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ASTEROIDAL NUMBER FOR PRODUCTS OF PATHS

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Abstract. In this paper, we find asteroidal numbers for Direct product $P_m \times P_n$ and Cartesian product $P_m \circ P_n$ of paths, considering all possible cases for m and n.

Keywords: asteroidal number; asteroidal sets; independence number; cartesian product; direct product.

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1.Introduction

Like studying subgraphs of graphs, study of vertex subsets with specified properties and parameters evolving out of these subsets plays a crucial role in the study of graphs. Domination numbers related to various domination concepts, independence number, chromatic number are few examples in this context. The notion of Asteroidal triples was introduced by Lekkerkerker and Boland [4]. Walter [6] generalized the concept of asteroidal triples to asteroidal sets. More investigation was carried out by Haiko Muller [3]. Hence we have yet another parameter - Asteroidal number. Asteroidal number of a graph is the maximum cardinality of an asteroidal

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set in the graph. The asteroidal number for some elementary graphs may be easily observed and are listed below [3]:

1. Path P_n : 2
2. Cycle C_n : $\lfloor \frac{n}{2} \rfloor$
3. Complete graph K_n : 1
4. Complete bipartite graph $K_{m,n}$: 2 where $max\{m,n\} \geq 2$

In this connection, we investigate direct product and cartesian product of paths.

2.Preliminaries

Definition 2.1 The **neighbourhood** of a vertex u in a graph G is the set $N(u)$ consisting of all vertices v which are adjacent with u. The closed neighbourhood is $N[u] = N(u) \cup \{u\}$.

Though we don't discuss asteroidal triples here, we give the definition just to show how this naturally paved the way for asteroidal sets. The following definition is from [4]. D.G.Corneil and others [1], Ekkehard Kohler [2] further studied asteroidal triples.

Definition 2.2 An **asteroidal triple** of a graph $G = (V,E)$ is a set of three vertices, such that there exists a path between any two of them avoiding the neighbourhood of the third. Graphs without asteroidal triples are called asteroidal triple free graphs.

The following generalization of asteroidal triples to asteroidal sets was given by J.R.Walter. [6]

Definition 2.3 Let $G = (V,E)$ be a graph. A subset A of vertices is called an **asteroidal set** of the graph G if for each vertex $a \in A$ all elements of A$\setminus \{a\}$ are contained in the same connected component of G-N$[a]$.

Definition 2.4 The **asteroidal number** of G denoted by $an(G)$ is defined as the maximum cardinality of an asteroidal set in G.

Definition 2.5 Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs. The **Cartesian product** of G and H, denoted by $G \circ H$, has $V(G \circ H) = \{(g,h)/g \in G; h \in H\}$ and $E(G \circ H) = \{(g_1,h_1)(g_2,h_2)/g_1 = g_2, h_1h_2 \in E(H) \ or \ g_1g_2 \in E(G), h_1 = h_2\}$. [5]

Definition 2.6 Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs. The **Direct product** of G and H, denoted by $G \times H$, has $V(G \times H) = \{(g,h)/g \in G; h \in H\}$ as the vertex set and $E(G \times H) = \{(g_1,h_1)(g_2,h_2)/g_1g_2 \in E(G) \textit{ and } h_1h_2 \in E(H)\}$. [5]

Definition 2.7 Let $G = (V, E)$ be a graph. A subset S of V is called an **independent set** if there is no edge between any two vertices of S.

Definition 2.8 The cardinality of a maximum independent set in a graph G is called **independence number** of G and is denoted by $\beta_0(G)$ or simply β_0.

Definition 2.9 Let $V = \{x_1, x_2, \ldots x_n\}$ and $E = \{(x_i, x_{i+1})/0 \leq i \leq n-1\}$. The graph (V, E) is called a **path** on n vertices and is denoted by P_n.

The following result is from [3].

Lemma 2.10 For a disconnected graph G, an(G) = $max\{an(G[C]) : C \in Comp(G)\}$, where Comp($G$) denotes the set of all components of G.

Some conventions : While drawing the product of paths P_m and P_n,when both m and n are either even or odd, the graph should be drawn in such a way that the number of columns is greater than the number of rows. When one of them is even and the other is odd, we draw the graph such that it has even number of columns and odd number of rows. Also vertices are labeled in such a way that (i, j) denotes the vertex in the i^{th} column , j^{th} row.

3. Main results

First we take up asteroidal number for direct product of paths.

Theorem 3.1. For $m, n \in N$,

$$an(P_m \times P_n) = \begin{cases} kl & if\ m = 2k+1\ ,\ n = 2l+1\ ;\ m,n \geq 5 \\ (k-1)(l-1)+2 & if\ m = 2k\ ,\ n = 2l\ ;\ m,n \geq 6 \\ an(P_{m-1} \times P_n)+1 & if\ m = 2k\ ,\ n = 2l+1\ ;\ m \geq 6\ n \geq 5 \end{cases}$$

Proof. $P_m \times P_n$ has two components. Let G_1 be the component containing (1,1) and G_2 be the other component. We first construct a maximum asteroidal set Γ for G_1 , considering each case.

Case(i): $m, n - odd$; $m = 2k+1$, $n = 2l+1$

We construct Γ as follows:

Leaving the pendent vertices, collect all the corner vertices in G_1 for Γ. By corner vertices, we mean those vertices having 1,m or n as one of the co-ordinates. Maximum number of vertices that can be chosen from the first row is $\lfloor \frac{m-2}{2} \rfloor$, given by

$\{(3,1),(5,1),\ldots(m-2,1)\} = A$, say.

Maximum number of vertices that can be chosen from the first column is $\lfloor \frac{n-2}{2} \rfloor$, given by

$\{(1,3),(1,5),\ldots(1,n-2)\}$.

Maximum number of vertices that can be chosen from the last row is $\lfloor \frac{m-2}{2} \rfloor$, given by

$\{(3,n),(5,n),\ldots(m-2,n)\}$.

Maximum number of vertices that can be chosen from the last column is $\lfloor \frac{n-2}{2} \rfloor$, given by

$\{(m,3),(m,5),\ldots(m,n-2)\}$.

$$\begin{aligned} Thus,\ number\ of\ corner\ vertices\ in\ \Gamma &= 2\left\lfloor \frac{m-2}{2} \right\rfloor + 2\left\lfloor \frac{n-2}{2} \right\rfloor \\ &= m+n-6 \end{aligned}$$

None of the second row vertices of G_1 can be chosen for Γ since they are adjacent with those first row vertices lying in Γ. Apart from (1,3) and (m,3), none of the third row vertices of G_1 can be chosen for Γ, since removal of any other third row vertex and its neighbourhood will leave an element of A alone. The vertices $(4,4),(6,4)\ldots(m-3,4)$ from the fourth row can be added to the set Γ. Since each vertex in fifth row of G_1 is either in Γ or adjacent to a vertex in Γ, we skip the fifth row. The vertices $(4,6),(6,6),\ldots(m-3,6)$ from the sixth row can be added to the set Γ. Skip the seventh row for the same reason as for fifth. Continue this process till $(n-3)^{rd}$ row. Since each vertex in $(n-2)^{th}$ row of G_1 is either in Γ or adjacent to a vertex in Γ,we leave $(n-2)^{th}$ row. Vertices of $(n-1)^{th}$ row in G_1 are adjacent to those n^{th} row vertices lying in Γ and hence we leave out $(n-1)^{th}$ row too. Note that, always first three rows, first three columns, last three rows, last three columns are omitted while choosing non-corner vertices for Γ. The illustration for Case(i) is given in Fig.1

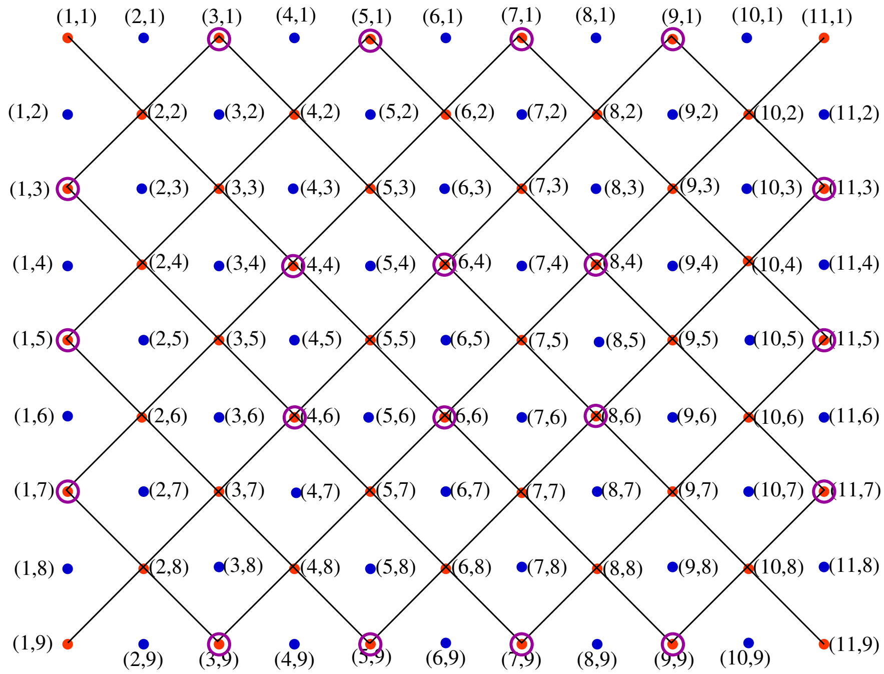

Fig. 1

Therefore, the set of non-corner vertices is

$$\begin{aligned}
&\{(4,4),(6,4),\ldots(m-3,4),\\
&\ (4,6),(6,6),\ldots(m-3,6),\\
&\ (4,8),(6,8),\ldots(m-3,8),\\
&\ \vdots\\
&\ (4,n-3),(6,n-3),\ldots(m-3,n-3)\}
\end{aligned}$$

$$\begin{aligned}
Number\ of\ non-corner\ vertices\ in\ \Gamma &= \left\lceil\frac{m-6}{2}\right\rceil\left\lceil\frac{n-6}{2}\right\rceil\\
&= \left(\frac{m-5}{2}\right)\left(\frac{n-5}{2}\right)\\
&= \frac{mn-5m-5n+25}{4}
\end{aligned}$$

Hence,

$$\begin{aligned}|\Gamma| &= m+n-6+\frac{mn-5m-5n+25}{4} \\ &= \frac{(m-1)}{2}\frac{(n-1)}{2} \\ &= kl\end{aligned}$$

Case(ii): $m,n-even$; $m=2k$, $n=2l$

Leaving the pendent vertices, collect all the corner vertices in G_1 from the first and last row for Γ.

Maximum number of vertices that can be chosen from the first row is $\left(\frac{n}{2}-1\right)$, given by $\{(3,1),(5,1),\dots(n-1,1)\}$.

Maximum number of vertices that can be chosen from the last row is $\left(\frac{n}{2}\right)$, given by $\{(2,m),(4,m),\dots(n-2,m)\}$.

Again, leaving the pendent vertices, collect those corner vertices in G_1 from the first and last columns, which are not adjacent to the already chosen ones for Γ.

Maximum number of vertices that can be chosen from the first column is $\left(\frac{m}{2}-1\right)$, given by $\{(1,3),(1,5),\dots(1,m-3)\}$.

Maximum number of vertices that can be chosen from the last column is $\left(\frac{m}{2}-1\right)$, given by $\{(n,4),(n,6),\dots(n,m-2)\}$.

$$\begin{aligned}Thus,\ number\ of\ corner\ vertices\ in\ \Gamma &= 2\left(\frac{n}{2}-1\right)+2\left(\frac{m}{2}-1\right) \\ &= m+n-6\end{aligned}$$

The procedure to choose non-corner vertices for Γ is the same as it is for the previous case, but for some minor changes. Here, first three rows, first three columns, last four rows and last four columns are omitted while choosing non-corner vertices for Γ. Thus the procedure continues till $(n-4)^{th}$ row. The illustration for this case is given in Fig.2.

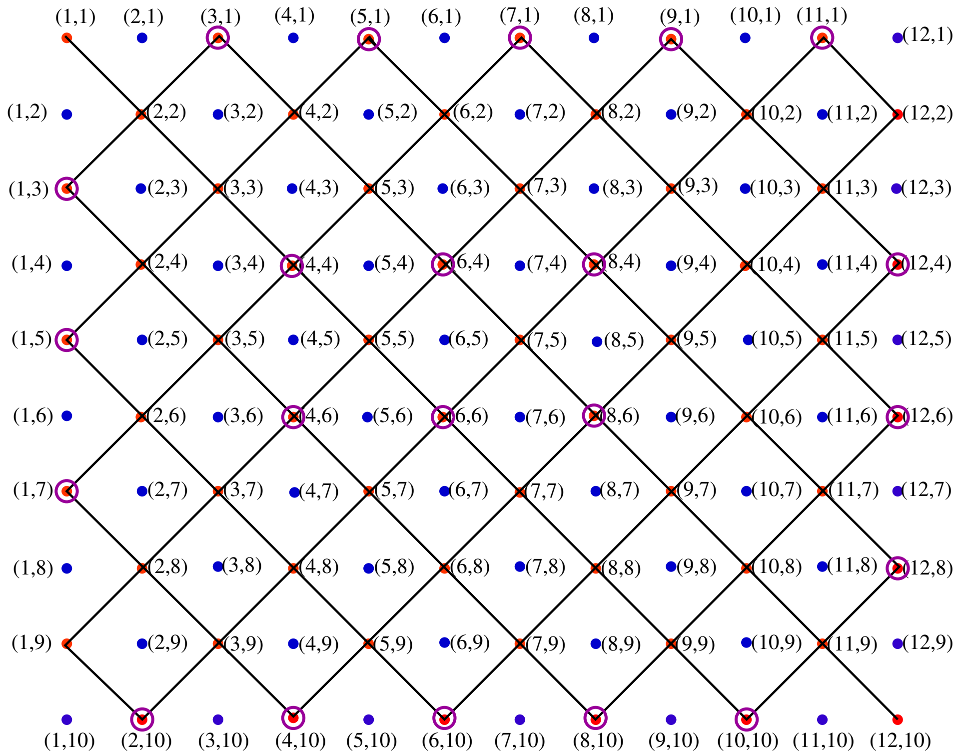

Fig. 2

Therefore, the set of non-corner vertices is

$$\begin{aligned}
&\{(4,4),(6,4),\dots(n-4,4),\\
&(4,6),(6,6),\dots(n-4,6),\\
&(4,8),(6,8),\dots(n-4,8),\\
&\vdots\\
&(4,m-4),(6,m-4),\dots(n-4,m-4)\}
\end{aligned}$$

$$\begin{aligned}
\mathit{Number\ of\ non-corner\ vertices\ in}\ \Gamma &= \left\lceil \frac{m-7}{2} \right\rceil \left\lceil \frac{n-7}{2} \right\rceil \\
&= \frac{mn-6m-6n+36}{4}
\end{aligned}$$

Hence,

$$\begin{aligned}|\Gamma| &= m+n-6+\frac{mn-6m-6n+36}{4}\\ &= \left(\frac{m}{2}-1\right)\left(\frac{n}{2}-1\right)+2\\ &= (k-1)(l-1)+2\end{aligned}$$

In Γ, the combinations of the replacements of $(2,m),(n-1,1)$ and non-corner vertices by $(1,m-1),(n,2)$ and

$$\begin{aligned}&\{(5,5),(7,5),\ldots(n-3,5),\\ &(5,7),(7,7),\ldots(n-3,7),\\ &(5,9),(7,9),\ldots(3,9),\\ &\vdots\\ &(5,m-3),(7,m-3),\ldots(n-3,m-3)\}\end{aligned}$$

correspondingly, give seven new maximum asteroidal sets excluding the one given above. The number of rows and columns to be omitted for choosing the non-corner vertices is the same but for the change that they change places.

Case(iii): $m-even$, $n-odd$; $m=2k$, $n=2l+1$

Procedure to choose corner vertices for Γ is the same as that for case(ii).

Maximum number of vertices that can be chosen from the first row is $\lfloor\frac{n}{2}\rfloor-1$, given by $\{(3,1),(5,1),\ldots(m-1,1)\}$.

Maximum number of vertices that can be chosen from the last row is $\lfloor\frac{n}{2}\rfloor$, given by $\{(3,n),(5,n),\ldots(m-1,n)\}$.

Again, leaving the pendent vertices, collect those corner vertices in G_1 from the first and last columns, which are not adjacent to the already chosen ones for Γ.

Maximum number of vertices that can be chosen from the first column is $\left(\frac{m}{2}-2\right)$, given by $\{(1,3),(1,5),\ldots(1,n-2)\}$.

Maximum number of vertices that can be chosen from the last column is $\left(\frac{m}{2}-2\right)$, given by

$\{(m,4),(m,6),\ldots(m,n-3)\}$.

$$Number\ of\ corner\ vertices\ in\ \Gamma = 2\left(\frac{m}{2}-2\right)+2\left\lfloor\frac{n}{2}\right\rfloor-1$$
$$= m+n-6$$

Procedure to choose non-corner vertices is the same as that of case (i) but for some minor changes. Here, first three rows, first three columns, last three rows and last four columns are omitted while choosing non-corner vertices for Γ. Thus, procedure continues till $(m-4)^{th}$ row. The illustration for this case is given in Fig.3.

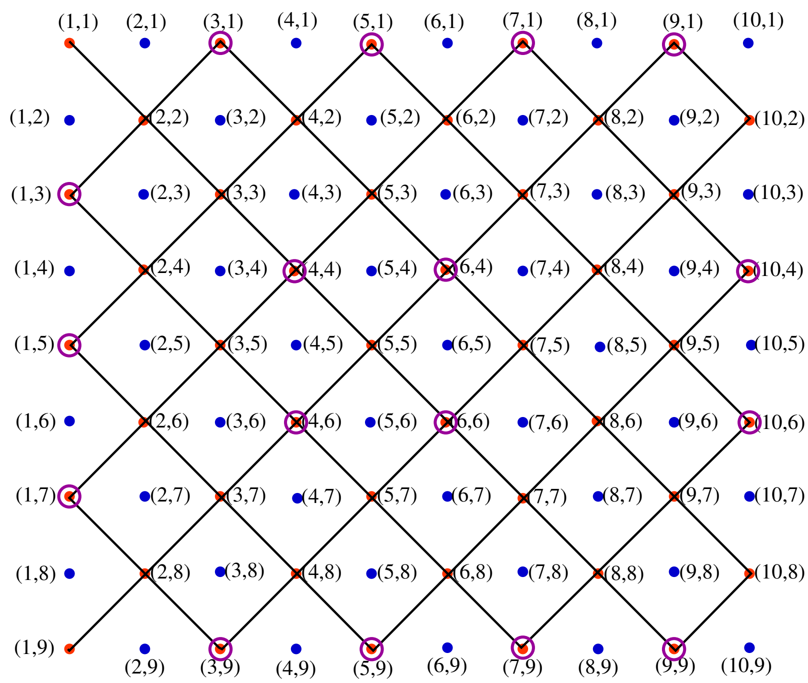

Fig. 3

The set of non-corner vertices is

$$\begin{aligned}&\{(4,4),(6,4),\ldots(m-4,4),\\ &(4,6),(6,6),\ldots(m-4,6),\\ &(4,8),(6,8),\ldots(n-3,8),\\ &\vdots\\ &(4,n-3),(6,n-3),\ldots(m-4,n-3)\}\end{aligned}$$

$$
\begin{aligned}
Number\ of\ non-corner\ vertices\ in\ \Gamma &= \left\lceil \frac{m-7}{2} \right\rceil \left\lceil \frac{n-6}{2} \right\rceil \\
&= \left(\frac{m-6}{2}\right)\left(\frac{n-5}{2}\right) \\
&= \frac{mn-6n-5m+30}{4}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|\Gamma| &= m+n-6+\frac{mn-6n-5m+30}{4} \\
&= \left(\frac{m}{2}-1\right)\left(\frac{n-1}{2}\right)+1 \\
&= (k-1)l+1 \\
&= an\left(P_{m-1}\times P_n\right)+1
\end{aligned}
$$

In Γ, the combinations of the replacements of $(m-1,n),(m-1,1)$ by $(m,2),(m,n-1)$ correspondingly, give three new maximum asteroidal sets excluding the one given above.

In case(ii) and case(iii), G_2 is the upside down image of G_1 of the respective cases. Therefore, an(G_1)=an(G_2) for case (ii) and case (iii). Hence by Lemma 2.10, the theorem is proved for case(ii) and case (iii). Now consider G_2 for case(i).

We construct a maximum asteroidal set X for G_2 using the same procedure as that of case(i). Since there is no pendent vertex, collect all the corner vertices such that they form an independent set. Non-corner vertices are also chosen by the same procedure. Here first three and last three rows, first four and last four columns are omitted while choosing non-corner vertices for X. See fig.4 for illustration.

$$
\begin{aligned}
Number\ of\ corner\ vertices\ in\ X &= 2\left\lfloor \frac{n}{2} \right\rfloor + 2\left(\left\lfloor \frac{m}{2} \right\rfloor - 2\right) \\
&= 2\left(\frac{n-1}{2}\right) + 2\left(\frac{m-1}{2}-2\right) \\
&= m+n-6
\end{aligned}
$$

$$Number\ of\ non-corner\ vertices\ in\ X = \left\lceil \frac{m-6}{2} \right\rceil \left\lceil \frac{n-8}{2} \right\rceil$$
$$= \frac{mn-5n-7m+35}{4}$$

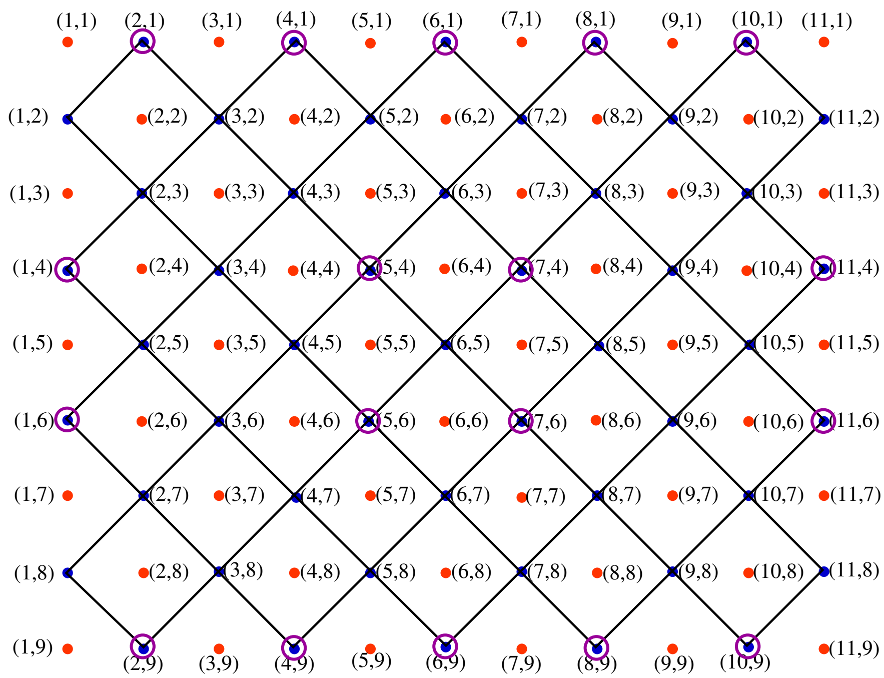

Fig. 4

$$|X| = m+n-6+\frac{mn-5n-7m+35}{4}$$
$$= \left(\frac{m-1}{2}\right)\left(\frac{n-1}{2}-1\right)+2$$
$$= k(l-1)+2$$

Therefore, $|X| \leq |\Gamma|$

Hence, by Lemma 2.10, an$(P_m \times P_n) = |\Gamma| = kl$ for m,n - odd.

Now we study asteroidal number for Cartesian product of paths.

Theorem 3.2 $an(P_m \circ P_n) = \begin{cases} \beta_0 - 1 & if\ m,n\ are\ odd \\ \beta_0 - 2 & if\ at\ least\ one\ of\ them\ is\ even \end{cases}$

Proof. Starting with (1,1) and choosing all the alternate vertices we get the maximum independent set. Thus $\beta_0(P_m \circ P_n) = \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil + \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor$. Starting with (2,1) and choosing all the alternate vertices along each column and row, we get the maximum asteroidal set Γ. The reason for not starting with (1,1) is that while removing (2,2) and its neighbourhood, (1,1) will become isolated.

Case(i): m,n-odd.

The maximum asteroidal set contains $\lfloor \frac{m}{2} \rfloor$ vertices from $\lceil \frac{n}{2} \rceil$ rows (from first, third, fifth, $\ldots n^{th}$ row) and $\lceil \frac{m}{2} \rceil$ vertices from $\lfloor \frac{n}{2} \rfloor$ rows (from second, fourth, sixth, $\ldots (n-1)^{th}$ row). For illustration, see Fig.5

$$\begin{aligned}
|\Gamma| &= \left\lfloor \frac{m}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor \\
&= \left(\frac{m-1}{2}\right)\left(\frac{n+1}{2}\right) + \left(\frac{m+1}{2}\right)\left(\frac{n-1}{2}\right) \\
&= \frac{mn+1}{2} - 1 \\
&= \beta_0 - 1
\end{aligned}$$

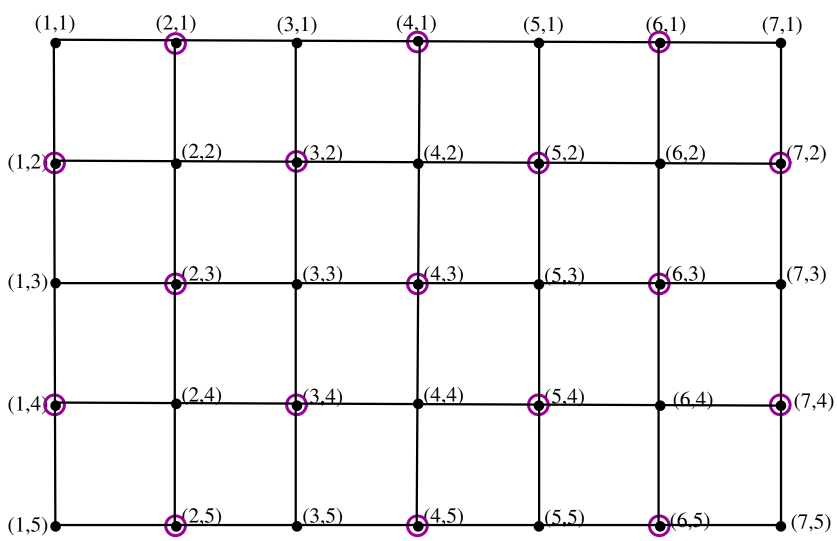

Fig. 5

Case(ii) : At least one of m,n is even.

Subcase(i) : Both m,n are even.

Let $n \leq m.$ Starting with (2,1) choose all the alternate vertices along each row and each column except $(1,n)$ *and* $(m,1)$. See Fig.6 for illustration.

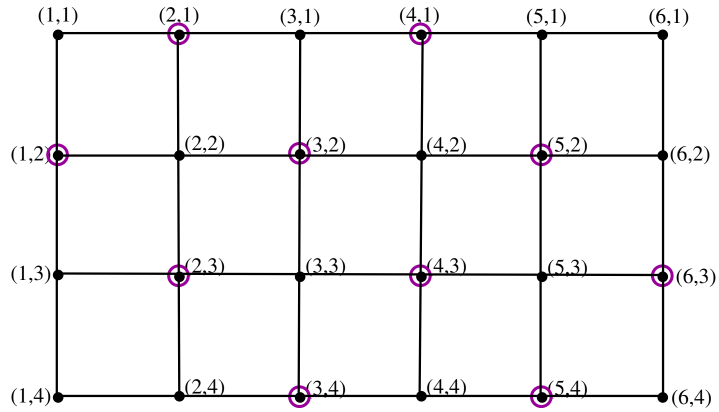

Fig. 6

Subcase(ii) : m is even and n is odd.

Starting with (2,1), choose all the alternate vertices along each row and each column except (m,n) *and* $(m,1)$.

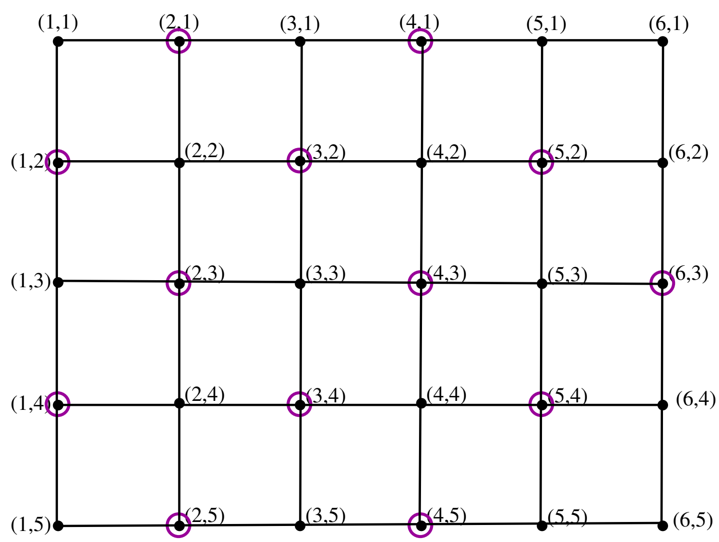

Fig. 7

Thus,$|\Gamma| = \beta_0 - 2$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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