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RULED SURFACE PAIR GENERATED BY A CURVE AND ITS NATURAL LIFT IN $\mathbb{R}_1^3$

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Abstract. In this study, firstly, the Frenet vector fields $\overline{T}, \overline{N}$,$\overline{B}$ of the natural lift $\overline{\alpha}$ of a curve α are calculated in terms of those of α in $\mathbb{R}_1^3$. Secondly, we obtained striction lines and distribution parameters of ruled surface pair generated by the curve α and its natural lift $\overline{\alpha}$. Finally, for α and $\overline{\alpha}$ those notions are compared with each other.

Keywords:Natural Lift, Ruled Surface, Striction Line, Distribution Parameter.

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1. Introduction and Preliminaries

Let Minkowski 3-space $\mathbb{R}_1^3$ be the vector space $\mathbb{R}^3$ equipped with the Lorentzian inner product g given by

$$g(X,X) = -x_1^2 + x_2^2 + x_3^2$$

where $X = (x_1, x_2, x_3) \in \mathbb{R}^3$. A vector $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ is said to be timelike if $g(X,X) < 0$, spacelike if $g(X,X) > 0$ and lightlike (or null) if $g(X,X) = 0$. Similarly, an arbitrary curve $\alpha = \alpha(t)$ in $\mathbb{R}_1^3$ where t is a pseudo-arclength parameter, can locally

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be timelike, spacelike or null (lightlike), if all of its velocity vectors $\alpha^{'}(t)$ are respectively timelike, spacelike or null (lightlike), for every $t \in I \subset \mathbb{R}$.

A lightlike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ $(resp.x_1 < 0)$ and a timelike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ $(resp.\ x_1 < 0)$.The norm of a vector X is defined by [4]

$$\|X\|_{IL} = \sqrt{|g(X,X)|}.$$

We denote by $\{T(t), N(t), B(t)\}$ the moving Frenet frame along the curve α. Then T, N and B are the tangent, the principal normal and the binormal vector of the curve α, respectively.

Let α be a unit speed timelike space curve with curvature κ and torsion τ. Let Frenet vector fields of α be $\{T, N, B\}$. In this trihedron, T is timelike vector field, N and B are spacelike vector fields. Then, Frenet formulas are given by [8]

$$T^{'} = \kappa N \;\; N^{'} = \kappa T + \tau B \;\; B^{'} = -\tau N.$$

Let α be a unit speed spacelike space curve with a spacelike binormal. In this trihedron, we assume that T and B are spacelike vector fields and N is a timelike vector field .Then, Frenet formulas are given by [8]

$$T^{'} = \kappa N \;\; N^{'} = \kappa T + \tau B \;\; B^{'} = \tau N.$$

Let α be a unit speed spacelike space curve with a timelike binormal. In this trihedron, we assume that T and N are spacelike vector fields and B is a timelike vector field. Then, Frenet formulas are given by [8]

$$T^{'} = \kappa N \;\; N^{'} = -\kappa T + \tau B \;\; B^{'} = \tau N.$$

Lemma1.1. *Let X and Y be nonzero Lorentz orthogonal vectors in $\mathbb{R}^3_1$. If X is timelike, then Y is spacelike* [10].

Lemma1.2. *Let X and Y be pozitive (negative) timelike vectors in $\mathbb{R}_1^3$. Then*

$$g(X,Y) \leq \|X\| \|Y\|$$

whit equality if and only if X and Y are linearly dependent [10].

Lemma1.3.

i) *Let X and Y be pozitive (negative) timelike vectors in $\mathbb{R}_1^3$. By the Lemma 1.2, there is unique nonnegative real number $\varphi(X,Y)$ such that*

$$g(X,Y) = \|X\| \|Y\| \cosh \varphi(X,Y)$$

the Lorentzian timelike angle between X and Y is defined to be $\varphi(X,Y)$.

ii) *Let X and Y be spacelike vectors in $\mathbb{R}_1^3$ that span a spacelike vector subspace. Then we have*

$$|g(X,Y)| \leq \|X\| \|Y\| .$$

Hence, there is a unique real number $\varphi(X,Y)$ between 0 and π such that

$$g(X,Y) = \|X\| \|Y\| \cos \varphi(X,Y)$$

the Lorentzian spacelike angle between X and Y is defined to be $\varphi(X,Y)$.

iii) *Let X and Y be spacelike vectors in $\mathbb{R}_1^3$ that span a timelike vector subspace. Then we have*

$$g(X,Y) > \|X\| \|Y\| .$$

Hence, there is a unique pozitive real number $\varphi(X,Y)$ between 0 and π such that

$$|g(X,Y)| = \|X\| \|Y\| \cosh \varphi(X,Y)$$

the Lorentzian timelike angle between X and Y is defined to be $\varphi(X,Y)$.

iv) *Let X be a spacelike vector and Y be a pozitive timelike vector in $\mathbb{R}_1^3$. Then there is a unique nonnegative reel number $\varphi(X,Y)$ such that*

$$|g(X,Y)| = \|X\| \|Y\| \sinh \varphi(X,Y)$$

the Lorentzian timelike angle between X and Y is defined to be $\varphi(X,Y)$ [10].

Definition1.1. (**Unit Vector C of Direction W for Non-null Curves**):

i) *For the curve α with a timelike tanget, θ being a Lorentzian timelike angle between the spacelike binormal unit $-B$ and the Frenet instantaneous rotation vector W ,*

a) *If $|\kappa| > |\tau|$, then W is a spacelike vector. In this situation, from Lemma 1.3 iii) we can write*

$$\kappa = \|W\| \cosh \theta, \ \ \tau = \|W\| \sinh \theta$$

$\|W\|^2 = g(W, W) = \kappa^2 - \tau^2$ *and* $C = \frac{W}{\|W\|} = \sinh \theta T + \cosh \theta B$, *where C is unit vector of direction W.*

b) *If $|\kappa| < |\tau|$, then W is a timelike vector. In this situation, from Lemma 1.3 iv) we can write*

$$\kappa = \|W\| \sinh \theta, \ \ \tau = \|W\| \cosh \theta$$

$\|W\|^2 = -g(W, W) = -(\kappa^2 - \tau^2)$ *and* $C = \cosh \theta T + \sinh \theta B$.

ii) *For the curve α with a timelike principal normal, θ being an angle between the B and the W, if B and W spacelike vectors that span a spacelike vektor subspace then by the Lemma 1.3 ii) we can write*

$$\kappa = \|W\| \cos \theta, \ \ \tau = \|W\| \sin \theta$$

$\|W\|^2 = g(W, W) = \kappa^2 + \tau^2$ *and* $C = \sin \theta T - \cos \theta B$.

iii) *For the curve α with a timelike binormal, θ being a Lorentzian timelike angle between the $-B$ and the W,*

a) *If $|\kappa| < |\tau|$, then W is a spacelike vector. In this situation, from Lemma 1.3 iv) we can write*

$$\kappa = \|W\| \sinh \theta, \ \ \tau = \|W\| \cosh \theta$$

$\|W\|^2 = g(W, W) = \tau^2 - \kappa^2$ *and* $C = -\cosh \theta T + \sinh \theta B$.

b) *If $|\kappa| > |\tau|$, then W is a timelike vector. In this situation, from Lemma 1.3 i) we have*

$$\kappa = \|W\| \cosh \theta, \ \ \tau = \|W\| \sinh \theta$$

$\|W\|^2 = -g(W, W) = -(\tau^2 - \kappa^2)$ *and* $C = -\sinh \theta T + \cosh \theta B$.

Corollary 1.1. *Let α be a unit speed timelike space curve. Then the natural lift $\overline{\alpha}$ of α is a spacelike space curve* [5].

Corollary 1.2. *Let α be a unit speed spacelike space curve with a spacelike binormal. Then the natural lift $\overline{\alpha}$ of α is a timelike space curve* [5].

Corollary 1.3. *Let α be a unit speed spacelike space curve with a timelike binormal. Then the natural lift $\overline{\alpha}$ of α is a spacelike space curve* [5].

Corollary 1.4. *Let α be a unit speed timelike space curve and $\overline{\alpha}$ be the natural lift of α. Then*

$$\overline{T}(s) = N(s),\ \overline{N}(s) = -\frac{\kappa(s)}{\|W\|}T(s) - \frac{\tau(s)}{\|W\|}B(s)\ ,\ \overline{B}(s) = -\frac{\tau(s)}{\|W\|}T(s) - \frac{\kappa(s)}{\|W\|}B(s)\ \ [7].$$

Corollary 1.5. *Let α be a unit speed spacelike space curve with a spacelike binormal and $\overline{\alpha}$ be the natural lift of α. Then*

$$\overline{T}(s) = N(s),\ \overline{N}(s) = \frac{\kappa(s)}{\|W\|}T(s) + \frac{\tau(s)}{\|W\|}B(s)\ ,\ \overline{B}(s) = \frac{\tau(s)}{\|W\|}T(s) - \frac{\kappa(s)}{\|W\|}B(s)\ \ [7].$$

Corollary 1.6. *Let α be a unit speed spacelike space curve with a timelike binormal and $\overline{\alpha}$ be the natural lift of α. Then*

$$\overline{T}(s) = N(s),\ \overline{N}(s) = -\frac{\kappa(s)}{\|W\|}T(s) - \frac{\tau(s)}{\|W\|}B(s)\ ,\ \overline{B}(s) = \frac{\tau(s)}{\|W\|}T(s) + \frac{\kappa(s)}{\|W\|}B(s)\ \ [7].$$

Definition1.2. *Let M be a hypersurface in $\mathbb{R}_1^3$ and let $\alpha : I \longrightarrow M$ be a parametrized curve. α is called an integral curve of X if*

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s))\ \ (\textit{for all } s \in I)\ \ [4]$$

where X is a smooth tangent vector field on M. We have

$$TM = \bigcup_{P \in M} T_P M = \chi(M)$$

where $T_P M$ is the tangent space of M at P and $\chi(M)$ is the space of vector fields on M.

Definition1.3. *For any parametrized curve* $\alpha : I \longrightarrow M$, $\overline{\alpha} : I \longrightarrow TM$ *given by*

$$\overline{\alpha}(s) = \left(\alpha(s), \alpha'(s)\right) = \alpha'(s)\,|_{\alpha(s)}$$

is called the natural lift of α *on* TM [5]. *Thus, we can write*

$$\frac{d\overline{\alpha}}{ds} = \frac{d}{ds}\left(\alpha'(s)\,|_{\alpha(s)}\right) = D_{\alpha'(s)}\alpha'(s)$$

where D *is the Levi-Civita connection on* $\mathbb{R}^3_1$.

A ruled surface is generated by a one-parameter family of straight lines and it possesses a parametric representation

$$X(s,v) = \alpha(s) + ve(s),$$

where $\alpha(s)$ represents a space curve which is called the base curve and e is a unit vector representing the direction of a straight line.

The striction point on a ruled surface X is the foot of the common normal between two consecutive generators (or ruling). The set of striction points defines the striction curve given as

$$\beta(s) = \alpha(s) - \frac{g\left(\alpha', e'\right)}{g\left(e', e'\right)}e(s) \ \ [2].$$

The distribution parameter of the ruled surface X is defined by

$$P_e = \frac{\det\left(\alpha', e, e'\right)}{\|e'\|^2}[2].$$

The ruled surface is developable if and only if $P_e = 0$.

3. RuledSurface Pair Generated By a Curve and Its Natural Lift in $\mathbb{R}^3_1$

Let α be a unit speed timelike space curve. Then the natural lift $\overline{\alpha}$ of α is a spacelike space curve.

(i) Let X and $\overline{X}$ be two ruled surfaces which is given by

$$X(s,v)=\alpha(s)+vT(s),\ \ \overline{X}(s,v)=\overline{\alpha}(s)+v\overline{T}(s).$$

The striction curves of X and $\overline{X}$ are given by $\beta(s)=\alpha(s)-\lambda T(s)$ and $\overline{\beta}(s)=\overline{\alpha}(s)-\mu\overline{T}(s)$, respectively. Then we obtain

$$\lambda=0,\ \ \mu=0.$$

The distribution parameters of the ruled surfaces X and $\overline{X}$ are defined by $P_T=\frac{\det\left(\alpha',T,T'\right)}{\left\|T'\right\|^2}$ and $\overline{P}_{\overline{T}}=\frac{\det\left(\overline{\alpha}',\overline{T},\overline{T}'\right)}{\left\|\overline{T}'\right\|^2}$. Then we have

$$P_T=0,\ \ \overline{P}_{\overline{T}}=0.$$

Corollary 2.1. *Let the striction curves of* X *and* $\overline{X}$ *be given by* $\beta(s)=\alpha(s)-\lambda T(s)$ *and* $\overline{\beta}(s)=\overline{\alpha}(s)-\mu\overline{T}(s)$*, respectively. Then* $\beta(s)=\alpha(s)$ *and* $\overline{\beta}(s)=\overline{\alpha}(s)$*.*

Corollary 2.2. *If the ruled surface* X *is developable then the ruled surface* $\overline{X}$ *are also developable.*

(ii) Let X and $\overline{X}$ be two ruled surfaces which is given by

$$X(s,v)=\alpha(s)+vN(s),\ \ \overline{X}(s,v)=\overline{\alpha}(s)+v\overline{N}(s).$$

The striction curves of X and $\overline{X}$ are given by $\beta(s)=\alpha(s)-\lambda N(s)$ and $\overline{\beta}(s)=\overline{\alpha}(s)-\mu\overline{N}(s)$, respectively. Then we have

$$\lambda=\frac{\kappa}{\kappa^2-\tau^2},\ \ \mu=\frac{\kappa\left(-\kappa^2+\tau^2\right)\|W\|}{-\kappa'^2+\tau'^2+\left(-\kappa^2+\tau^2\right)^2}.$$

The distribution parameters of the ruled surfaces X and $\overline{X}$ are defined by $P_N = \frac{\det\left(\alpha', N, N'\right)}{\left\|N'\right\|^2}$ and $\overline{P}_{\overline{N}} = \frac{\det\left(\overline{\alpha}', \overline{N}, \overline{N}'\right)}{\left\|\overline{N}'\right\|^2}$. Then we obtain

$$P_N = \frac{\tau}{-\kappa^2 + \tau^2}, \ \overline{P}_{\overline{N}} = \frac{-\kappa^2\tau' + \kappa\tau\kappa'}{(-\kappa'^2 + \tau'^2) + (-\kappa^2 + \tau^2)^2}.$$

Corollary 2.3. *Let the striction curves of X and $\overline{X}$ be given by $\beta(s) = \alpha(s) - \lambda N(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu\overline{N}(s)$, respectively.*

(1) *If W is a spacelike vector,then $\mu = \frac{-\kappa\left(\frac{\kappa}{\lambda}\right)^{\frac{3}{2}}}{\left(-\kappa'^2+\tau'^2\right)+\left(-\frac{\kappa}{\lambda}\right)^2}$.*

(2) *If W is a timelike vector,then $\mu = \frac{-\kappa\left(-\frac{\kappa}{\lambda}\right)^{\frac{3}{2}}}{\left(-\kappa'^2+\tau'^2\right)+\left(-\frac{\kappa}{\lambda}\right)^2}$.*

Corollary 2.4. *Let the distribution parameters of the ruled surfaces X and $\overline{X}$ be P_N and $\overline{P}_{\overline{N}}$, respectively.*

(1) *If W is a spacelike vector,then $\overline{P}_{\overline{N}} = \frac{-\kappa^2\tau'+\kappa\tau\kappa'}{\left(-\kappa'^2+\tau'^2\right)+\left(-\frac{\tau}{P_N}\right)^2}$.*

(2) *If W is a timelike vector,then $\overline{P}_{\overline{N}} = \frac{-\kappa^2\tau'+\kappa\tau\kappa'}{\left(-\kappa'^2+\tau'^2\right)+\left(\frac{\tau}{P_N}\right)^2}$.*

Corollary 2.5. *If α is a planer curve, then the ruled surface X and $\overline{X}$ are developable.*

(iii) Let X and $\overline{X}$ be two ruled surfaces which are given by

$$X(s,v) = \alpha(s) + vB(s), \ \overline{X}(s,v) = \overline{\alpha}(s) + v\overline{B}(s).$$

The striction curves of X and $\overline{X}$ are given by $\beta(s) = \alpha(s) - \lambda B(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu\overline{B}(s)$, respectively. Then we obtain

$$\lambda = 0, \ \mu = 0.$$

The distribution parameters of the ruled surfaces X and $\overline{X}$ are defined by $P_B = \frac{\det\left(\alpha', B, B'\right)}{\left\|B'\right\|^2}$ and $\overline{P}_{\overline{B}} = \frac{\det\left(\overline{\alpha}', \overline{B}, \overline{B}'\right)}{\left\|\overline{B}'\right\|^2}$. Then we have

$$P_B = \frac{1}{\tau}, \overline{P}_{\overline{B}} = \frac{\kappa^2\tau' - \kappa\tau\kappa'}{\kappa'^2 - \tau'^2}.$$

Corollary 2.6. *Let the striction curves of X and $\overline{X}$ be given by $\beta(s)=\alpha(s)-\lambda B(s)$ and $\overline{\beta}(s)=\overline{\alpha}(s)-\mu\overline{B}(s)$, respectively. Then $\beta(s)=\alpha(s)$ and $\overline{\beta}(s)=\overline{\alpha}(s)$.*

Corollary 2.7. *Let the distribution parameters of the ruled surfaces X and $\overline{X}$ be P_B and $\overline{P}_{\overline{B}}$, respectively. Then $\overline{P}_{\overline{B}}=\frac{\kappa^2\tau'-\kappa\left(\frac{1}{P_B}\right)\kappa'}{\kappa'^2+\tau'^2}$.*

Let α be a unit speed spacelike space curve with a spacelike binormal. Then the natural lift $\overline{\alpha}$ of α is a timelike space curve.

(i) Let X and $\overline{X}$ be two ruled surfaces which is given by

$$X(s,v)=\alpha(s)+vT(s),\ \overline{X}(s,v)=\overline{\alpha}(s)+v\overline{T}(s).$$

The striction curves of X and $\overline{X}$ are given by $\beta(s)=\alpha(s)-\lambda T(s)$ and $\overline{\beta}(s)=\overline{\alpha}(s)-\mu\overline{T}(s)$,respectively. Then we obtain

$$\lambda=0,\ \mu=0.$$

The distribution parameters of the ruled surfaces X and $\overline{X}$ are defined by $P_T=\frac{\det\left(\alpha',T,T'\right)}{\left\|T'\right\|^2}$ and $\overline{P}_{\overline{T}}=\frac{\det\left(\overline{\alpha}',\overline{T},\overline{T}'\right)}{\left\|\overline{T}'\right\|^2}$. Then we have

$$P_T=0,\ \overline{P}_{\overline{T}}=0.$$

Corollary 2.8. *Let the striction curves of X and $\overline{X}$ be given by $\beta(s)=\alpha(s)-\lambda T(s)$ and $\overline{\beta}(s)=\overline{\alpha}(s)-\mu\overline{T}(s)$, respectively. Then $\beta(s)=\alpha(s)$ and $\overline{\beta}(s)=\overline{\alpha}(s)$.*

Corollary 2.9. *If the ruled surface X is developable then the ruled surface $\overline{X}$ are also developable.*

(ii) Let X and $\overline{X}$ be two ruled surfaces which is given by

$$X(s,v)=\alpha(s)+vN(s),\ \overline{X}(s,v)=\overline{\alpha}(s)+v\overline{N}(s).$$

The striction curves of X and $\overline{X}$ are given by $\beta(s) = \alpha(s) - \lambda N(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu\overline{N}(s)$, respectively. Then we have

$$\lambda = \frac{\kappa}{\kappa^2 + \tau^2}, \ \mu = \frac{-\kappa\left(\kappa^2 + \tau^2\right)\|W\|}{\left(\kappa'^2 + \tau'^2\right) - \left(\kappa^2 + \tau^2\right)^2}.$$

The distribution parameters of the ruled surfaces X and $\overline{X}$ are defined by $P_N = \frac{\det\left(\alpha', N, N'\right)}{\|N'\|^2}$ and $\overline{P}_{\overline{N}} = \frac{\det\left(\overline{\alpha}', \overline{N}, \overline{N}'\right)}{\|\overline{N}'\|^2}$. Then we obtain

$$P_N = \frac{\tau}{\kappa^2 + \tau^2}, \ \overline{P}_{\overline{N}} = \frac{-\kappa^2\tau' + \kappa\tau\kappa'}{\left(\kappa'^2 + \tau'^2\right) - \left(\kappa^2 + \tau^2\right)^2}.$$

Corollary 2.10. *Let the striction curves of X and $\overline{X}$ be given by $\beta(s) = \alpha(s) - \lambda N(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu\overline{N}(s)$, respectively. Then* $\mu = \frac{-\kappa\left(\frac{\kappa}{\lambda}\right)^{\frac{3}{2}}}{\left(\kappa'^2 + \tau'^2\right) - \left(\frac{\kappa}{\lambda}\right)^2}$.

Corollary 2.11. *Let the distribution parameters of the ruled surfaces X and $\overline{X}$ be P_N and $\overline{P}_{\overline{N}}$, respectively. Then* $\overline{P}_{\overline{N}} = \frac{-\kappa^2\tau' + \kappa\tau\kappa'}{\left(\kappa'^2 + \tau'^2\right) + \left(\frac{\tau}{P_N}\right)^2}$.

Corollary 2.12. *If α is a planer curve, then the ruled surfaces X and $\overline{X}$ are developable.*

(iii) Let X and $\overline{X}$ be two ruled surfaces which are given by

$$X(s, v) = \alpha(s) + vB(s), \ \overline{X}(s, v) = \overline{\alpha}(s) + v\overline{B}(s).$$

The striction curves of X and $\overline{X}$ are given by $\beta(s) = \alpha(s) - \lambda B(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu\overline{B}(s)$,respectively. Then we obtain

$$\lambda = 0, \ \mu = 0.$$

The distribution parameters of the ruled surfaces X and $\overline{X}$ are defined by $P_B = \frac{\det\left(\alpha',B,B'\right)}{\left\|B'\right\|^2}$ and $\overline{P}_{\overline{B}} = \frac{\det\left(\overline{\alpha}',\overline{B},\overline{B}'\right)}{\left\|\overline{B}'\right\|^2}$.Then we have

$$P_B = \frac{1}{\tau}, \overline{P}_{\overline{B}} = \frac{-\kappa^2\tau' + \kappa\tau\kappa'}{\kappa'^2 + \tau'^2}.$$

Corollary 2.13. *Let the striction curves of X and $\overline{X}$ be given by $\beta(s) = \alpha(s) - \lambda B(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu\overline{B}(s)$, respectively. Then $\beta(s) = \alpha(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s)$.*

Corollary 2.14. *Let the distribution parameters of the ruled surfaces X and $\overline{X}$ be P_B and $\overline{P}_{\overline{B}}$, respectively. Then $\overline{P}_{\overline{B}} = \frac{-\kappa^2\tau' + \kappa\left(\frac{1}{P_B}\right)\kappa'}{\kappa'^2 + \tau'^2}$.*

Let α be a unit speed spacelike space curve with a timelike binormal. Then the natural lift $\overline{\alpha}$ of α is a spacelike space curve.

(i) Let X and $\overline{X}$ be two ruled surfaces which is given by

$$X(s,v) = \alpha(s) + vT(s), \ \overline{X}(s,v) = \overline{\alpha}(s) + v\overline{T}(s).$$

The striction curves of X and $\overline{X}$ are given by $\beta(s) = \alpha(s) - \lambda T(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu\overline{T}(s)$, respectively. Then we obtain

$$\lambda = 0, \ \mu = 0.$$

The distribution parameters of the ruled surfaces X and $\overline{X}$ are defined by $P_T = \frac{\det\left(\alpha',T,T'\right)}{\left\|T'\right\|^2}$ and $\overline{P}_{\overline{T}} = \frac{\det\left(\overline{\alpha}',\overline{T},\overline{T}'\right)}{\left\|\overline{T}'\right\|^2}$. Then we have

$$P_T = 0, \ \overline{P}_{\overline{T}} = 0.$$

Corollary 2.15. *Let the striction curves of X and $\overline{X}$ be given by $\beta(s) = \alpha(s) - \lambda T(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu\overline{T}(s)$, respectively. Then $\beta(s) = \alpha(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s)$.*

Corollary 2.16. *If the ruled surface X is developable then the ruled surface $\overline{X}$ are also developable.*

(ii) Let X and $\overline{X}$ be two ruled surfaces which is given by

$$X(s,v)=\alpha(s)+vN(s),\ \overline{X}(s,v)=\overline{\alpha}(s)+v\overline{N}(s).$$

The striction curves of X and $\overline{X}$ are given by $\beta(s)=\alpha(s)-\lambda N(s)$ and $\overline{\beta}(s)=\overline{\alpha}(s)-\mu\overline{N}(s)$, respectively. Then we have

$$\lambda=\frac{-\kappa}{\kappa^2-\tau^2},\ \mu=\frac{\kappa(\kappa^2+\tau^2)\|W\|}{(\kappa'^2-\tau'^2)+(\kappa^2+\tau^2)^2}.$$

The distribution parameters of the ruled surfaces X and $\overline{X}$ are defined by $P_N=\frac{\det\left(\alpha',N,N'\right)}{\|N'\|^2}$ and $\overline{P}_{\overline{N}}=\frac{\det\left(\overline{\alpha}',\overline{N},\overline{N}'\right)}{\left\|\overline{N}'\right\|^2}$. Then we obtain

$$P_N=\frac{\tau}{\kappa^2-\tau^2},\ \overline{P}_{\overline{N}}=\frac{-\kappa^2\tau'+\kappa\tau\kappa'}{(\kappa'^2-\tau'^2)+(\kappa^2+\tau^2)^2}.$$

Corollary 2.17. *Let the striction curves of X and $\overline{X}$ be given by $\beta(s)=\alpha(s)-\lambda N(s)$ and $\overline{\beta}(s)=\overline{\alpha}(s)-\mu\overline{N}(s)$, respectively.*

(1) *If W is a spacelike vector,then* $\mu=\frac{-\kappa\left(\kappa^2+\tau^2\right)\left(\frac{\kappa}{\lambda}\right)^{\frac{1}{2}}}{\left(\kappa'^2-\tau'^2\right)+(\kappa^2+\tau^2)^2}$.

(2) *If W is a timelike vector,then* $\mu=\frac{-\kappa\left(\kappa^2+\tau^2\right)\left(-\frac{\kappa}{\lambda}\right)^{\frac{1}{2}}}{\left(\kappa'^2-\tau'^2\right)+(\kappa^2+\tau^2)^2}$.

Corollary 2.18. *Let the distribution parameters of the ruled surfaces X and $\overline{X}$ be P_N and $\overline{P}_{\overline{N}}$, respectively.*

(1) *If W is a spacelike vector,then*$\overline{P}_{\overline{N}}=\frac{-\kappa^2\tau'+\kappa\left(-P_N\|W\|^2\right)\kappa'}{\left(\kappa'^2-\tau'^2\right)+(\kappa^2+\tau^2)^2}$.

(2) *If W is a timelike vector,then* $\overline{P}_{\overline{N}}=\frac{-\kappa^2\tau'+\kappa\left(P_N\|W\|^2\right)\kappa'}{\left(\kappa'^2-\tau'^2\right)+(\kappa^2+\tau^2)^2}$.

Corollary 2.19. *If α is a planer curve, then the ruled surface X and $\overline{X}$ are developable.*

(iii) Let X and $\overline{X}$ be two ruled surfaces which are given by

$$X(s,v)=\alpha(s)+vB(s),\ \overline{X}(s,v)=\overline{\alpha}(s)+v\overline{B}(s).$$

The striction curves of X and $\overline{X}$ are given by $\beta(s) = \alpha(s) - \lambda B(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{B}(s)$, respectively. Then we obtain

$$\lambda = 0,\ \mu = \frac{2\kappa^2 \tau \|W\|}{(-\kappa'^2 + \tau'^2) + 4\kappa^2\tau^2}.$$

The distribution parameters of the ruled surfaces X and $\overline{X}$ are defined by $P_B = \frac{\det\left(\alpha', B, B'\right)}{\|B'\|^2}$ and $\overline{P}_{\overline{B}} = \frac{\det\left(\overline{\alpha}', \overline{B}, \overline{B}'\right)}{\|\overline{B}'\|^2}$. Then we have

$$P_B = -\frac{1}{\tau}, \overline{P}_{\overline{B}} = \frac{\kappa^2\tau' - \kappa\tau\kappa'}{(-\kappa'^2 + \tau'^2) + 4\kappa^2\tau^2}.$$

Corollary 2.20. *Let the striction curves of X and $\overline{X}$ be given by $\beta(s) = \alpha(s) - \lambda B(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu\overline{B}(s)$, respectively. Then $\beta(s) = \alpha(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \frac{2\kappa^2\tau\|W\|}{\left(-\kappa'^2+\tau'^2\right)+4\kappa^2\tau^2}\overline{B}(s)$.*

Corollary 2.21. *Let the distribution parameters of the ruled surfaces X and $\overline{X}$ be P_B and $\overline{P}_{\overline{B}}$, respectively. Then $\overline{P}_{\overline{B}} = \frac{\kappa^2\tau' + \kappa\left(\frac{1}{P_B}\right)\kappa'}{\left(-\kappa'^2+\tau'^2\right)+4\kappa^2\tau^2}$.*

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