),F(u,v)) \leq k_1d(gx,gu)+k_2d(F(x,y),gx)+k_3d(gu,gv)$$
$$+k_4d(F(u,v),gu)+k_5d(F(u,v),gu)+k_6d(F(u,v),gx)$$

for all $x,y,u,v \in X$, *where* $k_i \in K(i = 1,\cdots,6)$ *are generalized Lipschitz constants with* $\rho(k_1)+\rho(k_3)+\rho(k_2+k_4+k_5+k_6) < 1$, *if* k_1,k_3 *commutes with* $k_2+k_4+k_5+k_6$, *then Picards iteration is T-stable.*

Proof:by utilizing Theorem 3.1 and 3.2, we obtain that F *and* g *have a couple coincidence point* u *and* v *in* X*.Assume that* gx_n *satisfies the following condition:for each* $c \gg \theta$, *there exists*

N such that for all $n > N$, $d(F) \ll c$. *we have*

$$\begin{aligned}
d(gx_{n+1},gx) &= d(F(x_n,y_n),F(x,y)) \\
&\leq k_1d(gx_n,gx)+k_2d(F(x_n,y_n),gx_n)+k_3d(gy_n,gy) \\
&+ k_4d(F(x,y),gx)+k_5d(F(x_n,y_n),gx)+k_6d(F(x,y),gx_n) \\
&= k_1d(gx_n,gx)+k_2d(gx_{n+1},gx_n)+k_3d(gy_n,gy) \\
&+ k_4d(gx,gx)+k_5d(gx_{n+1},gx)+k_6d(gx,gx_n) \\
&\leq k_1d(gx_n,gx)+k_2d(gx_{n+1},gx)+k_2d(gx,gx_n)+k_3d(gy_n,gy) \\
&+ k_4d(gx,gx)+k_5d(gx_{n+1},gx)+k_6d(gx,gx_n) \\
&\leq (k_1+k_2+k_6)d(gx_n,gx)+(k_2+k_5)d(gx_{n+1},gx)+k_3d(gy,gy_n).
\end{aligned}$$

from which it follows

$$(e-k_2-k_5)d(gx_{n+1},gx) \leq (k_1+k_2+k_6)d(gx_n,gx)+k_3d(gy_n,gy). \tag{3.15}$$

Similarly,

$$\begin{aligned}
d(gx_{n+1},gx) &= d(F(x,y),F(x_n,y_n)) \\
&\leq k_1d(gx,gx_n)+k_2d(F(x,y),gx)+k_3d(gy,gy_n) \\
&+ k_4d(F(x_n,y_n),gx)+k_5d(F(x,y),gx_n)+k_6d(F(x_n,y_n),gx) \\
&= k_1d(gx,gx_n)+k_2d(gx,gx)+k_3d(gy,gy_n) \\
&+ k_4d(gx_{n+1},gx_n)+k_5d(gx,gx_n)+k_6d(gx_{n+1}),gx) \\
&\leq k_1d(gx,gx_n)+k_3d(gy,gy_n)+k_4d(gx_{n+1},gx) \\
&+ k_4d(gx,gx_n)+k_5d(gx,gx_n)+k_6d(gx_{n+1}),gx) \\
&\leq (k_1+k_4+k_5)d(gx_n,gx)+(k_4+k_6)d(gx_{n+1},gx)+k_3d(gy,gy_n).
\end{aligned}$$

from which it follows

$$(e-k_4-k_6)d(gx_{n+1},gx) \leq (k_1+k_4+k_5)d(gx_n,gx)+k_3d(gy_n,gy). \tag{3.16}$$

Add up (3.15) and (3.16) yields that

$$(2e-k_2-k_5-k_4-k_6)d(gx_{n+1},gx) \leq (2k_1+k_4+k_5+k_2+k_6)d(gx_n,gx)+2k_3d(gy_n,gy). \tag{3.17}$$

Put $k=k_4+k_5+k_2+k_6$, *then*

$$(2e-k)d(gx_{n+1},gx) \leq (2k_1+k)d(gx_n,gx)+2k_3d(gy_n,gy)$$

Based on the proof of Theorem 3.1, it is not hard to obtain that

$$d(gx_{n+1},gx) \leq hd(gx_n,gx)+2md(gy_n,gy).$$

where $h=(2e-k)^{-1}(2k_1+k), m=(2e-k)^{-1}k_3$, *and* $\rho(h)<1$ *Setting* $a_n=d(gx_{n+1},gx), c_n=d(gy_n,gy)$, *we can obtain that*

$$a_{n+1} \leq ha_n+mc_n$$

for each $\frac{c}{m} \gg \theta$, *there exists* N *such that for all* $n>N$, $c_n=d(gy_n,gy) \ll \frac{c}{m}$. *Then making the Lemma 2.10, we have* $a_n \ll c$.*That is to proof, the iteration is T-stable.The proof is over.*

Corollary 3.4. *Let* (X,d) *be a cone metric space over Banach algebra* $\mathcal{A}$ *and* $\mathcal{K}$ *be a solid cone in* $\mathcal{A}$.*Suppose that the mappings* $F:X\to X$ *and* $g:X\to X$ *satisfies the following contractive condition:*

$$\begin{aligned} d(F(x,y),F(u,v)) &\leq \alpha[d(gx,gu)+d(F(x,y),gx)]+\beta[d(gu,gv)+d(F(u,v),gu)] \\ &+ \gamma[d(F(u,v),gu)+d(F(u,v),gx)] \end{aligned}$$

for all $x,y,u,v\in X$, *where* α,β,γ *are generalized Lipschitz constants, then then* F *and* g *have a unique couple coincidence point in* X.*Moreover, for arbitrary* $x,y\in X$, *iterative sequence* $F^n(x,y)$ *converges to the fixed point.Further, the iteration sequence is T-stable.*

Corollary 3.5. *Let* (X,Y) *be a cone metric space over Banach algebra* $\mathcal{A}$ *and* $\mathcal{K}$ *be a solid cone in* $\mathcal{A}$.*Suppose that the mappings* $F:X\to X$ *and* $g:X\to X$ *satisfies the following contractive condition:*

$$d(F(x,y),F(u,v)) \leq \alpha d(F(x,y),x)+\beta d(F(u,v),u)$$

for all $x, y, u, v \in X$, where α, β are generalized Lipschitz constants with $\rho(\alpha) + \rho(\beta) < 1$, then then F and g have a unique couple coincidence point in X.Moreover, for arbitrary $x, y \in X$, iterative sequence $F^n(x, y)$ converges to the fixed point.Further, the iteration sequence is T-stable.

Competing Interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgment

The second author was supported by the Natural Science Foundation of Shandong Province of China (ZR2017LA001) and Youth Foundation of Linyi University (LYDX2016BS023).

REFERENCES

[1] L.-G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contrative mappings, J. Math. Anal. Appl., 332 (2007), 1468-1476.

[2] H. Liu, S.-Y. Xu, Cone metric spaces over Banach algebras and fixed point theorems of generalized lipschitz mappings, Fixed Point Theory Appl., 2013 (2013), Article ID 320.

[3] X. Xu, D. Jiang, W. Hu, D. O'Regan, R. P. Agarwal, Positive properties of Green's function for three-point boundary value problems of nonlinear fractional differential equations and its applications, Appl. Anal., 91(2012), 323-343.

[4] T.G. Bhashkar, V.lakshmikantham, Fixed point theorems in partially ordered cone metric spaces and applications. Nonliear Anal., TMA, 65 (2006), 825-832.

[5] M.Abbas, M.Ali Khan, S.Radenovic, Common coupled fixed point theorems in cone metric spaces for compatible mappings, Appl. Math. Comput., 217 (2010), 195-202.

[6] F. Sabetghadam, H.P. Masiha, A.H.Sanatpour, Some coupled fixed point theorems in cone metric space, Fixed Point Theory Appl., 2009 (2009), Article ID 125426, 8 pages.

[7] H. Liu, S.-Y. Xu, Fixed point theorems of quasi-contractions on cone metric spaces with Banach algebras, Abstr. Appl. anal., 2013 (2013), Article ID187348, 5 pages.

[8] H. Liu, S.-Y. Xu, Cone metric spaces over Banach algebras and fixed point theorems of generalized Lipschitz mappins, Fixed Point Theory Appl., 2013 (2013), Article ID 320.

[9] W.-S. Du, E. Karapimar, A note on cone b-metric and its related results:generalizations or equivalence?, Fixed Point Theory Appl., 2013 (2013), Article ID 210.

[10] Z. Kadelburg, S. Radenović, V.Rakocević, A note on equivalence of some metric and cone metric fixed point results, Appl. Math. Lett., 24 (2011), 370-374.

[11] M. Asadi, B.E. Rhoades, H.Soleimani, Some notes on the paper ”The equivalence of cone metric and metric spaces, Fixed Point Theory Appl., 2012 (2012), Article ID 87.

[12] Z. Ercan, On the end of the cone metric spaces,, Topology Appl., 166 (2014), 10-14.

[13] Y. Qing, B.E. Rhoades, T-stability of Picard iteration in metric spaces, Fixed Point Theory Appl., 2008 (2008), Article ID 418971, 4 pages.

[14] M. Asadi, H. Soleimani, S.M. Vaezpour, B.E. Rhoades, On T-stability of Picard iteration in cone metric spaces, Fixed Point Theory Appl., 2009 (2009), Article ID 751090, 6 pages.

[15] M. Arshad, A. Azam, P. Vetro, Some common fixed point results in cone metric spaces, Fixed Point Theory Appl., 2009 (2009), Article ID 493965, 11 pages.

[16] V. Lakshmikantham, Lj. Ćirić, Couple fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal.: TMA, 70(2009), 4341-4349.

[17] S.-Y. Xu, S. Radenović, Fixed point theorems for generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality, Fixed Point Theory Appl., 2014 (2014), Article ID 102.

[18] W. Rudin, Fuctional Analysis, McGraw-Hill, New York, 2nd edn, 1991.

[19] S. Janković, Z. Kadelburg, S. Radenović, On cone metric spaces: A survey, Nonlinear Anal., 74 (2011), 2591-2601.