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# ON UNIVALENT FUNCTIONS DEFINED BY A NEW GENERALIZED MULTIPLER DIFFERENTIAL OPERATOR

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**Abstract:** The object of this paper is to obtain some interesting properties of functions belonging to a new class $SW^m(\alpha,\beta,\gamma,\rho)$, defined by using a new generalised multiplier differential operator.

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## 1. Introduction.

Denote by $U$ the open unit disc of the complex plane, $U=\{z\in C;|z|<1\}$. Let $H(U)$ be the space of holomorphic functions in $U$. Let $A$ denote the family of functions in $H(U)$ of the form

(1.1) $$f(z)=z+\sum_{k=2}^{\infty}a_k z^k.$$

By S and K we denote the subclasses of functions in $A$, which are univalent and convex in $U$, respectively. Let P be the well-known Caratheodory class of normalized functions with positive real part in $U$. The convolution or Hadamard product of functions $f$ , given by (1.1) and $g(z)=z+\sum_{k=2}^{\infty}b_k z^k$ is defined as the power series

$$(f*g)(z)=z+\sum_{k=2}^{\infty}a_k b_k z^k, z\in U.$$

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We now define a new generalized multiplier differential operator.

**Definition 1.1**. Let $m \in N_0 = N \cup \{0\}, \beta \geq 0, \alpha$ and $\gamma$ are real numbers such that $\alpha + \beta > 0$ and $\gamma + \beta \geq 0.$ Then for $f \in A$, we define a new generalized multiplier operator $I^m_{\alpha,\beta,\gamma}$ by

$$I^0_{\alpha,\beta,\gamma} f(z) = f(z),$$

$$I^1_{\alpha,\beta,\gamma} f(z) = \frac{\alpha f(z) + \beta z f'(z) + \gamma z^2 f''(z)}{\alpha + \beta},$$

$\ldots,$

$$I^m_{\alpha,\beta,\gamma} f(z) = I_{\alpha,\beta,\gamma}(I^{m-1}_{\alpha,\beta,\gamma} f(z)).$$

**Remark 1.2.** If $f(z)$ is given by (1.1), then from the definition 1.1, we obtain

$$I^m_{\alpha,\beta,\gamma} f(z) = z + \sum_{k=2}^{\infty} A_k(\alpha,\beta,\gamma,m) a_k z^k, \tag{1.2}$$

where

$$A_k(\alpha,\beta,\gamma,m) = \left( \frac{\alpha + k\beta + k(k-1)\gamma}{\alpha + \beta} \right)^m. \tag{1.3}$$

From (1.2) it follows that $I^m_{\alpha,\beta,\gamma} f(z)$ can be written in terms of convolution as

$$I^m_{\alpha,\beta,\gamma} f(z) = (f * g)(z), \tag{1.4}$$

where

$$g(z) = z + \sum_{k=2}^{\infty} A_k(\alpha,\beta,\gamma,m) z^k. \tag{1.5}$$

It also follows from (1.2) that

$$I^m_{\alpha,0,0} f(z) = f(z),$$

$$(\alpha + \beta) I^{m+1}_{\alpha,\beta,\gamma} f(z) = \alpha I^m_{\alpha,\beta,\gamma} f(z) + \beta z (I^m_{\alpha,\beta,\gamma} f(z))' + \gamma z^2 (I^m_{\alpha,\beta,\gamma} f(z))''. \tag{1.6}$$

We note that

- $I^m_{\alpha,\beta,0} f(z) = I^m_{\alpha,\beta} f(z)$ (See [18] ).
- $I^m_{1-\beta,\beta,0} f(z) = D^m_\beta f(z), \beta \geq 0$ (See Al-Oboudi [1] ).

- $I^m_{l+1-\beta,\beta,0} f(z) = I^m_{l,\beta} f(z), l > -1, \beta \geq 0$ (See Catas [5]).
- $I^m_{1-\lambda+\delta,\lambda-\delta,\lambda\delta} f(z) = D^m_{\lambda,\delta} f(z), \lambda \geq (\delta/(\delta+1)), \delta \geq 0$ (See Raducanu et.al [14]).

**Remark 1.2. a)** $I^m_{1-\lambda+\delta,\lambda-\delta,\lambda\delta} f(z) = D^m_{\lambda,\delta} f(z)$ was investigated for $\lambda \geq \delta \geq 0$ in [11] and [15]. So our results in this paper are improvement of corresponding results proved earlier for $D^m_{\lambda,\delta} f(z)$, from $\lambda \geq \delta \geq 0$ to $\lambda \geq (\delta/(\delta+1)), \delta \geq 0.$

**b)** $D^m_1 f(z)$ was introduced by Salagean [16] and was considered for $m \geq 0$ in [3].

**Definition 1.3.** Let $m \in N_0 = N \cup \{0\}, \rho \in [0,1), \beta \geq 0, \alpha$ and $\gamma$ are real numbers such that $\alpha + \beta > 0$ and $\gamma + \beta \geq 0.$ Then a function $f \in A$ is said to be in the class $SW^m(\alpha,\beta,\gamma,\rho),$ if it satisfies the condition

$$\mathrm{Re}[I^m_{\alpha,\beta,\gamma} f(z)]' > \rho, z \in U.$$

The main object of this paper is to present a systematic investigation of the class $SW^m(\alpha,\beta,\gamma,\rho).$ In particular; we derive an inclusion result, structural formula, extreme points and other interesting results.

## 2. Preliminaries

In order to prove our results, we will make use of the following lemmas.

**Lemma 2.1**([13]). Let $\mathrm{A} \geq 0, h \in \mathrm{K}.$ Suppose that $B(z)$ and $D(z)$ are analytic in $U$, with $D(0) = 0$ and

$$\mathrm{Re}(B(z)) \geq \mathrm{A} + 4\left|\frac{D(z)}{h'(0)}\right|, z \in U.$$

If an analytic function $p$ with $p(0) = h(0)$ satisfies

$$\mathrm{A}z^2 p''(z) + B(z)zp'(z) + p(z) + D(z) \prec h(z), z \in U,$$

then $p(z) \prec h(z), z \in U.$

**Lemma 2.2** ([12]) . Let $q$ be a convex function in $U$ and let $h(z) = q(z) + \rho z q'(z),$ where $\rho > 0$. If $p \in H(U)$ with $p(z) = q(0) + p_1 z + p_2 z^2 \ldots$ and $p(z) + \rho z p'(z) \prec h(z), z \in U,$ then

$$p(z) \prec q(z), z \in U,$$

and this result is sharp.

**Lemma 2.3**([17]). If $p(z)$ is analytic in $U$, $p(0)=1$ and $\mathrm{Re}(p(z))>\frac{1}{2}$, then for any function $F$ analytic in $U$, the function $F*p$ takes values in the convex hull of $F(U)$.

Note that the symbol "$\prec$" stands for subordination throughout this paper.

**3. Main Results.**

**Theorem 3.1.** If $m\in N_0=N\cup\{0\}$, $\rho\in[0,1)$, $\beta\geq 0$, $\alpha$ and $\gamma$ are real numbers such that $\alpha+\beta>0$ and $\gamma+\beta\geq 0$, then $SW^{m+1}(\alpha,\beta,\gamma,\rho)\subset SW^{m}(\alpha,\beta,\gamma,\rho)$.

**Proof.** Let $f\in SW^{m+1}(\alpha,\beta,\gamma,\rho)$. By using the properties of the operator $I^m_{\alpha,\beta,\gamma}$, we get

$$(3.1)\qquad I^{m+1}_{\alpha,\beta,\gamma}f(z)=\frac{\alpha I^m_{\alpha,\beta,\gamma}f(z)+\beta z(I^m_{\alpha,\beta,\gamma}f(z))'+\gamma z^2(I^m_{\alpha,\beta,\gamma}f'(z))''}{\alpha+\beta}.$$

Differentiating (3.1) with respect to z and using (1.6), we obtain

$$(3.2)\qquad (I^{m+1}_{\alpha,\beta,\gamma}f(z))'=\left\{p(z)+\left(\frac{\beta+2\gamma}{\alpha+\beta}\right)zp'(z)+\left(\frac{\gamma}{\alpha+\beta}\right)z^2p''(z)\right\}$$

where

$$p(z)=(I^m_{\alpha,\beta,\gamma}f(z))'.$$

Since $f\in SW^{m+1}(\alpha,\beta,\gamma,\rho)$, by using Definition 1.3 and (3.2), we have

$$\mathrm{Re}\left\{p(z)+\left(\frac{\beta+2\gamma}{\alpha+\beta}\right)zp'(z)+\left(\frac{\gamma}{\alpha+\beta}\right)z^2p''(z)\right\}>\rho,\ z\in U,$$

which is equivalent to

$$\left\{p(z)+\left(\frac{\beta+2\gamma}{\alpha+\beta}\right)zp'(z)+\left(\frac{\gamma}{\alpha+\beta}\right)z^2p''(z)\right\}\prec\frac{1+(2\rho-1)z}{1+z}\equiv h(z).$$

From Lemma 2.1, with $\mathrm{A}=\left(\frac{\gamma}{\alpha+\beta}\right)$, $B(z)=\left(\frac{\beta+2\gamma}{\alpha+\beta}\right)$, and $D(z)=0$ we have $p(z)\prec h(z)$, which implies that $\mathrm{Re}[(I^m_{\alpha,\beta,\gamma}f(z))']>\rho, z\in U$. Hence $f\in SW^m(\alpha,\beta,\gamma,\rho)$ and the proof of the theorem is complete.

Clearly $SW^m(\alpha,\beta,\gamma,\rho)\subset SW^{m-1}(\alpha,\beta,\gamma,\rho)\subset\ldots\subset SW^0(\alpha,\beta,\gamma,\rho)\subset S$ (see [6, 8]) and one can easily show that the set $SW^m(\alpha,\beta,\gamma,\rho)$ is convex (see [11]).

**Theorem 3.2.** Let $q$ be convex function with $q(0)=1$ and let $h$ be a function of the form $h(z)=q(z)+zq^{'}(z), z\in U.$ If $f\in A$ satisfies the differential subordination $(I^{m}_{\alpha,\beta,\gamma}f(z))^{'}\prec h(z), z\in U$, then $(I^{m}_{\alpha,\beta,\gamma}f(z))/z\prec q(z)$ and the result is sharp.

**Proof.** If we let $p(z)=(I^{m}_{\alpha,\beta,\gamma}f(z))/z, z\in U,$ then we obtain $(I^{m}_{\alpha,\beta,\gamma}f(z))^{'}=p(z)+zp'(z).$ So the subordination $(I^{m}_{\alpha,\beta,\gamma}f(z))^{'}\prec h(z), z\in U$, becomes

$$p(z)+zp^{'}(z)\prec q(z)+zq^{'}(z), z\in U,$$

and hence from Lemma 2.2 we have $(I^{m}_{\alpha,\beta,\gamma}f(z))/z\prec q(z)$. The result is sharp.

We now obtain a structural formula, extreme points and coefficient bounds for functions in $SW^{m}(\alpha,\beta,\gamma,\rho).$

**Theorem 3.3.** A function $f\in A$ is in the class $SW^{m}(\alpha,\beta,\gamma,\rho)$ if and only if it can be expressed as

$$f(z)=\left[z+\sum_{k=2}^{\infty}\frac{1}{A_k(\alpha,\beta,\gamma,m)}z^k\right]*\int_{|\zeta|=1}\left[z+2(1-\rho)\overline{\zeta}\sum_{k=2}^{\infty}\frac{(\zeta z)^k}{k}\right]d\mu(\zeta), \tag{3.3}$$

where $A_k(\alpha,\beta,\gamma,m)$ is given by (1.3) and $\mu$ is a positive probability measure defined on the unit circle $E=\{\zeta\in C:|\zeta|=1\}$.

**Proof.** From Definition 1.3 it follows that $f\in SW^{m}(\alpha,\beta,\gamma,\rho)$ if and only if

$$\frac{[I^{m}_{\alpha,\beta,\gamma}f(z)]^{'}-\rho}{1-\rho}\in \mathrm{P}.$$

Using Hergoltz integral representation of functions in Caratheodory class $\mathrm{P}$ (see [7] and [9]), there exists a positive Borel probability measure $\mu$ such that

$$\frac{[I^{m}_{\alpha,\beta,\gamma}f(z)]^{'}-\rho}{1-\rho}=\int_{|\zeta|=1}\left(\frac{1+\zeta z}{1-\zeta z}\right)d\mu(\zeta), z\in U,$$

which is equivalent to

$$[I^{m}_{\alpha,\beta,\gamma}f(z)]^{'}=\int_{|\zeta|=1}\left(\frac{1+(1-2\rho)\zeta z}{1-\zeta z}\right)d\mu(\zeta), z\in U.$$

Integrating we obtain

(3.4) $$I^m_{\alpha,\beta,\gamma} f(z) = \int_0^z \left\{ \int_{|\zeta|=1} \left( \frac{1+(1-2\rho)\zeta u}{1-\zeta u} \right) d\mu(\zeta) \right\} du =$$

$$= \int_{|\zeta|=1} \left( z + 2(1-\rho)\overline{\zeta} \sum_{k=2}^{\infty} \frac{(\zeta z)^k}{k} \right) d\mu(\zeta).$$

Equality (3.3) follows now, from (1.4), (1.5) and (3.4). Since the converse of this deductive process is also true, we have proved our theorem.

**Corollary 3.4**. The extreme points of the class $SW^m(\alpha,\beta,\gamma,\rho)$ are

(3.5) $$f_\zeta(z) = z + 2(1-\rho)\overline{\zeta} \sum_{k=2}^{\infty} \frac{(\zeta z)^k}{kA_k(\alpha,\beta,\gamma,m)}, z \in U, |\zeta| = 1.$$

**Proof.** Consider the functions $g_\zeta(z) = z + 2(1-\rho)\overline{\zeta} \sum_{k=2}^{\infty} \frac{(\zeta z)^k}{k}$ and $g_\mu(z) = \int_{|\zeta|=1} g_\zeta(z) d\mu(\zeta).$

The assertion now follows from (3.3), (making use of (1.4), (1.5) and (3.4)), since the map $\mu \to g_\mu$ is one-to-one (see [4]).

**Corollary 3.5**. If $f \in SW^m(\alpha,\beta,\gamma,\rho)$ is given by (1.1), then $|a_k| \le \frac{2(1-\rho)}{kA_k(\alpha,\beta,\gamma,m)}, k \ge 2.$ The result is sharp.

**Proof.** The result follows from (3.5), since the coefficient bounds are maximized at an extreme point.

**Corollary 3.6.** If $f \in SW^m(\alpha,\beta,\gamma,\rho)$, then for $|z| = r < 1,$

$$r - 2(1-\rho)r^2 \sum_{k=2}^{\infty} \frac{1}{kA_k(\alpha,\beta,\gamma,m)} \le |f(z)| \le r + 2(1-\rho)r^2 \sum_{k=2}^{\infty} \frac{1}{kA_k(\alpha,\beta,\gamma,m)}$$

and

$$1 - 2(1-\rho)r \sum_{k=2}^{\infty} \frac{1}{A_k(\alpha,\beta,\gamma,m)} \le |f'(z)| \le 1 + 2(1-\rho)r \sum_{k=2}^{\infty} \frac{1}{A_k(\alpha,\beta,\gamma,m)}.$$

Next, we prove the analogue of the Polya- Schoenberg conjecture for the class $SW^m(\alpha,\beta,\gamma,\rho).$

**Theorem 3.7.** If $f \in SW^m(\alpha,\beta,\gamma,\rho)$ and $g \in$K, then $f * g \in SW^m(\alpha,\beta,\gamma,\rho).$

**Proof.** It is known that if $g \in K$, then $\mathrm{Re}\left(\frac{g(z)}{z}\right) > \frac{1}{2}$ (see[12]).Making use of the convolution properties, we have

$$\mathrm{Re}[I^m_{\alpha,\beta,\gamma}(f*g)(z)]' = \mathrm{Re}\left[(I^m_{\alpha,\beta,\gamma} f(z))' * \frac{g(z)}{z}\right].$$

The result now follows, by applying Lemma 2.3.

**Corollary 3.8.**The class $SW^m(\alpha,\beta,\gamma,\rho)$ is invariant under Bernardi integral operator.

**Proof.** Let $f \in SW^m(\alpha,\beta,\gamma,\rho)$ . The Bernardi integral operator is defined as (see [2]):

$$F_c(h)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt, (c > -1; h \in A).$$

It is easy to check that $F_c(f)(z) = (f*g)(z)$ where

$$g(z) = \sum_{k=1}^{\infty}\left(\frac{c+1}{c+k}\right)z^k = \frac{c+1}{z^c}\int_0^z \frac{t^c}{1-t}dt, z \in U, c > -1.$$

Since the function $\phi(z) = \frac{z}{1-z}, z \in U$ is convex, it follows (see [10]) that the function $g$ is also convex. From Theorem 3.7 we obtain $F_c(f) \in SW^m(\alpha,\beta,\gamma,\rho)$ . Therefore $F_c(SW^m(\alpha,\beta,\gamma,\rho)) \subset SW^m(\alpha,\beta,\gamma,\rho)$.

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