Available online at http://scik.org
J. Math. Comput. Sci. 7 (2017), No. 3, 485-494
ISSN: 1927-5307

A GENERALIZATION OF MULTIPLICATIVE (GENERALIZED)-DERIVATONS

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Abstract. Let R be a semiprime ring and L be a semigroup ideal of R. The main object in this paper is to study the following situations in semiprime rings: When F is a multiplicative $(\alpha,1)$-(generalized) derivation associated with a map d, (i) $F(xy)\pm\alpha(x)\alpha(y)=0$ for $x,y\in L$. (ii) $F(x)F(y)\pm\alpha(x)\alpha(y)=0$ for all $x,y\in L$. When F is a multiplicative $(1,\alpha)$-(generalized) derivation associated with a map d, (iii) $F(xy)\pm xy=0$ for all $x,y\in L$. (iv) $F(x)F(y)\pm xy=0$ for all $x,y\in L$.

Keywords: semiprime ring; multiplicative derivation; multiplicative generalized derivation; multiplicative (generalized)-derivation; multiplicative $(\alpha,1)$-(generalized) derivation; multiplicative $(1,\alpha)$-(generalized) derivation.

2010 AMS Subject Classification: 16N60, 16W25, 16U80.

1. Introduction

In this paper R denotes an assocative ring. A ring R is called *semiprime ring* if $aRa=(0)$ implies that $a=0$. A subset L is called a *left semigroup ideal* of R if $ra\in L$ for all $a\in L$ and for

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Received February 24, 2017; Published May 1, 2017

all $r \in R$. Obviously, every left ideal is a left semigroup ideal. An additive mapping $d : R \to R$ is called a *derivation* of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In 1991, Daif, M. N. [1] defined that a map D is called a *multiplicative derivation* of R if $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$. In 1997, this definition of multiplicative derivation was extended to multiplicative generalized derivation by Daif, M. N. and Tammam El-Sayid, M. S. [2] as follow: a map $F : R \to R$ is called a *multiplicative generalized derivation* if there exists a derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. In 2013, the definition of multiplicative generalized derivation was extended to multiplicative (generalized)-derivation by Dahara, B. and Ali, S. [3] as follow: a map $F : R \to R$ is called a *multiplicative (generalized)-derivation* if there exists a map $F : R \to R$ such that $F(xy) = F(x)y + xg(y)$ for all $x, y \in R$ where g is any mapping on R.

We introduce the notion of multiplicative two-sided α-(generalized) derivation of R as follows.

A map $F : R \to R$ is said to be a *multiplicative* $(\alpha, 1)$*-(generalized) derivation* if there exists maps $d, \alpha : R \to R$ such that

$$F(xy) = F(x)\alpha(y) + xd(y) \text{ for all } x, y \in R.$$

Similarly, if $F(xy) = F(x)y + \alpha(x)d(y)$ for all $x, y \in R$ than F is called a *multiplicative* $(1, \alpha)$*-(generalized) derivation*. A map $F : R \to R$ is called a *multiplicative two-sided* α*-(generalized) derivation* if F is a multiplicative $(\alpha, 1)$-(generalized) derivation as well as multiplicative $(1, \alpha)$-(generalized) derivation. It is clear that every multiplicative (generalized)-derivation is multiplicative two-sided α-(generalized) derivation on R. But the converse is not true. The following example justifies the fact:

Example 1. *Let S be a ring and* $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in S \right\}$. *Define the maps $d, \alpha, F : R \to R$ as follows:*

$$d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \alpha \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$

$$and \quad F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then it is easy to verify that F is a multiplicative two-sided α-(generalized) derivation associated with a map d but F is not a multiplicative (generalized)- derivation of R.

In this connection, our aim in the present paper is to generalize the study of Dahara, B. and Ali, S. [3] in the case of a left semigroup ideal, a multiplicative $(\alpha,1)$- and $(1,\alpha)$-(generalized) derivation and to investigate some properties satisfying certain differential identities.

Throughout this paper, R is a semiprime ring, L is a nonzero left semigroup ideal of R and α is an epimorphism of R.

The material in this work is a part of first author's Master's Thesis which is supervised by Prof. Dr. Neşet Aydın.

2. **Results**

Lemma 2.1. *Let R is a semiprime ring, L is a nonzero left semigroup ideal of R and $0 \neq a \in R$. If $aL = (0)$, then $La = (0)$.*

Proof. Since L is a semigroup ideal of R, $a(RL) = (0)$. This gives $(La)R(La) = (0)$. Due to primeness of R, $La = (0)$. ☐

Theorem 2.1. *Let R is a semiprime ring, L is a nonzero left semigroup ideal of R and F is a multiplicative $(\alpha,1)$-(generalized) derivation. If $F(xy) \pm \alpha(x)\alpha(y) = 0$ for all $x,y \in L$ then $Ld(L) = (0)$, $F(xy) = F(x)\alpha(y)$ for all $x,y \in L$ and $[F(x),\alpha(x)] = 0$ for all $x \in L$.*

Proof. By the hypothesis, we have

$$F(xy) - \alpha(x)\alpha(y) = 0 \tag{1}$$

for all $x,y \in L$. Replacing y by yz, $z \in L$ in (1), we get

$$F(xyz) - \alpha(x)\alpha(yz) = 0 \text{ for all } x,y,z \in L.$$

Since $F(xy) = F(x)\alpha(y) + xd(y)$ for all $x, y \in R$ and α is an epimorphism of R, we can rewrite the above equation

$$\begin{aligned} 0 &= F(xy)\alpha(z) + xyd(z) - \alpha(x)\alpha(y)\alpha(z) \\ &= (F(xy) - \alpha(x)\alpha(y))\alpha(z) + xyd(z) \end{aligned}$$

for all $x, y \in L$. By (1) that gives

$$xyd(z) = 0 \text{ for all } x, y, z \in L.$$

Taking $d(z)rx$, $r \in R$ instead of y in the last equation, we get

$$xd(z)rxd(z) = 0 \text{ for all } x, y, z \in L, r \in R.$$

In particular, $xd(z)Rxd(z) = (0)$ for all $x, z \in L$. Since R is a semiprime ring, the last expression forces that $xd(z) = 0$ for all $x, z \in L$. That is,

$$Ld(L) = (0).$$

Thus $F(xy) = F(x)\alpha(y) + xd(y) = F(x)\alpha(y)$ for all $x, y \in L$. From the equation (1), we get $0 = F(xy) - \alpha(x)\alpha(y) = F(x)\alpha(y) - \alpha(x)\alpha(y) = (F(x) - \alpha(x))\alpha(y)$ for all $x, y \in L$. That is,

$$(F(x) - \alpha(x))\alpha(L) = (0) \text{ for all } x \in L.$$

Considering L is a left semigroup ideal of R, α is an epimorphism of R and $\alpha(L)$ is a semigroup ideal of R together with Lemma 2.1, we have $\alpha(L)(F(x) - \alpha(x)) = (0)$ for all $x \in L$. Thus $(F(x) - \alpha(x))\alpha(L) = (0)$ and $\alpha(L)(F(x) - \alpha(x)) = (0)$ for all $x \in L$, together implies

$$[F(x) - \alpha(x), \alpha(L)] = (0) \text{ for all } x \in L.$$

This yields that $[F(x), \alpha(x)] = 0$ for all $x \in L$.

Similarly, we can prove that the same results for

$$F(xy) + \alpha(x)\alpha(y) = 0$$

for all $x, y \in L$. □

Corollary 2.1. *Let R is a semiprime ring, L is a nonzero left semigroup ideal of R and F is a multiplicative $(\alpha, 1)$-(generalized) derivation. If $F(xy) \pm \alpha(x)\alpha(y) = 0$ for all $x, y \in R$ then $d = 0$ and $F = \pm\alpha$.*

Proof. By Theorem 2.1, we get $d = 0$ and $F(xy) = F(x)\alpha(y)$ for all $x, y \in R$. From the hypothesis, $0 = F(x)\alpha(y) \pm \alpha(x)\alpha(y) = (F(x) \pm \alpha(x))\alpha(y)$ for all $x, y \in R$. That is,

$$(F(x) \pm \alpha(x))R = (0) \text{ for all } x \in R.$$

Since R is a semiprime ring, $F = \pm\alpha$. □

Theorem 2.2. *Let R is a semiprime ring, L is a nonzero left semigroup ideal of R and F is a multiplicative $(\alpha, 1)$-(generalized) derivation. If $F(x)F(y) \pm \alpha(x)\alpha(y) = 0$ for all $x, y \in L$ then $Ld(L) = (0)$, $F(xy) = F(x)\alpha(y)$ for all $x, y \in L$ and $\alpha(L)[F(x), \alpha(x)] = (0)$ for all $x \in L$.*

Proof. By the assumption, we have

$$F(x)F(y) - \alpha(x)\alpha(y) = 0 \tag{2}$$

for all $x, y \in L$. Replacing y by yz, $z \in L$ in (2), we get

$$F(x)F(yz) - \alpha(x)\alpha(yz) = 0 \text{ for all } x, y, z \in L.$$

It holds that
$$\begin{aligned} 0 &= F(x)(F(y)\alpha(z) + yd(z)) - \alpha(x)\alpha(y)\alpha(z) \\ &= F(x)F(y)\alpha(z) + F(x)yd(z) - \alpha(x)\alpha(y)\alpha(z) \\ &= (F(x)F(y) - \alpha(x)\alpha(y))\alpha(z) + F(x)yd(z) \text{ for all } x, y, z \in L. \end{aligned}$$

By (2) it reduces

$$F(x)yd(z) = 0 \text{ for all } x, y, z \in L \tag{3}$$

Replacing x with ux, $u \in L$, we obtain $F(ux)yd(z) = 0$ for all $u, x, y, z \in L$. It follows that

$$\begin{aligned} 0 &= (F(u)\alpha(x) + ud(x))yd(z) \\ &= F(u)\alpha(x)yd(z) + ud(x)yd(z) \end{aligned}$$

Since L is a left semigroup ideal of R and by using (3) it gives

$$ud(x)yd(z) = 0 \text{ for all } x, y, z \in L \tag{4}$$

Replacing y by ry, $r \in R$ in (4), we get $ud(x)ryd(z) = 0$ for all $u, x, y, z \in L$ and $r \in R$. This implies that $ud(x)Ryd(z) = (0)$ for all $u, x, y, z \in L$. Taking $y = u$ and $z = x$, we obtain $yd(x)Ryd(x) = (0)$ for all $x, y \in L$. Since R is a semiprime ring, we have $yd(x) = 0$ for all $x, y \in L$. Namely,

$Ld(L) = (0)$. Thus $F(xy) = F(x)\alpha(y) + yd(z) = F(x)\alpha(y)$ for all $x, y \in L$. Replacing x by xy in (2), we get

$$F(x)\alpha(y)F(y) - \alpha(x)\alpha(y)^2 = 0 \tag{5}$$

for all $x, y \in L$. Equation(2) multiplied by $\alpha(y)$ from right, we get

$$F(x)F(y)\alpha(y) - \alpha(x)\alpha(y)^2 = 0 \tag{6}$$

for all $x, y \in L$. Substracting (5) from (6), we get

$$F(x)\,[F(y), \alpha(y)] = 0 \tag{7}$$

for all $x, y \in L$. Replacing x by xz, $z \in L$ in (7), we get

$$F(x)\alpha(z)\,[F(y), \alpha(y)] = 0$$

for all $x, y, z \in L$. This implies

$$\alpha(L)\,[F(x), \alpha(x)]\,R\alpha(L)\,[F(x), \alpha(x)] = (0).$$

Since R is a semiprime ring, it implies that $\alpha(L)\,[F(x), \alpha(x)] = (0)$ for all $x \in L$.

Similar way, we can prove that same conclusion for $F(x)F(y) + \alpha(x)\alpha(y) = 0$ for all $x, y \in L$. □

Corollary 2.2. *Let R is a semiprime ring, L is a nonzero left semigroup ideal of R and F is a multiplicative $(\alpha, 1)$-(generalized) derivation. If $F(x)F(y) \pm \alpha(x)\alpha(y) = 0$ for all $x, y \in R$ then $d = 0$ and $F(xy) = F(x)\alpha(y)$ for all $x, y \in R$*

Proof. Using Theorem 2.2, we come to a conclusion $d = 0$ and $F(xy) = F(x)\alpha(y)$ for all $x, y \in R$. □

Theorem 2.3. *Let R is a semiprime ring, L is a nonzero left semigroup ideal of R and F is a multiplicative $(1, \alpha)$-(generalized) derivation. If $F(xy) \pm xy = 0$ for all $x, y \in L$ then $\alpha(L)d(L) = (0)$, $F(xy) = F(x)y$ for all $x, y \in L$ and F is a commuting map on L.*

Proof. Assume that

$$F(xy) - xy = 0 \text{ for all } x, y \in L. \tag{8}$$

Taking yz, $z \in L$ instead of y in (8), $F(xyz) - xyz = 0$ for all $x, y, z \in L$. Since $F(xy) = F(x)y + \alpha(x)d(y)$ for all $x, y \in R$ and α is an epimorphism of R, it follows that

$$0 = F(xy)z + \alpha(xy)d(z) - xyz = (F(xy) - xy)z + \alpha(x)\alpha(y)d(z)$$

for all $x, y \in L$. By (8) it holds that

$$\alpha(x)\alpha(y)d(z) = 0 \text{ for all } x, y, z \in L.$$

Replacing y with rx, $r \in R$, we get $\alpha(x)\alpha(rx)d(z) = 0$. Since α is an epimorphism of R, it holds $\alpha(x)R\beta(x)d(z) = (0)$ for all $x, z \in L$. This implies

$$\alpha(x)d(z)R\alpha(x)d(z) = (0) \text{ for all } x, y, z \in L.$$

Since R is a semiprime ring, $\alpha(x)d(z) = 0$ for all $x, z \in L$. That is,

$$\alpha(L)d(L) = (0).$$

So, we obtain $F(xy) = F(x)y + \alpha(y)d(z) = F(x)y$ for all $x, y \in L$. Using (8), one obtains $0 = F(xy) - xy = F(x)y - xy = (F(x) - x)y$ for all $x, y \in L$. In paticular

$$(F(x) - x)L = (0) \text{ for all } x \in L.$$

Since L is a left semigroup ideal of R. By Lemma 2.1, we have

$$L(F(x) - x) = (0) \text{ for all } x \in L.$$

Thus $(F(x) - x)L = (0)$ and $L(F(x) - x) = (0)$ for all $x \in L$, together implies

$$[F(x) - x, L] = (0) \text{ for all } x \in L.$$

This yields that $[F(x), x] = 0$ for all $x \in L$. Thus, F is a commuting map on L.

In a similarly, we can prove that to achieve the same results for $F(xy) + xy = 0$ for all $x, y \in L$. □

Corollary 2.3. *Let R is a semiprime ring, L is a nonzero left semigroup ideal of R and F is a multiplicative $(1,\alpha)$-(generalized) derivation. If $F(xy) \pm xy = 0$ for all $x,y \in R$ then $d = 0$, $F(x) = \pm x$ and F is a commuting map on R.*

Proof. By Theorem 2.3 we have $d = 0$ and $F(xy) = F(x)y$ for all $x,y \in R$. From the hypothesis, we obtain $F(xy) \pm xy = 0$ for all $x,y \in R$.Since $F(xy) = F(x)y$,it implies that $(F(x) \pm x)y = 0$ for all $x,y \in R$. That is,

$$(F(x) \pm x)R = (0) \text{ for all } x \in R.$$

Since R is a semiprime ring, it follows that $F(x) = \pm x$.for all $x \in R$ □

Theorem 2.4. *Let R is a semiprime ring, L is a nonzero left semigroup ideal of R and F is a multiplicative $(1,\alpha)$-(generalized) derivation. If $F(x)F(y) \pm xy = 0$ for all $x,y \in L$ then $\alpha(L)d(L) = (0)$, $F(xy) = F(x)y$ for all $x,y \in L$ and $L[F(x),x] = (0)$ for all $x \in L$.*

Proof. First we consider that

$$F(x)F(y) - xy = 0 \tag{9}$$

for all $x,y \in L$.Substituting yz, $z \in L$ for y in (9), we get $F(x)F(yz) - xyz = 0$ for all $x,y,z \in L$. Since $F(xy) = F(x)y + \alpha(x)d(y)$ for all $x,y \in R$, it follows that

$$\begin{aligned} 0 &= F(x)(F(y)z + \alpha(y)d(z)) - xyz \\ &= F(x)F(y)z + F(x)\alpha(y)d(z) - xyz \\ &= (F(x)F(y) - xy)z + F(x)\alpha(y)d(z) \end{aligned}$$

By (9) it gives

$$F(x)\alpha(y)d(z) = 0 \text{ for all } x,y,z \in L \tag{10}$$

Replacing x with ux, $u \in L$, we get $F(ux)\alpha(y)d(z) = 0$ for all $u,x,y,z \in L$. Since $F(xy) = F(x)y + \alpha(x)d(y)$ for all $x,y \in R$, it follows that $0 = (F(u)x + \alpha(u)d(x))\alpha(y)d(z) = F(u)x\beta(y)d(z) +$

$\alpha(u)d(x)\alpha(y)d(z)$. Since L is a left semigroup ideal of R and α is an epimorphism of R, $\alpha(L)$ is a left semigroup ideal of R. By using (10), it gives

$$\alpha(u)d(x)\alpha(y)d(z) = 0 \tag{11}$$

for all $u,x,y,z \in L$. Replacing y by ry, $r \in R$ in (11), we get $\alpha(u)d(x)\alpha(ry)d(z) = 0$ for all $u,x,y,z \in L$ and $r \in R$. Since α is an epimorphism of R, it implies that $\alpha(u)d(x)R\beta(y)d(z) = (0)$ for all $u,x,y,z \in L$. Taking $y = u$ and $z = x$. We obtain

$$\alpha(y)d(x)R\beta(y)d(x) = (0) \text{ for all } x,y \in L.$$

Since R is a semiprime ring, we have $\alpha(y)d(x) = 0$ for all $x,y \in L$. That is, $\alpha(L)d(L) = (0)$. Thus $F(xy) = F(x)y + \alpha(y)d(z) = F(x)y$ for all $x,y \in L$. Replacing x by xy in (9), we get

$$F(x)yG(y) - xy^2 = 0 \tag{12}$$

for all $x,y \in L$. (9) multiplied by $\alpha(y)$ from right, we get

$$F(x)F(y)y - xy^2 = 0 \tag{13}$$

for all $x,y \in L$. Substracting (12) from (13), we get

$$F(x)[F(y),y] = 0 \tag{14}$$

for all $x,y \in L$. Replacing x by xz, $z \in L$ in (14), we get

$$F(x)z[F(y),y] = 0$$

for all $x,y,z \in L$. This implies $L[F(x),x]RL[F(x),x] = (0)$. Since R is a semiprime ring, it follows that $L[F(x),x] = (0)$ for all $x \in L$.

In the some way, we can prove the same results for $F(x)F(y) + xy = 0$ for all $x,y \in L$. □

Corollary 2.4. *Let R is a semiprime ring, L is a nonzero left semigroup ideal of R and F is a multiplicative $(1,\alpha)$-(generalized) derivation. If $F(x)F(y) \pm xy = 0$ for all $x,y \in R$ then $d = 0$, $F(xy) = F(x)y$ for all $x,y \in R$ and F is a commuting map on R*

Proof. By using Theorem 2.4, ,we conclude that $d = 0$, $F(xy) = F(x)y$ for all $x,y \in R$ and F is a commuting map on R □

Conflict of Interests

The authors declare that there is no conflict of interests.

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