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STABILITY AND BOUNDEDNESS OF SOLUTIONS OF CERTAIN NONLINEAR DELAY DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract. In this paper, we give some sufficient conditions to guarantee the asymptotic stability and uniform boundedness of certain vector second-order nonlinear delay differential equations with a continuous deviating argument by using a Lyapunov function as basic tool. In doing so we extends some of the existing results in the literature.

Keywords: stability; boundedness; Lyapunov function; delay differential equation; second-order nonlinear differential equations.

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1. Introduction

This paper consider the following second-order nonlinear delay differential equations

$$\ddot{X} + \Phi(\dot{X}) + H(X(t-r(t))) = P(t, X, X(t-r(t)), \dot{X}(t)) \tag{1.1}$$

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where $0 \leq r(t) \leq \rho$, $r'(t) \leq \eta$, $0 < \eta < 1$, ρ and η are some positive constants and ρ will be determined later, $X : \mathbb{R} \to \mathbb{R}^n$, $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, $H \in \mathscr{C}'(\mathbb{R}^n)$ and $P \in \mathscr{C}(\mathbb{R}^n)$ where $\mathscr{C}'(\mathbb{R}^n)$ is the set of all continuous function differentiable once on $\mathbb{R}^n$ and $\mathscr{C}(\mathbb{R}^n)$ is the set of all continuous function on $\mathbb{R}^n$. Let $\mathbb{R}$ denote the real line $-\infty < t < \infty$ and $\mathbb{R}^n$ denote the real n-dimensional Euclidean space equipped with the usual Euclidean norm which will be represented throughout the sequel by $\|.\|$. Moreover, the existence and uniqueness of solutions of (1.1) will be assumed. (See Picard-Lindelof theorem in [10]).

This paper is mainly concerned with the stability and boundedness of solutions of (1.1). For the special case in which (1.1) is a scalar equation (so that $n = 1$) with zero delay, a number of boundedness, stability and convergence of solutions results have been established by [4], [7], [12] and others. The conditions obtained in each of these previous investigations are generalizations in some form or the other of the conditions:

$a > 0$ and $b > 0$ for the scalar equation

$$\ddot{x} + a\dot{x} + bx = p(t)$$

with a, b constants, which conditions ensure the ultimate boundedness and convergence of all solutions if p is bounded.

In a series of papers [1, 5-6, 15-16], many authors have obtained n-dimensional analogue of some of the results reviewed in Ressig et al [11]. However, results for the second-order vector differential equations have appeared only rarely (see Afuwape and Omeike [2], Omeike et al [9] and Tejumola [14]). The Lyapunov's direct method was used with the aid of suitable differentiable auxiliary functions throughout the mentioned papers. Of recent there has been growing interest in the application of Lyapunov's direct method to systems of second-order ordinary differential equations with time lags, after effect, deviating argument, coupled circuits and oscillations that are of importance for certain technical problems. (See [8] and [14]). Consequently, this paper seeks to obtain an analogous result for second-order vector delay differential equation (1.1) by extending the arguments used in some of the papers mentioned above.

2. Notations and definitions

Given any $X,Y \in \mathbb{R}^n$ the symbol $\langle X,Y\rangle$ will be used to denote the usual scalar product in $\mathbb{R}^n$, that is $\langle X,Y\rangle = \sum_{i=1}^{n} x_i y_i$; thus $\|X\|^2 = \langle X,X\rangle$. The matrix A is said to be positive definite when $\langle AX,X\rangle > 0$ for all nonzero X in $\mathbb{R}^n$.

The following notations will be useful in subsequent sections. For $x \in \mathbb{R}^n$, $|x|$ is the norm of x. For a given $r > 0$, $t_1 \in \mathbb{R}$,

$$C(t_1) = \{\phi : [t_1 - r, t_1] \to \mathbb{R}^n \diagup \phi \;\; is \;\; continuous\}.$$

In particular, $C = C(0)$ denotes the space of continuous functions mapping the interval $[-r,0]$ into $\mathbb{R}^n$ and for $\phi \in C$, $\|\phi\| = Sup_{-r\le\theta\le 0}|\phi(0)|$. $C_{\mathbf{H}}$ will denote the set of ϕ such that $\|\phi\| \le \mathbf{H}$. For any continuous function $x(u)$ defined on $-h \le u < A$, $A > 0$, and any fixed t, $0 \le t < A$, the symbol x_t will be denote the restriction of $x(u)$ to the interval $[t-r,t]$, that is, x_t is an element of C defined by $x_t(\theta) = x(t+\theta)$, $-r \le \theta \le 0$.

3. Some preliminary results

We shall state for completeness, some standard results needed in the proofs of our results.

Lemma 1. *Let D be a real symmetric $n \times n$ matrices. Then for any $X \in \mathbb{R}^n$.*

$$\delta_d \|X\|^2 \le \langle DX,X\rangle \le \Delta_d \|X\|$$

where δ_d and Δ_d are the least and greatest eigenvalues of D, respectively.

Proof of lemma 1. see [1,6].

Secondly, we require the following lemma.

Lemma 2. *Let Q,D be real symmetric commuting $n \times n$ matrices. Then;*

(i): *the eigenvalues $\lambda_i(QD), (i = 1,2,\dots,n)$ of the product matrix QD are all real and satisfy*

$$\min_{1\le j,k\le n} \lambda_j(Q)\lambda_k(D) \le \lambda_i(QD) \le \max_{1\le j,k\le n} \lambda_j(Q)\lambda_k(D)$$

(ii): *the eigenvalues $\lambda_i(Q+D),(i=1,2,\ldots,n)$ of the sum of Q and D are all real and satisfy*

$$\left\{\min_{1\leq j,k\leq n}\lambda_j(Q)+\min_{1\leq j,k\leq n}\lambda_k(D)\right\}\leq\lambda_i(Q+D)\leq\left\{\max_{1\leq j,k\leq n}\lambda_j(Q)+\max_{1\leq j,k\leq n}\lambda_k(D)\right\}$$

where $\lambda_j(Q)$ and $\lambda_k(D)$ are respectively the eigenvalues of Q and D.

Proof of lemma 2. see [1,6].

Now, we will state the stability criteria for the general autonomous delay differential system. We consider:

$$\dot{x}=f(x_t),\ \ x_t(\theta)=x(t+\theta)\quad -r\leq\theta\leq 0,t\geq 0, \tag{3.1}$$

where $f:C_{\mathbf{H}}\longrightarrow\mathbb{R}^n$ is a continuous mapping,

$$f(0)=0,C_{\mathbf{H}}:=\{\phi\in(C[-r,0],\mathbb{R}^n):\|\phi\|\leq\mathbf{H}\}$$

and for $\mathbf{H}_1\leq\mathbf{H}$, there exists $L(H_1)>0$, with

$$|f(\phi)|\leq L(H_1)\quad when\quad \|\phi\|\leq\mathbf{H_1}.$$

Here, $\mathscr{C}([-r,0],\mathbb{R}^n)$ is the family of all vector functions mapping $[-r,0]$ into $\mathbb{R}^n$.

Definition 3.0.1. *([3,13]) An element $\psi\in C$ is in the ω-limit set of ϕ, say, $\Omega(\phi)$, if $x(t,0,\phi)$ is defined on $[0,\infty)$ and there is a sequence $\{t_n\}$, $t_n\to\infty$ as $n\to\infty$, with $\|x_{t_n}(\phi)-\psi\|\to 0$ as $n\to\infty$ where*

$$x_{t_n}(\phi)=x(t_n+\theta,0,\phi)\quad for\quad -r\leq\theta\leq 0.$$

$x(t;0,\phi)$ is a motion of a system at $t\in\mathbb{R}$ if and only if $x(0)=\phi$.

Definition 3.0.2. *([3,13]) A set $Q\in C_{\mathbf{H}}$ is an invariant set if for any $\phi\in Q$, the solution of (3.1), $x(t,0,\phi)$, is defined on $[0,\infty)$ and $x_t(\phi)\in Q$ for $t\in[0,\infty)$.*

Lemma 3. *([3,13]) An element $\phi\in C_{\mathbf{H}}$ is such that the solution $x_t(\phi)$ of (3.1) with $x_o(\phi)=\phi$ is defined on $[0,\infty)$ and $\|x_t(\phi)\|\leq\mathbf{H_1}<\mathbf{H}$ for $t\in[0,\infty)$, then $\Omega(\phi)$ (the ω-limit set of ϕ) is a non-empty, compact, invariant set and*

$$dist(x_t(\phi),\Omega(\phi))\to 0\quad as\quad t\to\infty.$$

Lemma 4. *([3,13]) Let $V(\phi) : C_{\mathbf{H}} \longrightarrow \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition. $V(0) = 0$, and such that:*

(i): $W_1|\phi(0)| \leq V(\phi) \leq W_2\|\phi(0)\|$ *where $W_1(r)$, $W_2(r)$ are wedges*

(ii): $\dot{V}_{(3.1)}(\phi) \leq 0$ *for $\phi \in C_{\mathbf{H}}$.*

Then the zero solution of (3.1) is uniformly stable. If we define $Z = \{\phi \in C_{\mathbf{H}} : \dot{V}_{(3.1)}(\phi) = 0\}$, then the zero solution of (3.1) is asymptotically stable provided that the largest invariant set in Z is $Q = \{0\}$.

Next, is the Boundedness criteria for the general autonomous delay differential system.

Lemma 5. *([3]) Let $V(t,\phi) : \mathbb{R} \times C_{\mathbf{H}} \longrightarrow \mathbb{R}$ be continuous and locally Lipschitz in ϕ. If*

(i): $W(|x(t)|) \leq V(t,x_t) \leq W_1(|x(t)|) + W_2\left(\int_{t-r(t)}^{t} W_3(|x(s)|)ds\right)$ *and*

(ii): $\dot{V}_{(3.1)} \leq -W_3((|x(s)|) + M$,

for some $M > 0$, where $W(r)$, $W_i (i = 1,2,3)$ are wedges, then the solutions of (3.1) are uniformly bounded and uniformly ultimately bounded for bound **B**.

It is convenient to consider equation(1.1) in equivalent system form

$$\begin{aligned}
&\dot{X} = Y \\
&\dot{Y} = -\Phi(Y) - H(X) + \int_{t-r(t)}^{t} JH(X(s))Y(s)ds + P(t,X,X(t-r(t)),Y).
\end{aligned} \tag{3.2}$$

4. Statement of results

Throughout the sequel $JH(X)$ and $J\Phi(Y)$ are the Jacobian matrices $\left(\frac{\partial h_i}{\partial x_j}\right), \left(\frac{\partial y_i}{\partial y_j}\right)$ corresponding to the continuous vector functions $H(X), \Phi(Y)$, respectively.

The following will be our main stability result (when $P = 0$) for (3.2).

Theorem 1. *Suppose that $\Phi(0) = 0 = H(0)$, and that*

(i): *The matrices $JH(X)$ and $J\Phi(Y)$ for all X,Y in $\mathbb{R}^n$ are symmetric and commute.*

(ii): *The matrices $JH(X)$ and $J\Phi(Y)$ are positive definite and let $\delta_\phi, \delta_h, \Delta_\phi$ and Δ_h be positive constants such that the eigenvalues $\lambda_i(J\Phi(Y)), \lambda_i(JH(X)) (i = 1,2,\ldots,n)$ of*

$J\Phi(Y)$ and $JH(X)$ respectively are continuous and satisfy

$$0 < \delta_\phi \leq \lambda_i(J\Phi(Y)) \leq \Delta_\phi \tag{4.1a}$$

$$0 < \delta_h \leq \lambda_i(JH(X)) \leq \Delta_h. \tag{4.1b}$$

Then the zero solution of (3.2) is asymptotically stable provided

$$\rho < \min\left\{\frac{2\xi_1\delta_h}{\Delta_h};\frac{5\xi_1'\delta_\phi}{2\lambda+\Delta_h}\right\}.$$

Proof:

Our main tool is the following Lyapunov functional $V = V(X_t, Y_t)$ defined as

$$\begin{aligned} 2V(X_t,Y_t) &= \langle\delta_\phi X,\delta_\phi X\rangle + 2\langle\delta_\phi X, Y\rangle + 2\langle\delta_h X, X\rangle + 2\langle Y,Y\rangle \\ &+ 2\lambda\int_{r(t)}^{0}\int_{t+s}^{t} Y(\theta)Y(\theta)d\theta ds, \end{aligned} \tag{4.2}$$

where λ is a positive constant which will be determined later.

Using Lemma 1 and Lemma 2, the Lyapunov functional (4.2) can be arranged in the form

$$\begin{aligned} 2V(X_t,Y_t) &= \|\delta_\phi X + Y\|^2 + 2\delta_h\|X\|^2 + \|Y\|^2 \\ &+ 2\lambda\int_{r(t)}^{0}\int_{t+s}^{t} Y(\theta)Y(\theta)d\theta ds. \end{aligned}$$

Since $\|\delta_\phi X + Y\|^2 \geq 0$ and $2\lambda\int_{r(t)}^{0}\int_{t+s}^{t} Y(\theta)Y(\theta)d\theta ds$ is non-negative,
we have that

$$2V(X_t,Y_t) \geq 2\delta_h\|X\|^2 + \|Y\|^2.$$

Also, from (4.2), it is clear that

$$\langle\delta_\phi X,\delta_\phi X\rangle \leq \delta_\phi^2\|X\|^2,$$

$$2\delta_h\langle X,X\rangle \leq 2\delta_h\|X\|^2,$$

$$2\langle Y,Y\rangle \leq 2\|Y\|^2$$

and by Schwartz's inequality, the term

$$2\delta_\phi \langle X, Y\rangle \leq \delta_\phi(\|X\|^2 + \|Y\|^2).$$

Combining these estimates above gives

$$2V(X_t, Y_t) \leq D_2(\|X\|^2 + \|Y\|^2).$$

Hence, there exist positive constants $D_1 > 0$, $D_2 > 0$ such that

$$D_1(\|X\|^2 + \|Y\|^2) \leq 2V(X_t, Y_t) \leq D_2(\|X\|^2 + \|Y\|^2), \tag{4.3}$$

where $D_1 = \min\{2\delta_h; 1\}$ and $D_2 = \max\{2\delta_h + \delta_\phi(1+\delta_\phi); 2+\delta_\phi\}$.

Also, we are to show that $V(X_t, Y_t)$ satisfies the second condition of Lemma 4. Thus, using the Lyapunov function (4.2), we get

$$\dot{V}(X_t, Y_t) = -V_1 - V_2 - V_3 + V_4, \tag{4.4}$$

where

$$\begin{aligned}
V_1 &= \xi_1\langle \delta_\phi X, H(X)\rangle + \xi_1'\langle Y, \{4\Phi(Y) - \frac{3}{2}\delta_\phi Y\}\rangle \\
V_2 &= \xi_2\langle \frac{1}{2}\delta_\phi X, H(X)\rangle + \xi_2'\langle Y, \{\Phi(Y) - \frac{3}{4}\delta_\phi Y\}\rangle + \langle 2Y, \{H(X) - \delta_h X\}\rangle \\
V_3 &= \xi_3\langle \frac{1}{2}\delta_\phi X, H(X)\rangle + \xi_3'\langle Y, \{\Phi(Y) - \frac{3}{4}\delta_\phi Y\}\rangle + \langle \delta_\phi X, \{\Phi(Y) - \delta_\phi Y\}\rangle \\
V_4 &= \lambda r(t)\langle Y, Y\rangle - \lambda(1 - r(t))\int_{t-r(t)}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta \\
&+ \langle \delta_\phi X, \int_{t-r(t)}^{t}\langle JH(X(s)), Y(s)ds\rangle + \langle Y, \int_{t-r(t)}^{t}\langle JH(X(s)), Y(s)ds\rangle,
\end{aligned}$$

with ξ_i, ξ_i', $(i = 1, 2, 3)$ are strictly positive constants such that

$$\sum_{i=1}^{3}\xi_i = 1 \quad and \quad \sum_{i=1}^{3}\xi_i' = 1.$$

For strictly positive constants μ_1, μ_2 that will be chosen later, it is clear that

$$\langle 2Y, \{H(X) - \delta_h X\}\rangle$$

$$\begin{aligned} &= \|\mu_1 2^{\frac{1}{2}} Y + 2^{-\frac{1}{2}}\mu_1^{-1}\{H(X)-\delta_h X\}\|^2 - \langle 2\mu_1^2 Y, Y\rangle \\ &- \langle 2^{-1}\mu_1^{-2} H(X)-\delta_h X, H(X)-\delta_h X\rangle \end{aligned}$$

and

$$\langle \delta_\phi X, \{\Phi(Y)-\delta_\phi Y\}\rangle$$

$$\begin{aligned} &= \|\mu_2 \delta_\phi^{\frac{1}{2}} X + 2^{-1}\mu_2^{-1}\delta_\phi^{\frac{1}{2}}\{\Phi(Y)-\delta_\phi Y\}\|^2 - \langle \delta_\phi \mu_2^2 X, X\rangle \\ &- \langle \delta_\phi 2^{-2}\mu_2^{-2}\Phi(Y)-\delta_\phi Y, \Phi(Y)-\delta_\phi Y\rangle. \end{aligned}$$

In view of the assumptions of the Theorem 1, (4.1a) and (4.1b), we have

$$\begin{aligned} V_2 &\geq \|\mu_1 2^{\frac{1}{2}} Y + 2^{-\frac{1}{2}}\mu_1^{-1}\{H(X)-\delta_h X\}\|^2 \\ &+ \xi_2'\langle Y, \{\Phi(Y)-\frac{3}{4}\delta_\phi Y\}\rangle - \langle 2\mu_1^2 Y, Y\rangle \\ &+ \xi_2\langle \frac{1}{2}\delta_\phi X, H(X)\rangle - \langle 2^{-1}\mu_1^{-2}H(X)-\delta_h X, H(X)-\delta_h X\rangle \end{aligned}$$

That is,

$$\begin{aligned} V_2 &\geq \|\mu_1 2^{\frac{1}{2}} Y + 2^{-\frac{1}{2}}\mu_1^{-1}\{H(X)-\delta_h X\}\|^2 \\ &+ \left(\frac{\xi_2'}{4}\delta_\phi - 2\mu_1^2\right)\|Y\|^2 \\ &+ \left(\frac{\xi_2}{2}\delta_\phi\delta_h - \frac{1}{2}\mu_1^{-2}[\Delta_h-\delta_h]^2\right)\|X\|^2 \end{aligned}$$

Thus,

$$V_2 \geq 0, \quad \forall \quad X_t, Y_t \in \mathbb{R}^n,$$

if

$$\mu_1^2 \leq \frac{\xi_2'}{8}\delta_\phi$$

with

$$\delta_\phi \geq \frac{4(\Delta_h-\delta_h)}{\sqrt{\xi_2'\xi_2\delta_h}}.$$

Similarly,

$$\begin{aligned} V_3 &\geq \|\mu_2\delta_\phi^{\frac{1}{2}}X + 2^{-1}\mu_2^{-1}\delta_\phi^{\frac{1}{2}}\{\Phi(Y)-\delta_\phi Y\}\|^2 \\ &+ \left(\frac{\xi_3}{2}\delta_\phi\delta_h - \delta_\phi\mu_2^2\right)\|X\|^2 \\ &+ \left(\frac{\xi_3'}{4}\delta_\phi - \frac{1}{4}\mu_2^{-2}\delta_\phi[\Delta_\phi-\delta_\phi]^2\right)\|Y\|^2 \end{aligned}$$

$$V_3 \geq 0, \quad \forall \quad X_t, Y_t \in \mathbb{R}^n,$$

if

$$\mu_2^2 \leq \frac{\xi_3}{2}\delta_h$$

with

$$\delta_h \geq \frac{2(\Delta_\phi-\delta_\phi)^2}{\xi_3'\xi_3}.$$

We are left with the estimates for V_1 and V_4. From (4.4), we have

$$V_1 \geq \xi_1\delta_\phi\delta_h\|X\|^2 + \frac{5\xi_1'}{2}\delta_\phi\|Y\|^2$$

and

$$\begin{aligned} V_4 &= \lambda r(t)\langle Y,Y\rangle - \lambda(1-r'(t))\int_{t-r(t)}^t \langle Y(\theta),Y(\theta)\rangle d\theta \\ &+ \int_{t-r(t)}^t \langle \delta_\phi X, JH(X(s)Y(s))\rangle ds + \int_{t-r(t)}^t \langle Y, JH(X(s)Y(s))\rangle ds. \end{aligned}$$

Since

$$\int_{t-r(t)}^t \langle \delta_\phi X, JH(X(s)Y(s))\rangle ds \leq \frac{1}{2}\Delta_h r(t)\|X\|^2 + \frac{1}{2}\delta_\phi\Delta_h\int_{t-r(t)}^t \langle Y(s)Y(s)\rangle ds$$

and

$$\int_{t-r(t)}^t \langle Y, JH(X(s)Y(s))\rangle ds \leq \frac{1}{2}\Delta_h r(t)\|Y\|^2 + \frac{1}{2}\Delta_h\int_{t-r(t)}^t \langle Y(s)Y(s)\rangle ds.$$

It follows that

$$\begin{aligned} \dot{V}(X_t,Y_t) &\leq -\left(\xi_1\delta_\phi\delta_h - \frac{1}{2}\delta_\phi\Delta_h r(t)\right)\|X\|^2 - \left(\frac{5}{2}\xi_1'\delta_\phi - \lambda r(t) - \frac{1}{2}\Delta_h r(t)\right)\|Y\|^2 \\ &+ \left(\frac{1}{2}\delta_\phi\Delta_h + \frac{1}{2}\Delta_h - \lambda(1-r'(t))\right)\int_{t-r(t)}^t \langle Y(\theta)Y(\theta)\rangle d\theta \end{aligned}$$

$$\begin{aligned}\dot{V}(X_t,Y_t) &\leq -\left(\xi_1\delta_\phi\delta_h - \frac{1}{2}\delta_\phi\Delta_h\rho\right)\|X\|^2 - \left(\frac{5}{2}\xi_1'\delta_\phi - \lambda r(t) - \frac{1}{2}\Delta_h\rho\right)\|Y\|^2 \\ &+ \left(\frac{1}{2}\delta_\phi\Delta_h + \frac{1}{2}\Delta_h - \lambda(1-\eta)\right)\int_{t-r(t)}^{t}\langle Y(\theta)Y(\theta)\rangle d\theta.\end{aligned}$$

If we choose,

$$\lambda = \frac{(\delta_\phi+1)\Delta_h}{2(1-\eta)},$$

$$\dot{V}(X_t,Y_t) \leq -\left(\xi_1\delta_\phi\delta_h - \frac{1}{2}\delta_\phi\Delta_h\rho\right)\|X\|^2 - \left(\frac{5}{2}\xi_1'\delta_\phi - \lambda r(t) - \frac{1}{2}\Delta_h\rho\right)\|Y\|^2$$

and choosing

$$\rho < \min\left\{\frac{2\xi_1\delta_h}{\Delta_h};\frac{5\xi_1'\delta_\phi}{2\lambda+\Delta_h}\right\}.$$

We have

$$\dot{V}(X_t,Y_t) \leq -D_3(\|X\|^2+\|Y\|^2), \tag{4.5}$$

for some $D_3 > 0$.

It is obvious that the largest invariant set in Z is $Q = \{0\}$, where

$$Z = \{\phi \in C_{\mathbf{H}} : \dot{V}(\phi) = 0\}.$$

It follows that $\dot{V}(X_t,Y_t) = 0$ if and only if $X_t = Y_t = 0$, $\dot{V}(\phi) < 0$ for $\phi \in C_{\mathbf{H}}$ and for $V \geq U(|\phi(0)|) \geq 0$. Thus, (4.3) and (4.5) and the last statement agreed with Lemma 4. This shows that the trivial solution of (1.1) is uniformly asymptotically stable.

Hence, the proof of Theorem 1 is now complete.

5. Boundedness of Solutions

Theorem 2. *We assume that all the assumption of Theorem 1 and*

$$\|P(t,X,X(t-r(t)),Y)\| \leq \Delta_o(\|X\|^2+\|Y\|^2)^{\frac{1}{2}},$$

hold, where Δ_o satisfies $\Delta_o < \varepsilon$ and $\varepsilon > 0$ is a finite constant, then the solutions of (3.2) are uniformly bounded and uniformly ultimately bounded provided ρ satisfies

$$\rho < \min\left\{\frac{2\xi_1\delta_h}{\Delta_h}; \frac{5\xi_1'\delta_\phi}{2\lambda+\Delta_h}\right\}.$$

Proof:

As in Theorem 1, the proof of of Theorem 2 depends on the scalar differentiable Lyapunov function $V(X_t,Y_t)$ defined in (4.2).

Since $P \neq 0$, in (1.1).

In view of (4.5),

$$\dot{V}(X_t,Y_t) \leq \dot{V}_{(3.2)}(X_t,Y_t) + (\delta_\phi\|X\| + \|Y\|)\|P(t,X,X(t-r(t)),Y)\|.$$

Since $\dot{V}_{(3.2)} \leq 0$ for all t,X,Y, thus

$$\begin{aligned}\dot{V}(X_t,Y_t) &\leq -D_3(\|X\|^2+\|Y\|^2) + (\delta_\phi\|X\|+\|Y\|)\Delta_o(\|X\|^2+\|Y\|^2)^{\frac{1}{2}}\\ &\leq -D_3(\|X\|^2+\|Y\|^2) + \sqrt{2}\delta_\phi\Delta_o(\|X\|^2+\|Y\|^2)\\ &\leq -(D_3-D_4\Delta_o)(\|X\|^2+\|Y\|^2).\end{aligned}$$

Let ε be now fixed as: $\varepsilon = D_3D_4^{-1} > 0$. Thus, Δ_o satisfies $\Delta_o < \varepsilon$, then there exists a constant $D_5 > 0$ such that

$$\dot{V}(X_t,Y_t) \leq -D_5(\|X\|^2+\|Y\|^2).$$

The condition of (ii) of Lemma 5 is immediate if, provided $\Delta_o < \varepsilon$.

This completes the proof of Theorem 2.

Conflict of Interests

The authors declare that there is no conflict of interests.

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