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A RELATED FIXED POINT THEOREM IN THREE INTUITIONISTIC FUZZY METRIC SPACES

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Abstract. We prove a related fixed point theorem in three complete intuitionistic fuzzy metric spaces using an implicit relation which generalizes results of Aliouche and Fisher [2] and Rao et al. [20].

Keywords: Fuzzy metric space; implicit relation; Intuitionistic fuzzy metric space; related fixed point.

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1. Introduction and Preliminaries

The theory of fuzzy sets was introduced by Zadeh [24] in 1965. Since then, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. for example, Deng [5], Ereeg [11], George and Veeramani [12], Kramosil and Michalek [14] have introduced the concept of fuzzy metric spaces in different ways. One of the most important problems in fuzzy topology is to obtain an appropriate concept of intuitionistic fuzzy metric space. This notion has been introduced and studied by Park [18]. Alaca et al. [1] have redefined the concept of intuitionistic fuzzy metric spaces, according to concept of fuzzy metric spaces and proved Intuitionistic

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fuzzy Banach and Intuitionistic fuzzy Edelstein contraction theorems, with the different definition of Cauchy sequences and completeness.

Recently, Merghadi and Aliouche [17] Aliouche and Fisher [2], Aliouche et.al [3] and Rao et.al [20] proved some related fixed point theorems in compact metric spaces and sequentially compact fuzzy metric spaces. Inspired by a work due to Popa [19], we have remarked that proving common fixed point theorems using an implicit relation covers several contractive conditions.

In this paper, we prove a related fixed point theorem for three mappings in three complete intuitionistic fuzzy metric spaces using an implicit relation which generalizes results of Aliouche and Fisher [2] and Rao et al. [20].

Definition 1.1 [22]**.** A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous $t-$norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,

(2) $*$ is continuous,

(3) $a * 1 = a$ for all $a \in [0,1]$,

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of a continuous $t-$norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.2 [22]**.** A binary operation $\Diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous $t-$conorm if it satisfies the following conditions

(1) $\Diamond$ is associative and commutative,

(2) $\Diamond$ is continuous,

(3) $a \Diamond 0 = a$ for all $a \in [0,1]$,

(4) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Examples of a continuous $t-$conorm are $a \Diamond b = \max\{a, b\}$ and $a \Diamond b = \min\{1, a + b\}$.

The concept of intuitionistic fuzzy metric space is defined by Park [18].

Definition 1.3. A 5$-$tuple $(X, M, \mathcal{N}, *, \Diamond)$ is called an intuitionistic fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, $\Diamond$ a continuous t-conorm and $M, \mathcal{N}$ are fuzzy sets on $X^2 \times]0, +\infty[$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$

(1) $M(x,y,t) + \mathcal{N}(x,y,t) \leq 1$;

(2) $M(x,y,t) > 0$;

(3) $M(x,y,t) = 1$ if and only if $x = y$;

(4) $M(x,y,t) = M(y,x,t)$;

(5) $M(x,y,t) * M(\ y,z,s) \leq M(x,z,t+s)$;

(6) $M(x,y,\cdot) :]0,+\infty[\to [0,1]$ is continuous;

(7) $\mathcal{N}(x,y,t) = 0$ if and only if $x = y$;

(8) $\mathcal{N}(x,y,t) = \mathcal{N}(x,y,t)$;

(9) $\mathcal{N}(x,y,t) \Diamond \mathcal{N}(y,z,t) \geq \mathcal{N}(x,z,t+s)$;

(10) $\mathcal{N}(x,y,t) :]0,+\infty[\to [0,1]$ is continuous.

Then $(M,\mathcal{N})$ is called an intuitionistic fuzzy metric on X. The functions $M(x,y,t)$, $\mathcal{N}(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Every fuzzy metric space $(X,M,*)$ is an intuitionistic fuzzy metric space of the form $(X,M,1-M,*,\Diamond\})$ such that t-norm $*$ and t-conorm $\Diamond$ are associated [16], i.e., $x\Diamond y = 1-((1-x)*(1-y))$ for any $x,y \in X$.

Lemma 1.4 [18]**.** In intuitionistic fuzzy metric space X, $M(x,y,\cdot)$ is non-decreasing and $\mathcal{N}(x,y,\cdot)$ is non-increasing for all $x,y \in X$.

Example 1.5 [18]**.** Let (X,d) be a metric space. Denote $a*b = ab$ and $a\Diamond b = \min\{1,a+b\}$ for all $a,b \in [0,1]$ and let M_d and $\mathcal{N}_d$ be fuzzy sets on $X^2 \times]0,+\infty[$ defined as follows:

$$M_d(x,y,t) = \frac{t}{t+d(x,y)}, \quad \mathcal{N}_d(x,y,t) = \frac{d(x,y)}{t+d(x,y)}$$

Then $(X,M_d,\mathcal{N}_d,*,\Diamond)$ is an intuitionistic fuzzy metric space. We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric.

Note that the above example holds even with the t-norm $a*b = \min\{a,b\}$ and the t-conorm $a\Diamond b = \max\{a,b\}$and hence $(X,M_d,\mathcal{N}_d,*,\Diamond)$ is an intuitionistic fuzzy metric with respect to any continuous t-norm and continuous t-conorm.

Let $(X, M, \mathcal{N}, *, \Diamond)$ be an intuitionistic fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r \text{ and } \mathcal{N}(x, y, t) < r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let $\tau_{(M,\mathcal{N})}$ denote the family of all open subsets of X. Then $\tau_{(M,\mathcal{N})}$ is called the topology on X induced by the intuitionistic fuzzy metric $(M, \mathcal{N})$. This topology is Hausdorff and first countable. The topology τ_d induced by the metric d and the topology $\tau_{(M,\mathcal{N})}$ induced by the intuitionistic fuzzy metric (M, N) are the same [18].

Definition 1.6 [18]. Let $(X, M, \mathcal{N}, *, \Diamond)$ be an intuitionistic fuzzy metric space.

1) A sequence $\{x_n\}$ in X converges to x if for any $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$M(x_n, x, t) > 1 - \varepsilon$ and $\mathcal{N}(x_n, x_m, t) < \varepsilon$ for each $n \geq n_0$, i.e., $M(x_n, x, t) \to 1$ and $\mathcal{N}(x_n, x, t) \to 0$ as $n \to \infty$ for each $t > 0$.

2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for any $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$M(x_n, x_m, t) > 1 - \varepsilon$ and $\mathcal{N}(x_n, x_m, t) < \varepsilon$ for each $n, m \geq n_0$., i.e; $M(x_n, x_m, t) \to 1$ and $\mathcal{N}(x_n, x_m, t) \to 0$ as $n, m \to \infty$ for each $t > 0$.

3) The intuitionistic fuzzy metric space $(X, M, \mathcal{N}, *, \Diamond)$ is said to be complete if every Cauchy sequence is convergent.

Implicit relation

We denote by Φ, Ψ respectively, sets of all functions $\varphi, \psi : [0,1]^6 \longrightarrow [0,1]$ such that

(i) $\phi \in \Phi$, $\psi \in \Psi$ and ϕ, ψ are upper semi continuous in each coordinate variable,

(ii) ϕ, ψ are non-increasing in the second and the third variable,

(iii) For all $u, v \in (0,1)$, if either $\phi(u, 1, u, v, v, 1) > 0$ or $\phi(u, 1, u, v, 1, v) > 0$ or $\phi(u, u, 1, 1, v, v) > 0$ or $\phi(u, u, 1, v, 1, v) > 0$ then $u \geq v$.

Furthermore, for all $u, v \in (0,1)$, if either $\psi(u, 0, u, v, v, 0) < 0$ or $\psi(u, 0, u, v, 0, v) < 0$ or $\psi(u, u, 0, 0, v, v) < 0$ or $\psi(u, u, 1, v, 1, v) < 0$ then

$u \leq v$.

Example1.7. Let $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\}$. Then $\phi \in \Phi$.

Example1.8. Let $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\{t_2, t_3, t_4, t_5, t_6\}$. Then $\psi \in \Psi$.

Example1.9.

$$\begin{aligned} \phi\left(t_2, t_3, t_4, t_5, t_6\right) &= t_1 - \eta\left(\min\left\{t_2, t_3, t_4, t_5, t_6\right\}\right) \\ \psi\left(t_2, t_3, t_4, t_5, t_6\right) &= t_1 - \varphi\left(\max\left\{t_2, t_3, t_4, t_5, t_6\right\}\right) \end{aligned}$$

where $\eta, \varphi : [0,1] \to [0,1]$ is a increasing and continuous function respectively, with $\eta(t) \geq t$ and $\varphi(t) \leq t$ for $0 \leq t \leq 1$. For example $\eta(t) = \sqrt{t}$ or $\eta(t) = t^h$ for $0 < h < 1$ and $\phi(t) = \dfrac{t}{2}$.

We need the following lemma of [15].

Lemma 1.10. Let $\{x_n\}$ be a sequence in intuitionistic fuzzy metric space $(X, M, \mathcal{N}, *, \diamondsuit)$ with $M(x, y, t) \to 1$ and $\mathcal{N}(x, y, t) \to 0$ as $t \to \infty$ for all $x, y \in X$. If there exists a number $k \in]0, 1[$ such that

$$\begin{aligned} M\left(x_{n+1}, x_n, kt\right) &\geq M\left(x_n, x_{n-1}, t\right), \\ \mathcal{N}\left(x_{n+1}, x_n, kt\right) &\leq \mathcal{N}\left(x_n, x_{n-1}, t\right). \end{aligned}$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 1.11 [15]**.** Let $(X, M, , \mathcal{N}, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. If there exists $k \in (0, 1)$ such that $M(x, y, kt) \geq M(x, y, t)$ and $N(x, y; kt) \leq N(x, y; t)$ for $x, y \in X$, then $x = y$.

2. Main results

Theorem 2.1. *Let $(X_i, M_i, \mathcal{N}_i, \theta_i, \gamma_i)_{1 \leq i \leq 3}$, be three complete intuitionistic fuzzy metric spaces with $M_i(x, x_i, t) \longrightarrow 1$ and $\mathcal{N}_i(x, x_i, t) \to 0$ as $t \longrightarrow \infty$ for all $x, x_i \in X_i$ and let $\{A_i\}_{i=1}^{i=3}$ be 3-mappings such that $A_i : X_i \to X_{i+1}$ for all $i = 1, 2$ and $A_3 : X_3 \to X_1$, satisfying the inequalities*

$$(2.1_M) \qquad \phi_1\begin{pmatrix} M_1\left(A_3A_2x_2, A_3A_2A_1x_1, kt\right), M_1\left(x_1, A_3A_2x_2, t\right), \\ M_1\left(x_1, A_3A_2A_1x_1, t\right), M_2\left(x_2, A_1x_1, t\right), \\ M_2\left(x_2, A_1A_3A_2x_2, t\right), M_2\left(A_1x_1, A_1A_3A_2x_2, t\right) \end{pmatrix} > 0$$

$$(2.1_{calN})\qquad \psi_1\left(\begin{array}{c}\mathcal{N}_1\left(A_3A_2x_2, A_3A_2A_1x_1, kt\right), \mathcal{N}_1\left(x_1, A_3A_2x_2, t\right), \\ \mathcal{N}_1\left(x_1, A_3A_2A_1x_1, t\right), \mathcal{N}_2\left(x_2, A_1x_1, t\right), \\ \mathcal{N}_2\left(x_2, A_1A_3A_2x_2, t\right), \mathcal{N}_2\left(A_1x_1, A_1A_3A_2x_2, t\right)\end{array}\right) < 0$$

for all $x_1 \in X_1$, $x_2 \in X_2$ *and* $t > 0$, *where* $\phi_1 \in \Phi$, $\psi_1 \in \Psi$ *and* $0 < k < 1$.

$$(2.2_{M})\qquad \phi_2\left(\begin{array}{c}M_2\left(A_1A_3x_3, .A_1A_3A_2x_2, kt\right), M_2\left(x_2, A_1A_3x_3, t\right), \\ M_2\left(x_2, A_1A_3A_2x_2, t\right), M_3\left(x_3, A_2x_2, t\right), \\ M_3\left(x_3, A_2A_1A_3x_3, t\right), M_3\left(A_2x_2, A_2A_1A_3x_3, t\right)\end{array}\right) > 0$$

$$(2.2_{calN})\qquad \psi_2\left(\begin{array}{c}\mathcal{N}_2\left(A_1A_3x_3, .A_1A_3A_2x_2, kt\right), \mathcal{N}_2\left(x_2, A_1A_3x_3, t\right), \\ \mathcal{N}_2\left(x_2, A_1A_3A_2x_2, t\right), \mathcal{N}_3\left(x_3, A_2x_2, t\right), \\ \mathcal{N}_3\left(x_3, A_2A_1A_3x_3, t\right), \mathcal{N}_3\left(A_2x_2, A_2A_1A_3x_3, t\right)\end{array}\right) < 0$$

for all $x_2 \in X_2$, $x_3 \in X_3$, $t > 0$, *where* $\phi_2 \in \Phi$, $\psi_2 \in \Psi$ *and* $0 < k < 1$.

$$(2.3_{M})\qquad \phi_3\left(\begin{array}{c}M_3\left(A_2A_1x_1, A_3A_1A_3x_3, kt\right), M_3\left(x_3, A_2A_1x_1, t\right), \\ M_3\left(x_3, A_2A_1A_3x_3, t\right), M_1\left(x_1, A_3x_3, t\right), \\ M_1\left(x_1, A_3A_2A_1x_1, t\right), M_1\left(A_3x_3, A_3A_2A_1x_1, t\right)\end{array}\right) > 0$$

$$(2.3_{calN})\qquad \psi_3\left(\begin{array}{c}\mathcal{N}_3\left(A_2A_1x_1, A_3A_1A_3x_3, kt\right), \mathcal{N}_3\left(x_3, A_2A_1x_1, t\right), \\ \mathcal{N}_3\left(x_3, A_2A_1A_3x_3, t\right), \mathcal{N}_1\left(x_1, A_3x_3, t\right), \\ \mathcal{N}_1\left(x_1, A_3A_2A_1x_1, t\right), \mathcal{N}_1\left(A_3x_3, A_3A_2A_1x_1, t\right)\end{array}\right) < 0$$

for all $x_1 \in X_1$, $x_3 \in X_3$ *and* $t > 0$, *where* $\phi_3 \in \Phi$, $\psi_3 \in \Psi$ *and* $0 < k < 1$. *Further, suppose that one of* A_1, A_2 *and* A_3 *is continuous on* X_i. *Then*

$$\begin{aligned} A_3A_2A_1 \textit{ has a unique fixed point } p_1 &\in X_1 \\ A_1A_3A_2 \textit{ has a unique fixed point } p_2 &\in X_2 \\ A_2A_1A_3 \textit{ has a unique fixed point } p_3 &\in X_3. \end{aligned}$$

Further, $A_ip_i = p_{i+1}$ *for* $i = 1, 2$ *and* $A_3p_3 = p_1$.

Proof. Let $\left\{x_r^{(1)}\right\}, \left\{x_r^{(2)}\right\}$ and $\left\{x_r^{(3)}\right\}$ be sequences in $X_1,\ X_2, X_3$ respectively and $x_0^{(1)}$ be an arbitrary point in X_1 We define the sequences $\left\{x_r^{(i)}\right\}$ for $i = 1, 2, 3$ and $r \in \mathbb{N}$ by

$$\begin{aligned} x_r^{(1)} &= (A_3A_2A_1)^r\, x_0^{(1)} = (A_3A_2A_1)\, x_{r-1}^{(1)} \\ x_r^{(2)} &= A_1\,(A_3A_2A_1)^r\, x_0^{(1)} = A_1 x_{r-1}^{(1)} \\ x_r^{(3)} &= A_2A_1\,(A_3A_2A_1)^r\, x_0^{(1)} = A_2 x_r^{(2)} \end{aligned}$$

We assume that $x_r^{(1)} \neq x_{r+1}^{(1)}$. Applying the inequalities (2.1_M) and $(2.1_{\mathcal{N}})$ for $x_2 = x_{r-1}^{(2)} = A_1 x_{r-1}^{(1)} = A_1\,(A_3A_2A_1)^{r-1}\, x_0^{(1)}$and $x_1 = x_r^{(1)} = (A_3A_2A_1)^r\, x_0^{(1)}$ we get

$$\phi_1 \begin{pmatrix} M_1\left(x_r^{(1)}, x_{r+1}^{(1)}, kt\right), 1, M_1\left(x_r^{(1)}, x_{r+1}^{(1)}, t\right), \\ M_2\left(x_{r-1}^{(2)}, x_r^{(2)}, t\right), M_2\left(x_{r-1}^{(2)}, x_r^{(2)}, t\right), 1 \end{pmatrix} > 0$$

$$\psi_1 \begin{pmatrix} \mathcal{N}_1\left(x_r^{(1)}, x_{r+1}^{(1)}, kt\right), 0, \mathcal{N}_1\left(x_r^{(1)}, x_{r+1}^{(1)}, t\right), \\ \mathcal{N}_2\left(x_{r-1}^{(2)}, x_r^{(2)}, t\right), \mathcal{N}_2\left(x_{r-1}^{(2)}, x_r^{(2)}, t\right), 0 \end{pmatrix} < 0$$

Using (ii) and (iii) of the implicit relation we have

$$(3.1_M) \qquad M_1\left(x_r^{(1)}, x_{r+1}^{(1)}, kt\right) \geq M_2\left(x_{r-1}^{(2)}, x_r^{(2)}, t\right),$$

$$(3.1_{calN}) \qquad \mathcal{N}_1\left(x_r^{(1)}, x_{r+1}^{(1)}, kt\right) \leq \mathcal{N}_2\left(x_{r-1}^{(2)}, x_r^{(2)}, t\right).$$

Applying the inequalities (2.2_M) and $(2.2_{\mathcal{N}})$ for $x_3 = x_{r-1}^{(3)}$ and $x_2 = x_r^{(2)}$, we obtain

$$\phi_2 \begin{pmatrix} M_2\left(x_r^{(2)}, x_{r+1}^{(2)}, kt\right), 1, M_2\left(x_r^{(2)}, x_{r+1}^{(2)}, t\right) \\ , M_3\left(x_{r-1}^{(3)}, x_r^{(3)}, t\right), M_3\left(x_{r-1}^{(3)}, x_r^{(3)}, t\right), 1 \end{pmatrix} > 0$$

$$\psi_2 \begin{pmatrix} \mathcal{N}_2\left(x_r^{(2)}, x_{r+1}^{(2)}, kt\right), 0, \mathcal{N}_2\left(x_r^{(2)}, x_{r+1}^{(2)}, t\right) \\ , \mathcal{N}_3\left(x_{r-1}^{(3)}, x_r^{(3)}, t\right), \mathcal{N}_3\left(x_{r-1}^{(3)}, x_r^{(3)}, t\right), 0 \end{pmatrix} < 0$$

From (ii) and (iii) of the implicit relation we get

$$(3.2_M) \qquad M_2\left(x_r^{(2)}, x_{r+1}^{(2)}, kt\right) \geq M_3\left(x_{r-1}^{(3)}, x_r^{(3)}, t\right)$$

$$(3.2_{calN}) \qquad \mathcal{N}_2\left(x_r^{(2)}, x_{r+1}^{(2)}, kt\right) \leq \mathcal{N}_3\left(x_{r-1}^{(3)}, x_r^{(3)}, t\right).$$

Applying the inequalities (2.3_M) and ($2.3_{\mathcal{N}}$) for $x_3 = x_r^{(3)}$and $x_1 = x_{r-1}^{(1)}$ we have

$$\phi_3 \left(\begin{array}{c} M_3 \left(x_r^{(3)}, x_{r+1}^{(3)}, kt \right), 1, M_3 \left(x_r^{(3)}, x_{r+1}^{(3)}, t \right), \\ M_1 \left(x_r^{(1)}, x_{r-1}^{(1)}, t \right), M_1 \left(x_r^{(1)}, x_{r-1}^{(1)}, t \right), 1 \end{array} \right) > 0$$

$$\psi_3 \left(\begin{array}{c} \mathcal{N}_3 \left(x_r^{(3)}, x_{r+1}^{(3)}, kt \right), 0, \mathcal{N}_3 \left(x_r^{(3)}, x_{r+1}^{(3)}, t \right), \\ \mathcal{N}_1 \left(x_r^{(1)}, x_{r-1}^{(1)}, t \right), \mathcal{N}_1 \left(x_r^{(1)}, x_{r-1}^{(1)}, t \right), 0 \end{array} \right) < 0$$

and so by (ii) and (iii) of the implicit relation we get

$$(3.3_M) \qquad M_3 \left(x_r^{(3)}, x_{r+1}^{(3)}, kt \right) \geq M_1 \left(x_r^{(1)}, x_{r-1}^{(1)}, t \right)$$

$$(3.3_{calN}) \qquad \mathcal{N}_3 \left(x_r^{(3)}, x_{r+1}^{(3)}, kt \right) \leq \mathcal{N}_1 \left(x_r^{(1)}, x_{r-1}^{(1)}, t \right)$$

It follows from (3.1_M), (3.2_M) and ($3,3_M$) that for n large enough and for all $i = 1, 2, 3$

$$\begin{aligned} M_1 \left(x_r^{(1)}, x_{r+1}^{(1)}, kt \right) &\geq M_2 \left(x_{r-1}^{(2)}, x_r^{(2)}, t \right) \\ M_i \left(x_r^{(i)}, x_{r+1}^{(i)}, t \right) &\geq M_{i+1} \left(x_{r-1}^{(i+1)}, x_r^{(i+1)}, \frac{t}{k} \right) \\ &\geq ... \\ &\geq M_n \left(x_{r+i-n}^{(n)}, x_{r+i-n+1}^{(n)}, \frac{t}{k^{n-i}} \right) \\ &\geq M_1 \left(x_{r+i-n-1}^{(1)}, x_{r+i-n}^{(1)}, \frac{t}{k^{n-i+1}} \right) \\ &\geq ..... \\ & \quad M_1 \left(x_{r+i-2n-1}^{(1)}, x_{r+i-2n}^{(1)}, \frac{t}{k^{2n-i+1}} \right) \\ &\geq .... \\ &\geq M_1 \left(x_{r+i-mn-1}^{(1)}, x_{r+i-mn}^{(1)}, \frac{t}{k^{mn-i+1}} \right) \\ &\geq \min \left\{ \begin{array}{c} M_1 \left(x_1^{(1)}, x_2^{(1)}, \frac{t}{k^{mn}} \right), M_1 \left(x_1^{(2)}, x_2^{(2)}, \frac{t}{k^{mn}} \right) \\ , M_3 \left(x_1^{(3)}, x_2^{(3)}, \frac{t}{k^{mn}} \right) \end{array} \right\} \end{aligned}$$

It follows from $(3.1_{\mathcal{N}}), (3.2_{\mathcal{N}})$ and $(3,3_{\mathcal{N}})$ that for large enough n and for all $i = 1, 2, 3$.

$$\begin{aligned}
\mathcal{N}_i\left(x_r^{(i)}, x_{r+1}^{(i)}, t\right) &\leq \mathcal{N}_{i+1}\left(x_{r-1}^{(i+1)}, x_r^{(i+1)}, \frac{t}{k}\right) \\
&\leq \ldots \leq \mathcal{N}_1\left(x_{r+i-2n-1}^{(1)}, x_{r+i-2n}^{(1)}, \frac{t}{k^{2n-i+1}}\right) \\
&\leq \ldots. \leq \mathcal{N}_1\left(x_{r+i-mn-1}^{(1)}, x_{r+i-mn}^{(1)}, \frac{t}{k^{mn-i+1}}\right) \\
&\leq \max\left\{\begin{array}{c} \mathcal{N}_1\left(x_1^{(1)}, x_2^{(1)}, \frac{t}{k^{mn}}\right), \mathcal{N}_2\left(x_1^{(2)}, x_2^{(2)}, \frac{t}{k^{mn}}\right) \\ , \mathcal{N}_3\left(x_1^{(3)}, x_2^{(3)}, \frac{t}{k^{mn}}\right) \end{array}\right\}
\end{aligned}$$

Since $0 < k < 1$, it follows from Lemma 1.10 that $\left\{x_r^{(i)}\right\}$ is a Cauchy sequence in X_i for $i = 1, 2, 3$ with limits

$$\begin{aligned}
p_1 &= \lim_{r\to\infty} x_r^{(1)} = \lim_{r\to\infty} (A_3A_2A_1)^r x_0^{(1)} \\
p_2 &= \lim_{r\to\infty} x_r^{(2)} = \lim_{r\to\infty} A_1 x_r^{(1)} \\
p_3 &= \lim_{r\to\infty} x_r^{(3)} = \lim_{r\to\infty} A_2A_1 x_r^{(1)} = \lim_{r\to\infty} A_2 x_r^{(2)}.
\end{aligned}$$

Using the inequality (2.1_M) for $x_1 = p_1$ and $x_2 = x_{r-1}^{(2)}$ we have

$$(4.1_M) \qquad \phi_1\left(\begin{array}{c} M_1\left(x_r^{(1)}, A_3A_2A_1p_1, kt\right), M_1\left(p_1, x_r^{(1)}, t\right), \\ M_1\left(p_1, A_3A_2A_1p_1, t\right), M_2\left(x_{r-1}^{(2)}, A_1p_1, t\right), \\ M_2\left(x_{r-1}^{(2)}, x_r^{(2)}, t\right), M_2\left(A_1p_1, x_r^{(2)}, t\right) \end{array}\right) > 0.$$

From (2.2_M) and for $x_2 = p_2$ and $x_3 = x_{r-1}^{(3)} = A_2A_1 (A_3A_2A_1)^{r-1} x_0^{(1)}$ we get

$$(4.2_M) \qquad \phi_2\left(\begin{array}{c} M_2\left(x_r^{(2)}, A_1A_3A_2p_2, kt\right), M_2\left(p_2, x_r^{(2)}, t\right), \\ M_2\left(p_2, A_1A_3A_2p_2, t\right), M_3\left(x_{r-1}^{(3)}, A_2p_2, t\right), \\ M_3\left(x_{r-1}^{(3)}, x_r^{(3)}, t\right), M_3\left(A_2p_2, x_r^{(3)}, t\right) \end{array}\right) > 0$$

Finally, using the inequality (2.3_M) for $x_3 = p_3$ and $x_1 = (A_3A_2A_1)^r x_0^{(1)} = x_r^{(1)}$ we obtain

$$(4.3_M) \qquad \phi_3\left(\begin{array}{c} M_3\left(x_r^{(3)}, A_3A_1A_3p_3, kt\right), M_3\left(p_3, x_r^{(3)}, t\right), \\ M_3\left(p_3, A_2A_1A_3p_3, t\right), M_1\left(x_r^{(1)}, A_3p_3, t\right), \\ M_1\left(x_r^{(1)}, x_{r+1}^{(1)}, t\right), M_1\left(A_3p_3, x_{r+1}^{(1)}, t\right) \end{array}\right) > 0$$

Letting $r \to \infty$ in (4.1_M), (4.2_M) and (4.3_M) and using (i) we have

$$\phi_1 \begin{pmatrix} M_1(p_1, A_3A_2A_1p_1, kt), 1, \\ M_1(p_1, A_3A_2A_1p_1, t), M_2(p_2, A_1p_1, t), \\ 1, M_2(p_2, A_1p_1, t) \end{pmatrix} > 0$$

$$\phi_2 \begin{pmatrix} M_2(p_2, A_1A_3A_2p_2, kt), 1, \\ M_2(p_2, A_1A_3A_2p_2, t), M_3(p_3, A_2p_2, t), \\ 1, M_3(p_3, A_2p_2, t) \end{pmatrix} > 0$$

$$\phi_3 \begin{pmatrix} M_3(p_3, A_2A_1A_3p_3, kt), 1, \\ M_3(p_3, A_2A_1A_3p_3, t), M_1(p_1, A_3p_3, t), \\ 1, M_1(p_1, A_3p_3, t) \end{pmatrix} > 0$$

It follows from (ii) and (iii) that

$$M_1(p_1, A_3A_2A_1p_1, kt) \geq M_2(p_2, A_1p_1, t)\,(5.1_M)$$

(1)

$\mathrm{M}_2(p_2, A_1A_3A_2p_2, kt) \geq M_3(p_3, A_2p_2, t)$

$\mathrm{M}_3(p_3, A_2A_1A_3p_3, kt) \geq M_1(p_1, A_3p_3, t)$Similarly

$$\mathcal{N}_1(p_1, A_3A_2A_1p_1, kt) \leq \mathcal{N}_2(p_2, A_1p_1, t)\,(5.1_{calN})$$

(2)

$\mathrm{N}_2(p_2, A_1A_3A_2p_2, kt) \leq \mathcal{N}_3(p_3, A_2p_2, t)$

$\mathrm{N}_3(p_3, A_2A_1A_3p_3, kt) \leq \mathcal{N}_1(p_1, A_3p_3, t)$ .Suppose that A_2 is continuous. Then

$$p_3 = A_2p_2. \tag{6.1}$$

Using the inequality (2.1_M) for $x_1 = x_r^{(1)}$ and $x_2 = p_2$ we have

$$\phi_1 \begin{pmatrix} M_1\left(A_3A_2p_2, x_{r+1}^{(1)}, kt\right), M_1\left(x_r^{(1)}, A_3A_2p_2, t\right), \\ M_1\left(x_r^{(1)}, x_{r+1}^{(1)}, t\right), M_2\left(p_2, A_1x_r^{(1)}, t\right), \\ M_2(p_2, A_1A_3A_2p_2, t), M_2\left(A_1x_r^{(1)}, A_1A_3A_2p_2, t\right) \end{pmatrix} > 0 \tag{6.1_M}$$

Applying (2.2_M) for $x_2 = x_r^{(2)}$ and $x_3 = p_3$ we get

$$(6.2_M) \qquad \phi_2 \begin{pmatrix} M_2\left(A_1A_3p_3, .A_1A_3A_2x_r^{(2)}, kt\right), M_2\left(x_r^{(2)}, A_1A_3p_3, t\right), \\ M_2\left(x_r^{(2)}, x_{r+1}^{(2)}, t\right), M_3\left(p_3, A_2x_r^{(2)}, t\right), \\ M_3\left(p_3, A_2A_1A_3p_3, t\right), M_3\left(A_2x_r^{(2)}, A_2A_1A_3p_3, t\right) \end{pmatrix} > 0$$

Finally, using the inequality (2.3_M) for $x_3 = p_3$ and $x_1 = x_r^{(1)}$ we obtain

$$(6.3_M) \qquad \phi_3 \begin{pmatrix} M_3\left(x_r^{(3)}, A_2A_1A_3p_3, kt\right), M_3\left(p_3, x_r^{(3)}, t\right), \\ M_3\left(p_3, A_2A_1A_3p_3, t\right), M_1\left(x_{r-1}^{(1)}, A_3p_3, t\right), \\ M_1\left(x_{r-1}^{(1)}, x_r^{(1)}, t\right), M_1\left(A_3p_3, x_r^{(1)}, t\right) \end{pmatrix} > 0$$

Letting $r \to \infty$ in (6.1_M), (6.2_M) and (6.3_M) and using (i) and (ii) we have

$$\phi_1 \begin{pmatrix} M_1\left(A_3p_3, p_1, kt\right), M_1\left(p_1, A_3p_3, kt\right), 1, 1, \\ M_2\left(p_2, A_1A_3p_3, t\right), M_2\left(p_2, A_1A_3A_2p_2, t\right) \end{pmatrix} > 0$$

$$\phi_2 \begin{pmatrix} M_2\left(A_1A_3p_3, p_2, kt\right), M_2\left(p_2, A_1A_3p_3, kt\right), 1, 1, \\ M_3\left(p_3, A_2A_1A_3p_3, t\right), M_3\left(p_3, A_2A_1A_3p_3, t\right) \end{pmatrix} > 0$$

$$\phi_3 \begin{pmatrix} M_3\left(p_3, A_2A_1A_3p_3, kt\right), 1, \\ M_3\left(p_3, A_2A_1A_3p_3, kt\right), M_1\left(p_1, A_3p_3, t\right), \\ 1, M_1\left(p_1, A_3p_3, t\right) \end{pmatrix} > 0$$

It follows from (iii) that

$$\begin{aligned} M_1\left(A_3p_3, p_1, kt\right) &\geq M_2\left(p_2, A_1A_3p_3, t\right) \\ M_2\left(p_2, A_1A_3p_3, kt\right) &\geq M_3\left(p_3, A_2A_1A_3p_3, t\right) \\ M_3\left(p_3, A_2A_1A_3p_3, kt\right) &\geq M_1\left(A_3p_3, p_1, t\right) \end{aligned}$$

In the same manner

$$\begin{aligned} \mathcal{N}_1\left(A_3p_3, p_1, kt\right) &\leq \mathcal{N}_2\left(p_2, A_1A_3p_3, t\right) \\ \mathcal{N}_2\left(p_2, A_1A_3p_3, kt\right) &\leq \mathcal{N}_3\left(p_3, A_2A_1A_3p_3, t\right) \\ \mathcal{N}_3\left(p_3, A_2A_1A_3p_3, kt\right) &\leq \mathcal{N}_1\left(A_3p_3, p_1, t\right) \end{aligned}$$

Then

$$M_1\left(A_3p_3, p1, kt\right) \geq M_1\left(A_3p_3, p_1, t\right) \text{ and } \mathcal{N}_1\left(A_3p_3, p_1, kt\right) \leq \mathcal{N}_1\left(A_3p_3, p_1, t\right)$$

By lemma 1.11 we get

$$A_3p_3 = p_1 \tag{6.2}$$

From the inequalities (6.1), (6.2), (5.1_M) and ($5.1_{\mathcal{N}}$) we have

$$\begin{aligned} A_1A_3A_2p_2 &= p_2 \\ A_2A_1A_3p_3 &= p_3 \\ A_1p_1 &= p_2 6.3 \end{aligned} \tag{3}$$

By (6.3), (5.1_M) and ($5.1_{\mathcal{N}}$) we obtain

$$A_3A_2A_1p_1 = p_1$$

To prove the uniqueness of the fixed point p_i in X_i, we assume that there exists $z_i \in X_i$ such that $z_i \neq p_i$, $A_iz_i = z_{i+1}$ for $i = 1, 2$ and $A_3z_3 = z_1$. Using (2.1_M), (2.2_M) and (2.3_M) we have for $i = 1, 2$

$$\phi_i\left(\begin{array}{c} M_i\left(z_i, p_i, kt\right), M_i\left(p_i, z_i\ , t\right), 1, \\ M_{i+1}\left(z_{i+1}, p_{i+1}, t\right), 1, M_{i+1}\left(p_{i+1}, z_{i+1}, t\right) \end{array}\right) > 0$$

and

$$\phi_3\left(\begin{array}{c} M_3\left(z_3, p_3, kt\right), M_3\left(p_3, z_3, t\right), 1, \\ M_1\left(z_1, p_1, t\right), 1, M_1\left(p_1, z_1, t\right) \end{array}\right) > 0$$

which imply

$$\begin{aligned} M_1\left(p_1, z_1, kt\right) &\geq M_2\left(p_2, z_2, t\right) \\ M_2\left(p_2, z_2, kt\right) &\geq M_3\left(p_3, z_3, t\right) \\ M_3\left(p_3, z_3, kt\right) &\geq M_1\left(p_1, z_1, t\right). \end{aligned}$$

Similarly

$$\begin{aligned}\mathcal{N}_1(p_1,z_1,kt) &\leq \mathcal{N}_2(p_2,z_2,t)\\ \mathcal{N}_2(p_2,z_2,kt) &\leq \mathcal{N}_3(p_3,z_3,t)\\ \mathcal{N}_3(p_3,z_3,kt) &\leq \mathcal{N}_1(p_1,z_1,t).\end{aligned}$$

Using lemma 1.11 we get $z_i = p_i$, $i = 1,2,3$. This proves the uniqueness of p_i in X_i for all $i = 1,2,3$. This complete the proof of the theorem.

Example 2.2. Let (M_i, X_i, θ_i), $i = 1,2,3$, be 3 an intuitionistic fuzzy metric spaces such that $M_d(x,y,t) = \dfrac{t}{t+d(x,y)}$, $\mathcal{N}_d(x,y,t) = \dfrac{d(x,y)}{t+d(x,y)}$, $X_1 = [0,1]$, $X_2 = [1,2]$ and $X_3 = [2,3]$. Define $A_i : X_i \to X_{i+1}$ for $i = 1,2$ and $A_3 : X_3 \to X_1$ by

$$\begin{aligned}A_1x_1 &= \frac{3}{2} \text{ if } x_1 \in [0,1],\ A_2x_2 = \begin{cases}\frac{9}{4} \text{ if } x_2 \in \left[1,\frac{5}{4}\right[\\ \frac{5}{2} \text{ if } x_2 \in \left[\frac{5}{4},2\right]\end{cases}\\ A_3x_3 &= \begin{cases}\frac{3}{4} \text{ if } x_3 \in \left[2,\frac{9}{4}\right[\\ 1 \text{ if } x_3 \in \left[\frac{9}{4},3\right]\end{cases}\end{aligned}$$

Let $\phi_1 = \phi_2 = \phi_3 = \phi$ such that $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \min\{t_2,t_3,t_4,t_5,t_6\}$

and let $\psi_1 = \psi_2 = \psi_3 = \psi$ such that $\psi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \max\{t_2,t_3,t_4,t_5,t_6\}$.

Note that there exists w_i in X_i such that $(A_{i-1}A_{i-2}..A_1A_n...A_i)w_i = w_i$, $\forall i = 1,2,3$ and $n = 3$.

(a) If $i = 3$ we get $(A_2A_1A_3)w_3 = w_3$ if $w_3 = 3 - \frac{1}{2} = \frac{5}{2}$ because

$$\begin{aligned}&(A_2A_1A_3)\left(\frac{5}{2}\right)\\ =\ & A_2A_1(1) = A_2\left(\frac{3}{2}\right)\\ =\ & \frac{5}{2}\end{aligned}$$

(b) If $i = 2$ we find $A_1 A_3 A_2 w_2 = w_2$ and if $w_2 = \frac{3}{2} \in \left[\frac{5}{4}, 2\right[$;

$$\begin{aligned} & (A_1 A_3 A_2)\left(\frac{3}{2}\right) \\ = \; & A_1 A_3 \left(\frac{5}{2}\right) = A_1 \left(\frac{3}{2}\right) \\ = \; & \frac{3}{2} \end{aligned}$$

(c) If $i = 1$ we find $A_3 A_2 A_1 w_1 = w_1$ and if $w_1 = 1 \in [0, 1]$;

$$\begin{aligned} & (A_3 A_2 A_1)(1) \\ = \; & A_3 A_2 \left(\frac{3}{2}\right) = A_3 \left(\frac{5}{2}\right) \\ = \; & 1 \end{aligned}$$

Hence, all conditions of Theorem 2.1 are satisfied.

Remark. In the theorem 2.8 of [20], the inequalities (1) and (2) should be $>$ (greater than) in order to obtain a contradiction in example 2.5 of [20].

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