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SEQUENCE SPACES OF FUZZY NUMBERS DEFINED BY A SEQUENCE OF MODULI

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Abstract. In this paper, we introduce to certain class of sequence spaces of fuzzy numbers defined by a sequence of modulii and study the topology that arises on the said spaces.

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1. Introduction

The fuzzy theory has emmerged as the most active area of research in many branches of science and engineering. Among various developments of the theory of fuzzy sets[16] a progressive development has been made to find the fuzzy analogues of the classical set theory. In fact the fuzzy theory has become an area of active research for the last 50 years. It has a wide range of applications in the field of science and engineering, e.g. application of fuzzy topology in quantum particle physics, elecronic engineering, chaos control, computer programming,

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electrical engineering, nonlinear dynamical system, population dynamics and biological engineering etc.

In [9], Nanda studied the spaces of bounded and convergent sequences of fuzzy numbers and shown that they are complete metric spaces with the metric

$$\overline{d}(X,Y) = \sup_{0\leq\alpha\leq 1} d(X^{\alpha},Y^{\alpha}).$$

By using the metric , many spaces of fuzzy sequences have been built and published in famous math journals. By reviewing the literature, one can reach them easily,
(e. g., see, [1-6], [9], [10], [12-16] and the references there in.) Here we give the preliminaries.

Let D denote the set of all closed bounded intervals A=[$\underline{A}$, $\overline{A}$] on the real line R.
For A, B $\in$ D define A$\leq$ B iff $\underline{A} \leq \underline{B}$ and $\overline{A} \leq \overline{B}$,

$$d(A,B) = max(|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|)$$

It is easy to see that d defines a metric on D and (D, d) is a complete metric space. Also $\leq$ is a partial order on D.
A fuzzy number is a fuzzy subset of the real line R which is bounded, convex and normal. Let L(R) denote the set of all fuzzy numbers which are upper semicontinuous and have compact support. In other words, if X$\in$L(R) then for any α $\in$[0,1] X^{α} is compact where

$$X^{\alpha} = \begin{cases} t : X(t) \geq \alpha, & \text{if } \alpha \in (0,1], \\ t : X(t) > 0, & \text{if } \alpha = 0. \end{cases}$$

Define a map $\overline{d} : L(R) \times L(R) \to$R by $\overline{d}(X,Y) = \sup_{0\leq\alpha\leq 1} d(X^{\alpha}, Y^{\alpha})$.

For X, Y $\in$ L(R) define X$\leq$Y iff $X^{\alpha} \leq Y^{\alpha}$ for any α $\in$[0, 1].

Now we quote the following definitions which will be needed in the sequel

Definition 1.1. A sequence X=$\{X_k\}$ of fuzzy numbers is a function X from the set N of all positive integers into L(R). The fuzzy number $\{X_k\}$ denotes the value of the function at $k \in N$ and is called the k^{th} term of the sequence.

Definition 1.2. A sequence X=$\{X_k\}$ of fuzzy numbers is said to be convergent to a fuzzy number l, if for every $\varepsilon > 0$ there exists a positive integer n_0 such that

$$\overline{d}(X_k, l) < \varepsilon \text{ for all } k > n_0.$$

Definition 1.3. A sequence X=$\{X_k\}$ of fuzzy numbers is said to be Cauchy if for every $\varepsilon > 0$ there exists a positive integer n_0 such that

$$\overline{d}(X_k, X_m) < \varepsilon \text{ for all } k, m > n_0.$$

By C(F) we denote the set of all Cauchy sequences of fuzzy numbers.

Let $C(R^n) = \{A \subset R^n : \text{ A is compact and convex}\}$. The space $C(R^n)$ has a linear structure induced by the operations

$$A + B = \{a + b, a \in A, b \in B\}$$

$$\mu A = \{\mu a, \mu \in R\}$$

where $A, B \in C(R^n)$ and $\mu \in R$.

The Hausdorff distance between A and B is defined as

$$\delta_\infty(A, B) = max\{\sup_{a\in A} \inf_{b\in B} ||a - b||, \sup_{b\in B} \inf_{a\in A} ||a - b||\}$$

where the $||.||$ denotes the usual Euclidean norm in R^n. It is known that $(C(R^n), \delta_\infty)$ is a complete metric space.

A fuzzy number is a fuzzy subset of the real line R, i.e., a mapping

$$X : R^n \to [0, 1]$$

which is bounded, convex, normal, upper semicontinuous and have compact support. Let $L(R^n)$ denote the the set of all fuzzy numbers. The linear structure of $L(R^n)$ induces addition X+Y and scalar multiplication μX, $\mu \in R$ in terms of α-level sets, by

$$[X+Y]^{\alpha} = [X]^{\alpha} + [Y]^{\alpha}$$

and

$$[\mu X]^{\alpha} = \mu [X]^{\alpha}$$

Clearly $d_{\infty}(X,Y) = \sup_{0 \le \alpha \le 1} \delta_{\infty}(X^{\alpha}, Y^{\alpha})$ is a mertic on $L(R^n)$ and moreover $(L(R^n), d_{\infty})$ is a complete metric space.

By $\omega(F)$, we denote the set of all sequences of fuzzy numbers.

By $\ell_{\infty}(F)$, $c(F)$, $c_0(F)$ and $\ell_p(F)$ we denote the set of all bounded, convergent, null and absolutely p-summable sequences of fuzzy numbers respectively.

$$\ell_{\infty}(F) = \{(X_k) \in \omega(F) : \sup_{k \in N} \overline{d}(X_k, \overline{0}) < \infty\},$$

$$c(F) = \{(X_k) \in \omega(F) : \lim_{k \to \infty} \overline{d}(X_k, l) = 0\},$$

$$c_0(F) = \{(X_k) \in \omega(F) : \lim_{k \to \infty} \overline{d}(X_k, \overline{0}) = 0\},$$

$$\ell_p(F) = \{(X_k) \in \omega(F) : \sum_k \overline{d}(X_k, \overline{0})^p < \infty\}.$$

Here we give a brief account of the recent developments in this direction.

Talo and Basar[12-14] used the idea of modulus function f to define the following fuzzy sequence spaces

$$\ell_{\infty}(F,f) = \{X = (X_k) \in \omega(F) : \sup_{k \in N} f[\overline{d}(X_k, \overline{0})] < \infty\},$$

$$c(F,f) = \{X = (X_k) \in \omega(F) : \lim_{k \to \infty} f[\overline{d}(X_k, l)] = 0\},$$

$$c_0(F,f) = \{X = (X_k) \in \omega(F) : \lim_{k \to \infty} f[\overline{d}(X_k, \overline{0})] = 0\},$$

$$\ell_p(F,f,s) = \{X = (X_k) \in \omega(F) : \sum_k \frac{\{f[\overline{d}(X_k, \overline{0})]\}^p}{k^s} < \infty\}.$$

Hazarika[6] used the Orlicz function M and $\Lambda = (\gamma_k)$ a sequence of non zero scalars to define the following fuzzy sequence spaces

$$c^F(M,\Lambda) = \{(X_k) \in \omega^F : M(\frac{\lambda(\gamma_k X_k, X_0)}{\mu}) \to 0 \text{ and } M(\frac{\rho(\gamma_k X_k, X_0)}{\mu}) \to 0 \text{ as } k \to \infty\},$$

$$c_0^F(M,\Lambda) = \{(X_k) \in \omega^F : M(\frac{\lambda(\gamma_k X_k, \overline{0})}{\mu}) \to 0 \text{ and } M(\frac{\rho(\gamma_k X_k, \overline{0})}{\mu}) \to 0 \text{ as } k \to \infty\},$$

$$\ell_\infty^F(M,\Lambda) = \{(X_k) \in \omega^F : \sup_k M(\frac{\lambda(\gamma_k X_k, \overline{0})}{\mu}) < \infty \text{ and } \sup_k M(\frac{\rho(\gamma_k X_k, \overline{0})}{\mu}) < \infty.\}$$

for some $\mu > 0$. Here λ(X, Y)= $\sup_{0\leq\alpha\leq 1} \lambda_\alpha(X^\alpha, Y^\alpha)$ and ρ(X, Y)= $\sup_{0\leq\alpha\leq 1} \rho_\alpha(X^\alpha, Y^\alpha)$.

Recently Mursaleen and Noman[7-8] introduced the notion of λ-convergent sequences. Let ω be the set of all complex sequences $x = (x_k)$ and $\lambda = (\lambda_k)_{k=1}^\infty$ be a strictly increasing sequence of positive real numbers tending to infinity.

A sequence $x = (x_k) \in \omega$ is λ-convergent to the number L called the λ-limit of x if $\Lambda_m(x) \to L$ as $m \to \infty$, where

$$\Lambda_m(x) = \frac{1}{\lambda_m}\sum_{k=1}^m (\lambda_k - \lambda_{k-1})x_k.$$

It is well known that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \frac{1}{\lambda_m}(\sum_{k=1}^m (\lambda_k - \lambda_{k-1})|x_k - a|) = 0.$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m \frac{1}{\lambda_m}|\sum_{k=1}^m (\lambda_k - \lambda_{k-1})(x_k - a)| = 0.$$

Let C denotes the space whose elements are the sets of distinct positive integers. Given any element σ of C, we denote by c(σ) the sequence $\{c_n(\sigma)\}$ such that

$c_n(\sigma) = 1$ if $n \in \sigma$, $c_n(\sigma) = 0$ otherwise.

$$C_s = \{\sigma \in C : \sum_{n=1}^\infty c_n(\sigma) \leq s\}$$

Sargent[11] defined the sequence space

$$m(\varphi) = \{x = (x_k) \in \omega : \sup_{s\geq 1}\sup_{\sigma\in C_s}\frac{1}{\varphi_s}(\sum_{k\in\sigma}|x_k|) < \infty\}.$$

where $\{\varphi_k\}$ is a real sequence.

Alotaibi, Mursaleen, Sharma and Mohiuddine[1] used the Musielak-Orlicz function $M= (M_k)$, $p = (p_k)$ a bounded sequence of positive real numbers and σ one- to- one mapping from the set of positive integers into it self such that $\sigma^k(n) = \sigma(\sigma^{k-1}(n))$ to define the following fuzzy sequence spaces

$$\ell_\infty^F(M,\Lambda,\sigma,p) = \{X = (X_k) \in \omega(F) : \sup_{k,n}[M_k(\frac{d(\Lambda_k X_{\sigma^k(n)},\overline{0})}{\rho})]^{p_k} < \infty\},$$

$$\ell_1^F(M,\Lambda,\sigma,p) = \{X = (X_k) \in \omega(F) : \sup_n \sum_k [M_k(\frac{d(\Lambda_k X_{\sigma^k(n)},\overline{0})}{\rho})]^{p_k} < \infty\},$$

$$m^F(M,\Lambda,\varphi,\sigma,p) = \{X = (X_k) \in \omega(F) : \sup_n \sup_{s\geq 1,\sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} [M_k(\frac{d(\Lambda_k X_{\sigma^k(n)},\overline{0})}{\rho})]^{p_k} < \infty\}.$$

2. Main results

In this article we introduce the following classes of sequences of fuzzy numbers using the sequence of modulii $F = (f_k)$

$$\ell_\infty^F(F,\Lambda,\sigma,p) = \{X = (X_k) \in \omega(F) : \sup_{k,n}[f_k(d(\Lambda_k X_{\sigma^k(n)},\overline{0}))]^{p_k} < \infty\},$$

$$\ell_1^F(F,\Lambda,\sigma,p) = \{X = (X_k) \in \omega(F) : \sup_n \sum_k [f_k(d(\Lambda_k X_{\sigma^k(n)},\overline{0}))]^{p_k} < \infty\},$$

$$m^F(F,\Lambda,\varphi,\sigma,p) = \{X = (X_k) \in \omega(F) : \sup_n \sup_{s\geq 1,\sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} [f_k(d(\Lambda_k X_{\sigma^k(n)},\overline{0}))]^{p_k} < \infty\}.$$

When $\sigma(n) = \sigma(n+1)$ we obtain the following classes of sequences of fuzzy numbers

$$\ell_\infty^F(F,\Lambda,p) = \{X = (X_k) \in \omega(F) : \sup_{k,n}[f_k(d(\Lambda_k X_{k+n},\overline{0}))]^{p_k} < \infty\},$$

$$\ell_1^F(F,\Lambda,\sigma,p)=\{X=(X_k)\in\omega(F):\sup_n\sum_k[f_k(d(\Lambda_kX_{k+n},\overline{0}))]^{p_k}<\infty\},$$

$$m^F(F,\Lambda,\varphi,\sigma,p)=\{X=(X_k)\in\omega(F):\sup_n\sup_{s\geq1,\sigma\in C_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}[f_k(d(\Lambda_kX_{k+n},\overline{0}))]^{p_k}<\infty\}.$$

If $p=(p_k)=1$ then we have the following classes of sequences of fuzzy numbers

$$\ell_\infty^F(F,\Lambda,\sigma,p)=\{X=(X_k)\in\omega(F):\sup_{k,n}[f_k(d(\Lambda_kX_{\sigma^k(n)},\overline{0}))]<\infty\},$$

$$\ell_1^F(F,\Lambda,\sigma,p)=\{X=(X_k)\in\omega(F):\sup_n\sum_k[f_k(d(\Lambda_kX_{\sigma^k(n)},\overline{0}))]<\infty\},$$

$$m^F(F,\Lambda,\varphi,\sigma,p)=\{X=(X_k)\in\omega(F):\sup_n\sup_{s\geq1,\sigma\in C_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}[f_k(d(\Lambda_kX_{\sigma^k(n)},\overline{0}))]<\infty\}.$$

Theorem 2.1. $\ell_\infty^F(F,\Lambda,\sigma,p),\ell_1^F(F,\Lambda,\sigma,p)$ and $m^F(F,\Lambda,\varphi,\sigma,p)$ are linear spaces over the field C of complex numbers.

Proof. We prove the result for $m^F(F,\Lambda,\varphi,\sigma,p)$.
Let $X=(X_k),Y=(Y_k)\in m^F(F,\Lambda,\varphi,\sigma,p)$ and $\alpha,\beta\in C$
Then we have

$$\sup_n\sup_{s\geq1,\sigma\in C_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}[f_k(d(\Lambda_kX_{\sigma^k(n)},\overline{0}))]^{p_k}<\infty;$$

and

$$\sup_n\sup_{s\geq1,\sigma\in C_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}[f_k(d(\Lambda_kY_{\sigma^k(n)},\overline{0}))]^{p_k}<\infty.$$

Since $F=(f_k)$ is non decreasing and continuous, we have

$$\sup_n\sup_{s\geq1,\sigma\in C_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}[f_k(d(\Lambda_k(\alpha X_{\sigma^k(n)}+\beta Y_{\sigma^k(n)}),\overline{0}))]^{p_k}$$

$$\leq\sup_n\sup_{s\geq1,\sigma\in C_s}\frac{1}{\varphi_s}\sum_{k\in\sigma}[f_k(\alpha d(\Lambda_kX_{\sigma^k(n)},\overline{0})+\beta d(\Lambda_kY_{\sigma^k(n)},\overline{0}))]^{p_k}$$

$$\leq \alpha \sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} [f_k(d(\Lambda_k X_{\sigma^k(n)}, \overline{0}))]^{p_k}$$
$$+\beta \sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} [f_k(d(\Lambda_k Y_{\sigma^k(n)}, \overline{0}))]^{p_k} < \infty$$

. Hence $m^F(F,\Lambda,\varphi,\sigma,p)$ is a linear space. Similarly we can prove that $\ell_\infty^F(F,\Lambda,\sigma,p)$ and $\ell_1^F(F,\Lambda,\sigma,p)$ are linear spaces.

Theorem 2.2. Let $F=(f_k)$ be a sequence of modulii and $p=(p_k)$ be a bounded sequence of positive real numbers, then the sequence space $m^F(F,\Lambda,\varphi,\sigma,p)$ is a complete metric space, with the metric defined by

$$g(X,Y) = \sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} [f_k(d(\Lambda_k(X_{\sigma^k(n)}, Y_{\sigma^k(n)}))))]^{p_k}$$

Proof. Let (X^i) be a Cauchy sequence in $m^F(F,\Lambda,\varphi,\sigma,p)$. Then,

$$g(X^i,X^j) = \sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} [f_k(d(\Lambda_k(X^i_{\sigma^k(n)}, X^j_{\sigma^k(n)}))))]^{p_k} \to 0 \text{ as i, j} \to \infty$$

Hence

$$[f_k(d(\Lambda_k(X^i_{\sigma^k(n)}, X^j_{\sigma^k(n)}))))]^{p_k} \to 0 \text{ as i, j} \to \infty, \text{ for all n.}$$

Therefore (X^i) is a Cauchy sequence in $L(R^n)$. Since $L(R^n)$ is complete, it is convergent so that $\lim_{i\to\infty} X_k^i = X_k$, for each $k\in N$. since (X^i) is a Cauchy sequence for each $\varepsilon>0$, there exists $n_0=n_0(\varepsilon)$ such that

$$g(X^i,X^j) < \varepsilon \text{ for all i, j} \geq n_0$$

So we have

$$\lim_j \sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} [f_k(d(\Lambda_k(X^i_{\sigma^k(n)}, X^j_{\sigma^k(n)}))))]^{p_k}$$
$$= \sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} [f_k(d(\Lambda_k(X^i_{\sigma^k(n)}, X_{\sigma^k(n)}))))]^{p_k}$$
$$< \varepsilon \text{ for all i} \geq n_0.$$

This implies that $g(X^i,X)<\varepsilon$ for all $i\geq n_0$ i.e $X^i\to X$ as $i\to\infty$.

Since

$$\sup_n \sup_{s\geq 1,\sigma\in C_s} \frac{1}{\varphi_s}\sum_{k\in\sigma}[f_k(d(\Lambda_k(X^{n_0}_{\sigma^k(n)},X_0)))]^{p_k}$$

$$\leq \sup_n \sup_{s\geq 1,\sigma\in C_s} \frac{1}{\varphi_s}\sum_{k\in\sigma}[f_k(d(\Lambda_k(X^{n_0}_{\sigma^k(n)},X_{\sigma^k(n)})))]^{p_k}$$

$$+\sup_n \sup_{s\geq 1,\sigma\in C_s} \frac{1}{\varphi_s}\sum_{k\in\sigma}[f_k(d(\Lambda_k(X_{\sigma^k(n)},X_0)))]^{p_k}$$

Thus we obtain $X=(X_k)\in m^F(F,\Lambda,\varphi,\sigma,p)$.

Hence $m^F(F,\Lambda,\varphi,\sigma,p)$ is a complete metric space.

Theorem 2.3. Let $F=(f_k)$ be a sequence of modulii and $p=(p_k)$ be a bounded sequence of positive real numbers, then the sequence space

(a) $\ell_1^F(F,\Lambda,\sigma,p)$ is a complete metric space, with the metric defined by

$$g(X,Y)=\sup_n\sum_k[f_k(d(\Lambda_k(X_{\sigma^k(n)},Y_{\sigma^k(n)})))]^{p_k}$$

(b)$\ell_\infty^F(F,\Lambda,\sigma,p)$ is a complete metric space, with the metric defined by

$$g(X,Y)=\sup_{k,n}[f_k(d(\Lambda_k(X_{\sigma^k(n)},Y_{\sigma^k(n)})))]^{p_k}$$

Proof. The proof is analogous to Theorem 2.2, so we omit the details.

Theorem 2.4. Let $F=(f_k)$ be a sequence of modulii and $p=(p_k)$ be a bounded sequence of positive real numbers, then $m^F(F,\Lambda,\varphi,\sigma,p)\subset m^F(F,\Lambda,\psi,\sigma,p)$ if and only if $\sup_{s\geq 1}\frac{\varphi_s}{\psi_s}<\infty$

Proof. Let $\sup_{s\geq 1}\frac{\varphi_s}{\psi_s}<\infty$ and $(X_k)\in m^F(F,\Lambda,\varphi,\sigma,p)$ Then,

$$\sup_n \sup_{s\geq 1,\sigma\in C_s} \frac{1}{\varphi_s}\sum_{k\in\sigma}[f_k(d(\Lambda_kX_{\sigma^k(n)},\overline{0}))]^{p_k}<\infty$$

This implies that

$$\sup_n \sup_{s\geq 1,\sigma\in C_s} \frac{1}{\psi_s}\sum_{k\in\sigma}[f_k(d(\Lambda_kX_{\sigma^k(n)},\overline{0}))]^{p_k}$$

$$\leq \{\sup_{s\geq 1} \frac{\varphi_s}{\psi_s}\} \sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} [f_k(d(\Lambda_k X_{\sigma^k(n)}, \overline{0}))]^{p_k} < \infty$$

Therefore $(X_k) \in m^F(F, \Lambda, \psi, \sigma, p)$. Hence $m^F(F, \Lambda, \varphi, \sigma, p) \subset m^F(F, \Lambda, \psi, \sigma, p)$.

Conversely, let $m^F(F, \Lambda, \varphi, \sigma, p) \subset m^F(F, \Lambda, \psi, \sigma, p)$. Suppose that $\sup_{s\geq 1} \frac{\varphi_s}{\psi_s} = \infty$, then there exists a sequence of natural numbers (s_i) such that $\lim_{i\to\infty} \frac{\varphi_{si}}{\psi_{si}} = \infty$

Let $(X_k) \in m^F(F, \Lambda, \varphi, \sigma, p)$ Then,

$$\sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} [f_k(d(\Lambda_k X_{\sigma^k(n)}, \overline{0}))]^{p_k} < \infty$$

We have

$\sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\psi_s} \sum_{k\in\sigma} [f_k(d(\Lambda_k X_{\sigma^k(n)}, \overline{0}))]^{p_k}$

$$\geq \{\sup_{i\geq 1} \frac{\varphi_{si}}{\psi_{si}}\} \sup_n \sup_{i\geq 1, \sigma\in C_s} \frac{1}{\varphi_{si}} \sum_{k\in\sigma} [f_k(d(\Lambda_k X_{\sigma^k(n)}, \overline{0}))]^{p_k} = \infty$$

Therefore $(X_k) \notin m^F(F, \Lambda, \psi, \sigma, p)$ which is a contradiction. Therefore $\sup_{s\geq 1} \frac{\varphi_s}{\psi_s} < \infty$. This completes the proof of the theorem.

Theorem 2.5. Let $F = (f_k)$ be a sequence of modulii and $p = (p_k)$ be a bounded sequence of positive real numbers, then $m^F(F, \Lambda, \varphi, \sigma, p) = m^F(F, \Lambda, \psi, \sigma, p)$ if and only if $\sup_{s\geq 1} \frac{\varphi_s}{\psi_s} < \infty$ and $\sup_{s\geq 1} \frac{\psi_s}{\varphi_s} < \infty$

Proof. The proof follows from Theorem 2.4

Theorem 2.6. Let $F = (f_k)$ be a sequence of modulii and $p = (p_k)$ be a bounded sequence of positive real numbers, then

$$\ell_1^F(F, \Lambda, \sigma, p) \subset m^F(F, \Lambda, \varphi, \sigma, p) \subset \ell_\infty^F(F, \Lambda, \sigma, p)$$

Proof. Let $(X_k) \in \ell_1^F(F,\Lambda,\sigma,p)$ then we have

$$\sup_n \sum_k [f_k(d(\Lambda_k X_{\sigma^k(n)},\overline{0}))]^{p_k} < \infty.$$

Since (φ_n) is monotonic increasing, so we have

$$\frac{1}{\varphi_s}\sum_{k\in\sigma}[f_k(d(\Lambda_k X_{\sigma^k(n)},\overline{0}))]^{p_k} \le \frac{1}{\varphi_1}\sum_{k\in\sigma}[f_k(d(\Lambda_k X_{\sigma^k(n)},\overline{0}))]^{p_k}$$

$$\le \frac{1}{\varphi_1}\sum_k [f_k(d(\Lambda_k X_{\sigma^k(n)},\overline{0}))]^{p_k} < \infty.$$

Hence

$$\sup_n \sup_{s\ge 1,\sigma\in C_s} \frac{1}{\varphi_s}\sum_{k\in\sigma}[f_k(d(\Lambda_k X_{\sigma^k(n)},\overline{0}))]^{p_k} < \infty$$

Thus $(X_k) \in m^F(F,\Lambda,\varphi,\sigma,p)$. Therefore $\ell_1^F(F,\Lambda,\sigma,p) \subset m^F(F,\Lambda,\varphi,\sigma,p)$

Next let $(X_k) \in m^F(F,\Lambda,\varphi,\sigma,p)$

Then we have

$$\sup_n \sup_{s\ge 1,\sigma\in C_s} \frac{1}{\varphi_s}\sum_{k\in\sigma}[f_k(d(\Lambda_k X_{\sigma^k(n)},\overline{0}))]^{p_k} < \infty$$

Thus

$$\sup_{k,n} \frac{1}{\varphi_1}\sum_{k\in\sigma}[f_k(d(\Lambda_k X_{\sigma^k(n)},\overline{0}))]^{p_k} < \infty, \text{(on taking cardinality of } \sigma \text{ to be 1).}$$

Therefore $(X_k) \in \ell_\infty^F(F,\Lambda,\sigma,p)$

Hence $m^F(F,\Lambda,\varphi,\sigma,p) \subset \ell_\infty^F(F,\Lambda,\sigma,p)$.

Therefore we have

$$\ell_1^F(F,\Lambda,\sigma,p) \subset m^F(F,\Lambda,\varphi,\sigma,p) \subset \ell_\infty^F(F,\Lambda,\sigma,p)$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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