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# COMMON FIXED POINT THEOREMS IN INTUITIONISTIC FUZZY METRIC SPACES USING OCCASIONALLY WEAKLY COMPATIBLE MAPS

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**Abstract.** In this paper, we obtain common fixed point theorems in intuitionistic fuzzy metric spaces using occasionally weakly compatible maps. Our results are the intuitionistic fuzzy version of some fixed point theorems for occasionally weakly compatible mappings on different metric spaces.

**Keywords**: Intuitionistic fuzzy metric space, Occasionally weakly compatible mappings, Common fixed point.

**2010 AMS Subject Classification:** 47H10; 54H25

## 1. Introduction

Atanassove[4] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets[11]. In 2004, Park[9] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms. Recently, in 2006, Alaca et al.[1] using the idea of intuitionistic fuzzy sets, defined the notion of

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intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek[5] . In the literature, many results have been proved for contraction maps in different settings such as fuzzy metric spaces [7], intuitionistic fuzzy metric spaces [6]. In this paper, we obtain common fixed point theorems in intuitionistic fuzzy metric spaces using occasionally weakly compatible maps. Our results are the intuitionistic fuzzy version of some fixed point theorems for occasionally weakly compatible mappings on different metric spaces.

## 2. Preliminaries

The concepts of triangular norms ($t$-norms) and triangular conorms ($t$-conorms) are known as the axiomatic skelton that we use are characterization fuzzy intersections and union respectively. These concepts were originally introduced by Menger [8] in study of statistical metric spaces.

**Definition 2.1[10].** A binary operation * : [0,1]×[0,1] → [0,1] is continuous $t$-norm if * satisfies the following conditions:

(i) * is commutative and associative;

(ii) * is continuous;

(iii) $a * 1 = a$ for all $a \in [0,1]$ ;

(iv) $a * b \leq c * d$ whenever a $\leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$ .

**Definition 2.2[10].** A binary operation ⋄ : [0,1]×[0,1] → [0,1] is continuous $t$-conorm if ⋄ satisfies the following conditions:

(i) ⋄ is commutative and associative;

(ii) ⋄ is continuous;

(iii) $a \diamond 0 = a$ for all $a \in [0,1]$ ;

(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$ .

Alaca et al. [1] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous $t$-norm and continuous $t$-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [5] as :

**Definition 2.3[1].** A 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous $t$-norm, $\diamond$ is a continuous $t$-conorm and $M, N$ are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;

(ii) $M(x, y, 0) = 0$ for all $x, y \in X$ ;

(iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y \in X$ and $s, t > 0$;

(vi) for all $x, y \in X$ , $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous;

(vii) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;

(viii) $N(x, y, 0) = 1$ for all $x, y \in X$ ;

(ix) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if x = y;

(x) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(xi) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$ for all $x, y \in X$ and $s, t > 0$;

(xii) for all $x, y \in X$, N(x, y, .) : $[0, \infty) \to [0, 1]$ is right continuous;

(xiii) $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$.

Then $(M, N)$ is called an intuitionistic fuzzy metric space on $X$. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ w.r.t. $t$ respectively.

**Remark 2.1.** Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \diamond)$ such that $t$-norm $*$ and $t$-conorm $\diamond$ are associated as $x \diamond y = 1 - ((1 - x) * (1 - y))$ for all $x, y \in X$ .

**Remark 2.2.** In intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$, $M(x, y, *)$ is non-decreasing and $N(x, y, \diamond)$ is non-increasing for all $x, y \in X$.

Alaca, Turkoglu and Yildiz [1] introduced the following notions:

**Definition 2.4[1].** Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in $X$ is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$,

$\lim_{n\to\infty}$ M($x_{n+p}$,$x_n$, t) = 1 and $\lim_{n\to\infty}$ N($x_{n+p}$,$x_n$, t) = 0,

(b) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$,

$\lim_{n\to\infty}$ M($x_n$,$x$,t)= 1 and $\lim_{n\to\infty}$ N($x_n$,$x$,t) = 0.

**Definition 2.5[1].** An intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be complete if and only if every Cauchy sequence in $X$ is convergent.

In 1976, Jungck [2] introduced the notion of weakly compatible maps as follows:

**Definition 2.6[2].** A pair of self mappings $(f, g)$ of a metric space is said to be weakly compatible if they commute at the coincidence points i.e. $fu = gu$ for some $u \in X$ , then $fgu = gfu$.

**Definition 2.7.**[2] Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. $f$ and $g$ be self maps on $X$. A point $x$ in $X$ is called a coincidence point of $f$ and $g$ iff $fx = gx$. In this case, $w = fx = gx$ is called a point of coincidence of $f$ and $g$.

**Definition 2.8.**[2] A pair of self mappings $(f, g)$ of intuitionistic fuzzy metric space is said to be weakly compatible if they commute at the coincidence points i.e., if $fu = gu$ for some $u \in X$ , then $fgu = gfu$.

It is easy to see that two compatible maps are weakly compatible but converse is not true.

**Definition 2.9[3].** Two self mappings $f$ and $g$ of intuitionistic fuzzy metric space are said to be occasionally weakly compatible (owc) iff there is a point $x$ in $X$ which is coincidence point of $f$ and $g$ at which $f$ and $g$ commute.

**Lemma 2.1[3].** Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. $f$ and $g$ be self maps on $X$ and let $f$ and $g$ have a unique point of coincidence, $w = fx = gx$, then $w$ is the unique common fixed point of $f$ and $g$.

**Proof.** Since $f$ and $g$ are owc, there exists a point $x$ in $X$ such that $fx = gx = w$ and $fgx = gfx$. Thus, $ffx = fgx = gfx$, which says that $fx$ is also a point of coincidence of $f$ and $g$. Since the point of coincidence $w = fx$ is unique by hypothesis, gfx = ffx = fx, and $w = fx$ is a common fixed point of $f$ and $g$.

Moreover, if $z$ is any common fixed point of $f$ and $g$, then $z = fz = gz = w$ by the uniqueness of the point of coincidence.

Alaca [1] proved the following results:

**Lemma 2.2[1].** Let $(X, M, N, *, \diamond)$ be intuitionistic fuzzy metric space and for all $x, y \in X$, $t > 0$ and if for a number $k \in (0, 1)$ such that $M(x, y, kt) \geq M(x, y, t)$ and $N(x, y, kt) \leq N(x, y, t)$. Then, $x = y$.

## 3. Main results

**Theorem 3.1.** Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space with continuous $t$-norm * and continuous $t$-conorm $\diamond$ . Let $A, B, S$ and $T$ be self mappings of $X$. Let the pairs $(A, S)$ and $(B, T)$ be owc. If there exist $k \in (0, 1)$ such that

(3.1)

M(Ax,By,kt) $\geq$ min {M(Sx,Ty,t),M(By,Sx,t),M(Sx,Ax,t).

M(By,Ty,t),M(Ax,Ty,t).$(\frac{1+M(Sx,Ax,t)}{1+M(By,Ty,t)})$}

and

N(Ax,By,kt) $\leq$ max {N(Sx,Ty,t),N(By,Sx,t),N(Sx,Ax,t).

N(By,Ty,t),N(Ax,Ty,t).$(\frac{1+N(Sx,Ax,t)}{1+N(By,Ty,t)})$}

for all $x, y \in X$ and $t > 0$. Then, there is a unique common fixed point of $A, B, S$ and $T$.

**Proof.** As the pairs $(A, S)$ and $(B, T)$ are owc, so there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. By (3.1), we have,

M(Ax,By,kt) $\geq$ min{M(Sx,Ty,t),M(By,Sx,t),M(Sx,Ax,t).

M(By,Ty,t),M(Ax,Ty,t).$(\frac{1+M(Sx,Ax,t)}{1+M(By,Ty,t)})$}

M(Ax,By,kt) $\geq$ min{M(Ax,By,t),M(By,Ax,t),M(Ax,Ax,t).

M(By,By,t),M(Ax,By,t).$(\frac{1+M(Ax,Ax,t)}{1+M(By,By,t)})$}

M(Ax,By,kt) $\geq$ min {M(Ax,By,t),M(By,Ax,t),1,M(Ax,By,t)}= M(Ax,By,t)

and

N(Ax,By,kt) $\leq$ max {N(Sx,Ty,t),N(By,Sx,t),N(Sx,Ax,t).

N(By,Ty,t),N(Ax,Ty,t).$(\frac{1+N(Sx,Ax,t)}{1+N(By,Ty,t)})$}

N(Ax,By,kt) $\leq$ max{N(Ax,By,t),N(By,Ax,t),N(Ax,Ax,t).

N(By,By,t),N(Ax,By,t).$(\frac{1+N(Ax,Ax,t)}{1+N(By,By,t)})$}

N(Ax,By,kt) $\leq$ max {N(Ax,By,t),N(By,Ax,t),0,N(Ax,By,t)} = N(Ax,By,t)

therefore, by lemma 2.2, $Ax = By$ i.e., $Ax = Sx = By = Ty$. Suppose that, there is another point $z$ such that $Az = Sz$ then by inequality (3.1), we have $Az = Sz = By = Ty$ so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of $A$ and $S$. By lemma 2.1, $w$ is the only common fixed point of $A$ and $S$. Similarly, there is a unique point $z$ in $X$ such that $z = Bz = Tz$. We now show that $z = w$. By (3.1), we have,

M(w,z,kt)= M(Aw,Bz,kt) $\geq$ min {M(Sw,Tz,t),M(Bz,Sw,t),M(Sw,Aw,t).

M(Bz,Tz,t),M(Aw,Tz,t).$(\frac{1+M(Sw,Aw,t)}{1+M(Bz,Tz,t)})$}

= min {M(w,z,t),M(z,w,t),M(w,w,t).M(z,z,t),M(w,z,t).$(\frac{1+M(w,w,t)}{1+M(z,z,t)})$}

= min {M(w,z,t),M(z,w,t),1,M(w,z,t)}= M(w,z,t)

and

N(w,z,kt)= N(Aw,Bz,kt) $\leq$ max {N(Sw,Tz,t),N(Bz,Sw,t),N(Sw,Aw,t).

N(Bz,Tz,t),N(Aw,Tz,t).($\frac{1+N(Sw,Aw,t)}{1+N(Bz,Tz,t)}$)}

= max {N(w,z,t),N(z,w,t),N(w,w,t).N(z,z,t),N(w,z,t).($\frac{1+N(w,w,t)}{1+N(z,z,t)}$)}

= max {N(w,z,t),N(z,w,t),0,N(w,z,t)}= N(w,z,t).

therefore, by lemma 2.2, we have $w = z$, hence $z$ is a common fixed point of $A$, $B$, $S$ and $T$. For uniqueness, let $u$ be another common fixed point of $A$, $B$, $S$ and $T$. Then, by (3.1), we have

M(z,u,kt)= M(Az,Bu,kt) $\geq$ min {M(Sz,Tu,t),M(Bu,Sx,t),M(Sz,Az,t).

M(Bu,Tu,t),M(Az,Tu,t).($\frac{1+M(Sz,Az,t)}{1+M(Bu,Tu,t)}$)}

= min {M(z,u,t),M(u,z,t),M(z,z,t).M(u,u,t),M(z,u,t).($\frac{1+M(z,z,t)}{1+M(u,u,t)}$)}

= min{M(z,u,t),M(u,z,t),1,M(z,u,t)}= M(z,u,t)

and

N(z,u,kt)= N(Az,Bu,kt) $\leq$ max {N(Sz,Tu,t),N(Bu,Sx,t),N(Sz,Az,t).

N(Bu,Tu,t),N(Az,Tu,t).($\frac{1+N(Sz,Az,t)}{1+N(Bu,Tu,t)}$)}

= max {N(z,u,t),N(u,z,t),N(z,z,t).N(u,u,t),N(z,u,t).($\frac{1+N(z,z,t)}{1+N(u,u,t)}$)}

= max{N(z,u,t),N(u,z,t),0,N(z,u,t)}= N(z,u,t).

Therefore, by lemma 2.2, we have $z = u$. Hence, z is unique common fixed point of $A$, $B$, $S$ and $T$.

**Theorem 3.2.** Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space with continuous t-norm * and continuous t-conorm $\diamond$. Let $A$, $B$, $S$ and $T$ be self mappings of $X$. Let the pairs $(A, S)$ and $(B, T)$ be owc. If there exist $k \in (0, 1)$ such that

(3.2)

M(Ax,By,kt) $\geq$ $\phi$ (min {M(Sx,Ty,t),M(By,Sx,t),M(Sx,Ax,t).

M(By,Ty,t),M(Ax,Ty,t).($\frac{1+M(Sx,Ax,t)}{1+M(By,Ty,t)}$)})

and

N(Ax,By,kt) $\leq \psi$ (max {N(Sx,Ty,t),N(By,Sx,t),N(Sx,Ax,t).

N(By,Ty,t),N(Ax,Ty,t).($\frac{1+N(Sx,Ax,t)}{1+N(By,Ty,t)}$)})

for all $x, y \in X$, $t > 0$ and $\phi, \psi$:[0,1]$\rightarrow$ [0,1] such that $\phi$(t)> t,$\psi$(t)<t for all $t \in (0,1)$ . Then, there is a unique common fixed point of $A$, $B$, $S$ and $T$.

**Proof.** The proof follows on the lines of theorem 3.1.

**Theorem 3.3.** Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space with continuous t-norm * and continuous t-conorm $\diamond$ . Let $A$, $B$, $S$ and $T$ be self mappings of X. Let the pairs $(A, S)$ and $(B, T)$ be owc. If there exist $k \in (0, 1)$ such that

(3.3)

M(Ax,By,kt) $\geq \phi$ (M(Sx,Ty,t),M(By,Sx,t),M(Sx,Ax,t).

M(By,Ty,t),M(Ax,Ty,t).($\frac{1+M(Sx,Ax,t)}{1+M(By,Ty,t)}$)) and

N(Ax,By,kt) $\leq \psi$ (N(Sx,Ty,t),N(By,Sx,t),N(Sx,Ax,t).

N(By,Ty,t),N(Ax,Ty,t).($\frac{1+N(Sx,Ax,t)}{1+N(By,Ty,t)}$))

for all $x, y \in X$, $t > 0$ and $\phi, \psi$:[0,1]$^4\rightarrow$ [0,1] such that $\phi$(t,t,1,t)> t,$\psi$(t,t,0,t)<t for all $t \in (0,1)$ . Then, there is a unique common fixed point of $A$, $B$, $S$ and $T$.

**Proof.** The proof follows on the lines of theorem 3.1.

**Theorem 3.4.**Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space with a*b = min {a,b} and $a\diamond$b = max {a,b} for all $a, b \in [0, 1]$. Let $A$, $B$, $S$ and $T$ be self mappings of X. Let the pairs $(A, S)$ and $(B, T)$ be owc. If there exist $k \in (0, 1)$ such that

(3.4)

M(Ax,By,kt)$\geq$ M(Sx,Ty,t)*M(By,Sx,t)*M(Sx,Ax,t).

M(By,Ty,t)*M(Ax,Ty,t).($\frac{1+M(Sx,Ax,t)}{1+M(By,Ty,t)}$) and

N(Ax,By,kt) $\leq$ N(Sx,Ty,t)$\diamond$N(By,Sx,t)$\diamond$N(Sx,Ax,t).

N(By,Ty,t)$\diamond$N(Ax,Ty,t).($\frac{1+N(Sx,Ax,t)}{1+N(By,Ty,t)}$)

for all $x, y \in X$, $t > 0$. Then, there is a unique common fixed point of $A$, $B$, $S$ and $T$.

**Proof.** The proof follows on the lines of theorem 3.1.

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