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HYBRID METHOD FOR SOLVING NONLINEAR VOLTERRA-FREDHOLM INTEGRO DIFFERENTIAL EQUATIONS

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Abstract. The modified Laplace Adomian decomposition method (LADM) has been developed to find the analytic approximation solution of the nonlinear Volterra-Fredholm integro differential equations under the initial or boundary conditions. We prove the convergence of LADM applied to the Volterra-Fredholm integro differential equations. In this paper, some examples will be examined to support the proposed analysis.

Keywords: Adomian decomposition method; Laplace transform; Volterra-Fredholm integro differential equation; convergence analysis.

2010 AMS Subject Classification: 47H30, 45A05, 34A12, 34K05.

1. Introduction

The modified form of Laplace decomposition method has been introduced by Khuri and Wazwaz [15, 19]. Yusufoglu [22] solved the Duffing equation by this method. This method generates a solution in the form of a series whose terms are determined by a recursive relation

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using the Adomian polynomials. The nonlinear Volterra-Fredholm integro differential equations as follows [20]:

$$y^{(j)}(x) = f(x) + \int_a^x K_1(x,t)G_1(y(t))dt + \int_a^b K_2(x,t)G_2(y(t))dt, \tag{1}$$

with the initial or boundary conditions

$$\begin{aligned} y^{(r)}(a) &= \alpha_r, \quad r = 0,1,...,(m-1), \\ y^{(r)}(b) &= \beta_r, \quad r = m,(m+1),...,(j-1). \end{aligned} \tag{2}$$

where $y^{(j)}(x)$ is the j^{th} derivative of the unknown function $y(x)$ that will be determined, $K_i(x,t)$, $i = 1,2$, be the kernels of the integro differential equation, $f(x)$ is an analytic function, $G_1(y)$ and $G_2(y)$ are nonlinear functions of y, α_r, and β_r are real finite constants. The modified Laplace decomposition method have applied for solving some partial differential equations Khan et. al, in [7]. Recently, the authors have used different methods for the numerical or the analytical solution of linear and nonlinear Fredholm and Volterra integral and integro differential equations of the second kind [1, 14, 16, 17]. This type of equations was introduced by Volterra for the first time in early 1900. Volterra investigated the population growth, focussing his study on the hereditary influences, where through his research work the topic of integro differential equations was established.[13, 20] More details about the sources where these equations arise can be found in physics, biology, and engineering applications as well as in advanced integral equations. Some works based on an iterative scheme have been focusing on the development of more advanced and efficient methods for integral equations and integro differential equations such as the variational iteration method (VIM) which is a simple and Adomian decomposition method (ADM) [11, 13, 20, 21], and the modified decomposition method (MDM) for solving Volterra-Fredholm integral and integro differential equations which is a simple and powerful method for solving a wide class of nonlinear problems [19, 20]. The Taylor polynomial solution of integro differential equations has been studied in [23, 18]. The use of Lagrange interpolation in solving integro differential equations was investigated by Marzban [16]. The VIM has been successfully applied for solving integral and integro differential equations [8, 11, 13, 20]. Wazwaz [19], used the modified decomposition method and the traditional methods for solving nonlinear integral equations. A variety of powerful methods has been presented, such as the homotopy analysis method [20], homotopy perturbation method [10], the triangular-function method [14], variational iteration method [11, 20] and the Adomian decomposition method [1, 5, 20], and many methods for solving integro differential equations [2, 4, 17, 19, 20]. By using the LADM we obtain analytical solutions for the integro-differential equations. Some fundamental works on various aspects of modifications of the Adomian's decomposition method are given by Araghi [1]. The modified form of Laplace decomposition method has been introduced by Manafianheris [9]. Babolian et. al, [3], applied the new direct method to solve nonlinear Volterra-Fredholm integral and integro differential equation using operational matrix with block-pulse functions. The Laplace transform method with the Adomian decomposition method to establish exact solutions or approximations of the nonlinear Volterra integro differential equations, Wazwaz [21]. Elgasery [6], applied the Laplace decomposition method for the solution of Falkner Skan equation. This paper deals with one of the most applied problems in the engineering sciences. This technique basically illustrates how the Laplace transform may be used to approximate the solutions of the nonlinear Volterra-Fredholm integro differential equations by manipulating the decomposition method. Our aim in this paper is to obtain the analytical solutions by using the modified Laplace Adomian decomposition method. The remainder of the paper is organized as follows: In Section **2**, preliminaries and describe the basic formulation of ADM. In Section **3**, a brief discussion for the modified Laplace Adomian decomposition method is presented. We present and describe the basic formulation of this method. In Section **4**, applications of this method and the exact solutions for some examples are obtained. In Section **5**, we prove the convergence of LADM applied to the Volterra-Fredholm integro differential equations. Finally, we will give report on our paper and a brief conclusion is given in Section **6**.

2. Preliminaries

The Adomian decomposition method is applied to the following general nonlinear equation:

$$Ly+Ry+Ny=g(x) \tag{3}$$

where y is the unknown function, L is the highest-order derivative which is assumed to be easily invertible, R is a linear differential operator of order less than L, Ny represents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of Eq. (3) and using the given conditions we obtain

$$y = f(x) - L^{-1}(Ry) - L^{-1}(Ny) \tag{4}$$

where the function $f(x)$ represents the terms arising from integrating the source term $g(x)$. The nonlinear operator $Ny = G(y)$ is decomposed as

$$G(y) = \sum_{n=0}^{\infty} A_n \tag{5}$$

where A_n; $n > 0$ are the Adomian polynomials determined formally as follows:

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty} \lambda^i y_i)]]_{\lambda=0} \tag{6}$$

The Adomian polynomials were introduced in [20, 21] as:

$$A_0 = G(y_0);\ \ A_1 = y_1 G^{'}(y_0);\ \ A_2 = y_2 G^{'}(y_0) + \frac{1}{2} y_1^2 G^{''}(y_0)$$
$$A_3 = y_3 G^{'}(y_0) + y_1 y_2 G^{''}(y_0) + \frac{1}{3} y_1^3 G^{'''}(y_0), \ldots$$

In recent years the Adomian decomposition method [20] has been applied to a wide class of functional equations and inverse problems such as integral equations [1, 12]. The standard decomposition technique represents the solution of y in Eq. 3 as the following series:

$$y = \sum_{i=0}^{\infty} y_i \tag{7}$$

where, the components $y_0, y_1, \ldots$ are usually determined recursively by

$$\begin{aligned} y_0 &= f(x), \\ y_{n+1} &= -L^{-1}(Ry_n) - L^{-1}(A_n), n \geq 0. \end{aligned} \tag{8}$$

Substituting 4 into 8 leads to the determination of the components of y. Having determined the components $y_0, y_1, \ldots$, the solution y in the series form defined by Eq. 7.

3. The Modified Laplace Adomian Decomposition Method

The nonlinear Volterra-Fredholm integro differential equation with difference kernels as follows:

$$y^{(j)}(x) = f(x) + \int_a^x K_1(x-t)G_1(y(t))dt + \int_a^b K_2(x-t)G_2(y(t))dt, \tag{9}$$

To solve the nonlinear Volterra-Fredholm integro differential Eq. (9) by using the Laplace transform method, we recall that the Laplace transforms of the derivatives of $y(x)$ are defined by

$$\pounds\{y^{(j)}(x)\} = s^j \pounds\{y(x)\} - s^{j-1}y(0) - s^{j-2}y'(0) - ... - y^{(j-1)}(0), \tag{10}$$

Applying the Laplace transform to both sides of Eq.(9) gives:

$$\begin{aligned} s^j \pounds\{y(x)\} - s^{j-1}y(0) - s^{j-2}y'(0) - ... - y^{(j-1)}(0) = \pounds\{f(x)\} \\ +\pounds\{K_1(x-t)\}\pounds\{G_1(y(t))\} + \pounds\{K_2(x-t)\}\pounds\{G_2(y(t))\}, \end{aligned} \tag{11}$$

This can be reduced to

$$\begin{aligned} \pounds\{y(x)\} &= \frac{1}{s}y(0) + \frac{1}{s^2}y'(0) + ... + \frac{1}{s^i}y^{(j-1)}(0) + \frac{1}{s^i}\pounds\{f(x)\} + \frac{1}{s^i}\pounds\{K_1(x-t)\} \\ & \pounds\{G_1(y(t))\} + \frac{1}{s^i}\pounds\{K_2(x-t)\}\pounds\{G_2(y(t))\}, \end{aligned} \tag{12}$$

The Adomian decomposition method and the Adomian polynomials can be used to handle Eq. (12) and to address the nonlinear term $G(y(x))$. We first represent the linear term $y(x)$ at the left side by an infinite series of components given by

$$y = \sum_{m=0}^{\infty} y_m(x). \tag{13}$$

where the components $y_m(x), m \geq 0$ will be determined recursively. However, the nonlinear terms $G_1(y(x))$ and $G_2(y(x))$ at the right side of Eq. (12) will be represented by an infinite series of the Adomian polynomials A_m and B_m in the form:

$$G_1(y(x)) = \sum_{m=0}^{\infty} A_m(x), \quad G_2(y(x)) = \sum_{m=0}^{\infty} B_m(x), \tag{14}$$

where A_m and B_m, $m \geq 0$ are defined by

$$A_m = \frac{1}{m!}[\frac{d^m}{d\lambda^m}[G_1(\sum_{i=0}^{m} \lambda^i y_i)]]_{\lambda=0}, \tag{15}$$

$$B_m = \frac{1}{m!}[\frac{d^m}{d\mu^m}[G_2(\sum_{i=0}^{m} \mu^i y_i)]]_{\mu=0}, \tag{16}$$

where the so-called Adomian polynomials A_m can be evaluated for all forms of nonlinearity. In other words, assuming that the nonlinear function is $G_1(y(x))$ therefore the Adomian polynomials are given by:

$$\begin{aligned}
A_0 &= G_1(y_0), \\
A_1 &= y_1 G_1'(y_0), \\
A_2 &= y_2 G_1'(y_0) + \frac{1}{2!} y_1^2 G_1''(y_0), \\
A_3 &= y_3 G_1'(y_0) + y_1 y_2 G_1''(y_0) + \frac{1}{3!} y_1^3 G_1'''(y_0), \\
A_4 &= y_4 G_1'(y_0) + (\frac{1}{2!} y_2^2 + y_1 y_3) G_1''(y_0) + \frac{1}{2!} y_1^2 y_2 G_1'''(y_0) + \frac{1}{4!} y_1^4 G_1^{(iv)}(y_0),
\end{aligned}$$

similarly, Adomian polynomials B_m can be evaluated for all forms of nonlinearity. In other words, assuming that the nonlinear function is $G_2(y(x))$, therefore the Adomian polynomials are given by

$$\begin{aligned}
B_0 &= G_2(y_0), \\
B_1 &= y_1 G_2'(y_0), \\
B_2 &= y_2 G_2'(y_0) + \frac{1}{2!} y_1^2 G_2''(y_0), \\
B_3 &= y_3 G_2'(y_0) + y_1 y_2 G_2''(y_0) + \frac{1}{3!} y_1^3 G_2'''(y_0), \\
B_4 &= y_4 G_2'(y_0) + (\frac{1}{2!} y_2^2 + y_1 y_3) G_2''(y_0) + \frac{1}{2!} y_1^2 y_2 G_2'''(y_0) + \frac{1}{4!} y_1^4 G_2^{(iv)}(y_0),
\end{aligned}$$

Substituting Eq.(13) and Eq.(14) into Eq.(12) leads to

$$\begin{aligned}
\pounds\{\sum_{m=0}^{\infty} y_m(x)\} &= \frac{1}{s} y(0) + \frac{1}{s^2} y'(0) + \ldots + \frac{1}{s^i} y^{(m-1)}(0) + \frac{1}{s^i} \pounds\{f(x)\} + \frac{1}{s^i} \pounds\{K_1(x-t)\} \\
&\pounds\{\sum_{m=0}^{\infty} A_m(y(t))\} + \frac{1}{s^i} \pounds\{K_2(x-t)\} \pounds\{\sum_{m=0}^{\infty} B_m(y(t))\}
\end{aligned} \tag{17}$$

The Adomian decomposition method presents the recursive relation

$$£\{y_0(x)\} = \frac{1}{s}y(0) + \frac{1}{s^2}y'(0) + ... + \frac{1}{s^i}y^{(m-1)}(0) + \frac{1}{s^i}£\{f(x)\},$$

$$£\{y_{k+1}(x)\} = \frac{1}{s^i}\left(£\{K_1(x-t)\}£\{A_k(y(t))\} + £\{K_2(x-t)\}£\{B_k(y(t))\}\right), k \geq 0 \tag{18}$$

Applying the inverse Laplace transform to the first part of Eq.(18) gives $y_0(x)$, that will define $A_0(x)$ and $B_0(x)$. Using $A_0(x)$ and $B_0(x)$ will enable us to evaluate $y_1(x)$. The determination of $y_0(x)$ and $y_1(x)$ leads to the determination of $A_1(x)$ and $B_1(x)$ that will allows us to determine $y_2(x)$, and so on. This in turn will lead to the complete determination of the components of $y_k(x)$, $k \geq 0$ upon using the second part of Eq.(18). The series solution follows immediately after using Eq.(13). The obtained series solution may converge to an exact solution if such a solution exists. Otherwise, the series solution can be used for numerical purposes. The combined modified Laplace Adomian decomposition method for solving nonlinear Volterra-Fredholm integro differential equations of the second kind is illustrated by studying the following examples in the section **4**.

4. Applications

In order to elucidate the solution procedure of the modified Laplace Adomian decomposition method for solving the nonlinear Volterra-Fredholm integro differential equations is illustrated in the four examples in this section which shows the effectiveness and generalization of our proposed method given above.

Example 4.1

Consider the nonlinear Volterra-Fredholm integro differential equation with:

$f(x) = -xe^x,\ k_1 = e^{x-3t},\ k_2 = e^{x-2t},\ y(0) = 1,$

we can write Eq. (1)

$$y'(x) = -xe^x + \int_0^x e^{x-3t}y^3(t)dt + \int_0^1 e^{x-2t}y^2(t)dt,\ y(0) = 1. \tag{19}$$

Taking Laplace transform of both sides of Eq. (19) gives

$$£\{y'(x)\} = £\{-xe^{x}\} + £\{e^{x-3t} * y^{3}(x)\} + £\{e^{x-2t} * y^{2}(x)\}$$

so that

$$sY(s) - y(0) = \frac{-1}{(s-1)^2} + \frac{1}{(s-1)}£\{y^3(x)\} + \frac{1}{(s-1)}£\{y^2(x)\}$$

or equivalently

$$Y(s) = \frac{1}{s} - \frac{1}{s(s-1)^2} + \frac{1}{s(s-1)}[£\{y^3(x)\} + £\{y^2(x)\}] \tag{20}$$

Substituting the series assumption for $Y(s)$ and the Adomian polynomials for $y^3(x)$ as given above in Eq.(13) and Eq.(14) respectively, and using the recursive relation Eq.(18) we obtain

$$\begin{aligned} Y_0(s) &= \frac{1}{s} - \frac{1}{s(s-1)^2} \\ £\{y_{k+1}(x)\} &= \frac{1}{s(s-1)}[£\{y^3(x)\} + £\{y^2(x)\}],\ k \geq 0. \end{aligned} \tag{21}$$

where $A_k(x)$ and $B_k(x)$ are the Adomian polynomials for the nonlinear term $y^3(x)$ and $y^2(x)$ respectively. The Adomian polynomials for $G_1(y(x)) = y^3(x)$ and $G_2(y(x)) = y^2(x)$ are given by

$$\begin{aligned} A_0 &= y_0^3, \\ A_1 &= 3y_1y_0^2, \\ A_2 &= 3y_2y_0^2 + 3y_1^2y_0, \\ A_3 &= 3y_3y_0^2 + 6y_0y_1y_2 + y_1^3, \end{aligned}$$

and

$$\begin{aligned} B_0 &= y_0^2, \\ B_1 &= 2y_1y_0, \\ B_2 &= 2y_2y_0 + y_1^2, \\ B_3 &= 2y_3y_0 + 2y_1y_2, \end{aligned} \tag{22}$$

Taking the inverse Laplace transform of both sides of the first part of Eq.(21), and using the recursive relation Eq.(21) gives

$$\begin{aligned} y_0 &= e^x - xe^x, \\ &= 1 - \frac{1}{2!}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - ..., \\ y_1 &= 2[\frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + ...], \\ &. \\ &. \\ &. \end{aligned}$$

that converges to the exact solution

$$y(x) = e^x$$

Example 4.2

Consider the following nonlinear Volterra-Fredholm integro differential equation of the first kind with:

$f(x) = -\frac{9}{5} - \frac{5}{2}x + \frac{1}{2}x^2 + 2e^x + \frac{1}{4}e^{2x} + xe^x$, $k_1 = (x-t)^2$, $k_2 = e^{x-t}$, $y(0) = 2$,

we can write Eq. (1)

$$(23) \qquad -\frac{9}{5} - \frac{5}{2}x + \frac{1}{2}x^2 + 2e^x + \frac{1}{4}e^{2x} + xe^x = \int_0^x (x-t)^2 y^2(t)dt + \int_0^1 e^{x-t} y'(t)dt.$$

Taking Laplace transform of both sides of Eq. (23) gives

$$-\frac{9}{5s} - \frac{5}{2s^2} + \frac{1}{s^3} + \frac{2}{s-1} + \frac{1}{4(s-2)} + \frac{1}{(s-1)^2} = \frac{1}{s^2}£\{y^2(s)\} + \frac{1}{s-1}(sY(s) - y(0))$$

so that

$$(24) Y(s) = \frac{2}{s} + \frac{s-1}{s}(-\frac{9}{5s} - \frac{5}{2s^2} + \frac{1}{s^3} + \frac{2}{s-1} + \frac{1}{4(s-2)} + \frac{1}{(s-1)^2}) - \frac{s-1}{s^3}£\{y^2(x)\}$$

Substituting the series assumption for $Y(s)$ and the Adomian polynomials for $y^2(x)$ as given above in Eq.(13) and Eq.(14) respectively, and using the recursive relation Eq.(18) we obtain

$$\begin{aligned} Y_0(s) &= \frac{2}{s}+\frac{s-1}{s}\left(-\frac{9}{5s}-\frac{5}{2s^2}+\frac{1}{s^3}+\frac{2}{s-1}+\frac{1}{4(s-2)}+\frac{1}{(s-1)^2}\right), \\ £\{y_{k+1}(x)\} &= -\frac{s-1}{s^3}£\{A_k(x)\},\ k\geq 0. \end{aligned} \tag{25}$$

where $A_k(x)$ are the Adomian polynomials for the nonlinear term $y^2(x)$. The Adomian polynomials for $G_1(y(x))=y^2(x)$ are given by

$$\begin{aligned} A_0 &= y_0^2, \\ A_1 &= 2y_1y_0, \\ A_2 &= 2y_2y_0+y_1^2, \\ A_3 &= 2y_3y_0+2y_1y_2, \end{aligned} \tag{26}$$

Taking the inverse Laplace transform of both sides of the first part of Eqs.(25), and using the recursive relation Eq.(25) gives

$$\begin{aligned} y_0 &= 2+x+\frac{5}{2}x^2+\frac{1}{6}x^3+\frac{1}{8}x^4+..., \\ y_1 &= -2x^2-\frac{3}{4}x^4-\frac{1}{10}x^5+..., \end{aligned} \tag{27}$$

and so on for other components. Using Eq. (13), the series solution is therefore given by

$$y(x)=2+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+..., \tag{28}$$

that converges to the exact solution

$$y(x)=1+e^x$$

Example 4.3

Consider the nonlinear integro differential equation with:

$f(x)=\frac{9}{4}-\frac{5}{2}x-\frac{1}{2}x^2-3e^{-x}-\frac{1}{4}e^{-2x},\ k_1=(x-t),\ k_2=0,\ j=1$

we can write Eq.(1)

$$y'(x) = \frac{9}{4} - \frac{5}{2}x - \frac{1}{2}x^2 - 3e^{-x} - \frac{1}{4}e^{-2x} + \int_0^x (x-t)y^2(t)dt, \; y(0) = 2. \tag{29}$$

Taking Laplace transform of both sides of Eq. (29) gives

$$\pounds\{y'(x)\} = \pounds\{\frac{9}{4} - \frac{5}{2}x - \frac{1}{2}x^2 - 3e^{-x} - \frac{1}{4}e^{-2x}\} + \pounds\{(x-t) * y^2(x)\} \tag{30}$$

so that

$$sY(s) - y(0) = \frac{9}{4s} - \frac{5}{2s^2} - \frac{1}{s^3} - \frac{3}{(s+1)} - \frac{1}{4(s+2)} + \frac{1}{s^2}\pounds\{y^2(x)\} \tag{31}$$

or equivalently

$$Y(s) = \frac{2}{s} + \frac{9}{4s^2} - \frac{5}{2s^3} - \frac{1}{s^4} - \frac{3}{s(s+1)} - \frac{1}{4s(s+2)} + \frac{1}{s^3}\pounds\{y^2(x)\} \tag{32}$$

Substituting the series assumption for $Y(s)$ and the Adomian polynomials for $y^2(x)$ as given above in Eq.(13) and Eq.(14) respectively, and using the recursive relation Eq.(18) we obtain

$$\begin{aligned} Y_0(s) &= \frac{2}{s} + \frac{9}{4s^2} - \frac{5}{2s^3} - \frac{1}{s^4} - \frac{3}{s(s+1)} - \frac{1}{4s(s+2)} \\ \pounds\{y_{k+1}(x)\} &= \frac{1}{s^3}\pounds\{A_k(x)\}, \; k \geq 0. \end{aligned} \tag{33}$$

where $A_k(x)$ are the Adomian polynomials for the nonlinear term $y^2(x)$. The Adomian polynomials for $G_1(y(x)) = y^2(x)$ are given by

$$\begin{aligned} A_0 &= y_0^2, \\ A_1 &= 2y_1y_0, \\ A_2 &= 2y_2y_0 + y_1^2, \\ A_3 &= 2y_3y_0 + 2y_1y_2. \end{aligned} \tag{34}$$

Taking the inverse Laplace transform of both sides of the first part of Eq.(33), and using the recursive relation Eq.(33) gives

$$\begin{aligned} y_0 &= 2 - x + \frac{1}{2!}x^2 - \frac{5}{3!}x^3 + \frac{5}{4!}x^4 - ..., \\ y_1 &= \frac{2}{3}x^3 - \frac{1}{3!}x^4 + \frac{1}{20}x^5 + ..., \end{aligned} \tag{35}$$

and so on for other components. Using Eq. (13), the series solution is therefore given by

$$y(x)=2-x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^4+...,$$

that converges to the exact solution

$$y(x)=1+e^{-x}$$

Example 4.4

Consider the nonlinear integro differential equation with:

$$y'(x)=1+\int_0^x y^2(t)dt+\int_0^1 y^2(t)dt, \quad y(0)=0. \tag{36}$$

Applying the Laplace transform and by using the initial condition we have

$$\pounds\{y'(x)\}=\pounds\{1\}+\pounds\{1*y^2(x)\}+\pounds\{1*y^2(x)\} \tag{37}$$

so that

$$sY(s)-y(0)=\frac{1}{s}+\frac{2}{s}\pounds\{y^2(x)\} \tag{38}$$

or equivalently

$$Y(s)=\frac{1}{s^2}+\frac{2}{s^2}\pounds\{y^2(x)\} \tag{39}$$

Substituting the series assumption for $Y(s)$ and the Adomian polynomials for $y^2(x)$ as given above in Eq.(13) and Eq.(14) respectively, and using the recursive relation Eq.(18) we obtain

$$\begin{aligned} Y_0(s) &= \frac{1}{s^2}, \\ \pounds\{y_{k+1}(x)\} &= \frac{2}{s^2}\pounds\{A_k(x)\},\ k\geq 0. \end{aligned} \tag{40}$$

where $A_k(x)$ are the Adomian polynomials for the nonlinear term $y^2(x)$. The Adomian polynomials for $G(y(x))=y^2(x)$ are given by

$$\begin{aligned} A_0 &= y_0^2, \\ A_1 &= 2y_1y_0, \\ A_2 &= 2y_2y_0+y_1^2, \\ A_3 &= 2y_3y_0+2y_1y_2, \end{aligned} \tag{41}$$

Taking the inverse Laplace transform of both sides of the first part of Eq.(40), and using the recursive relation Eq.(40) gives

$$\begin{aligned} y_0 &= x \\ y_1 &= \frac{1}{3}x^3 \\ &. \\ &. \\ &. \end{aligned} \tag{42}$$

and so on for other components. Using Eq. (13), the series solution is therefore given by

$$y(x) = x + \frac{1}{3}(x)^3 + \frac{2}{45}(x)^6 + ..., \tag{43}$$

5. Convergence Analysis

In this section, we will study the convergence analysis as the same manner in [12] of the LADM applied to the nonlinear Volterra-Fredholm integro differential equations. Let us consider the Hilbert space $\mathbb{H}$ which may define by $\mathbb{H} = L^2((\delta, \theta) \times [0, T])$, the set of applications: $y : ((\delta, \theta) \times [0, T]) \longrightarrow \mathbb{R}$ with $\int_{(\delta,\theta)\times[0,T]} y^2(x,s)dsd\tau < +\infty$. Now we consider the nonlinear integro differential equations in the light of above assumptions and let us denote

$$L(y) = \frac{\partial^n y}{\partial x^n}$$

then the nonlinear Volterra-Fredholm integro differential equations become in a operator form

$$L(y) = f(x) + \int_a^x K_1(x,t)[Ry(t) + Ny(t)]dt + \int_a^b K_2(x,t)[Ry(t) + Ny(t)]dt$$

The LADM is convergence if the following two hypotheses are satisfied:

(**H1**)$(L(y) - L(u), y - u) \geq K\|y - u\|^2; \quad \forall y, u \in \mathbb{H}.$

(**H2**) whatever may be $M > 0$, there exist a constant $\beta(M) > 0$ such that for $y, u \in \mathbb{H}$ with $\|y\| \leq M, \|u\| \leq M$, we have:

$$(L(y) - L(u), y - u) \geq \beta(M)\|y - u\|\|w\|$$

for every $w \in \mathbb{H}$, [12].

Theorem 5.1. *(Sufficient condition of convergence for Example 4.4). The Laplace Adomian decomposition method applied to the nonlinear Volterra -Fredholm integro differential equation as follows*

$$L(y) = \frac{\partial}{\partial x} y = 1 + \int_0^x y^2(t)dt + \int_0^1 y^2(t)dt$$

without initial condition, converges towards a particular solution.

Proof.

Now, we will verify the conditions (**H1**) and (**H2**) of convergence. We will start to verify the convergence hypotheses (**H1**) for the operator $L(y)$: $i.e, \exists k > 0, \forall y, u \in \mathbb{H}$, we have:

$$L(y) - L(u) = \int_0^x (y^2(t) - u^2(t))dt + \int_0^1 (y^2(t) - u^2(t))dt$$

Then we get

$$(L(y) - L(u), y - u) = (\int_0^x (y^2(t) - u^2(t))dt + \int_0^1 (y^2(t) - u^2(t))dt, y - u)$$

According the Schwartz inequality, we get

$$(\int_0^x (y^2(t) - u^2(t))dt + \int_0^1 (y^2(t) - u^2(t))dt, y - u) \leq \xi \quad \|y^2 - u^2\| \|y - u\|$$

Now we use the mean value theorem, then we have

$$\begin{aligned}
(\int_0^x (y^2(t) - u^2(t))dt + \int_0^1 (y^2(t) - u^2(t))dt, y - u) &\leq \xi \quad \|y^2 - u^2\| \|y - u\| \\
&= \frac{1}{3} \xi \eta^3 \|y - u\|^2 \\
&\leq \frac{1}{3} \xi M^3 \|y - v\|^2, \\
(-(\int_0^x (y^2(t) - u^2(t))dt + \int_0^1 (y^2(t) - u^2(t))dt), y - u) &\geq \frac{1}{3} \xi M^3 \|y - u\|^2,
\end{aligned}$$

where $y < \eta < u$ and $\|y\| \leq M, \|u\| \leq M$. Therefore:

$$(L(y) - L(u), y - u) \geq K \|y - u\|^2$$

where $k = \frac{1}{3} \xi M^3$. Hence, we find the hypothesis (**H1**). Now we verify the convergence hypotheses (**H2**) for the operator $L(y)$ which is for every $M > 0$, there exist a constant $\beta(M) > 0$

such that for $y,u\in\mathbb{H}$ with $\|u\|\le M, \|u\|\le M$, we have $(L(y)-L(u),y-u)\le\beta(M)\|y-u\|\|w\|$ for every $w\in\mathbb{H}$. For that we have:

$$\begin{aligned}(L(y)-L(u),w) &= (\int_0^x (y^2(t)-u^2(t))dt+\int_0^1 (y^2(t)-u^2(t))dt,w) \\ &\le M^3\|y-v\|\|w\| = \beta(M)\|y-u\|\|w\|\end{aligned}$$

where $\beta(M)=M^3$ and therefore (**H2**) is hold. The proof is complete.

6. Conclusion

A reliable method for obtaining approximate solutions of nonlinear Volterra-Fredholm integro differential using the modified Laplace Adomian decomposition method which avoids the tedious work needed by traditional techniques has been developed. Exact solutions were easily obtained. We carefully applied a reliable modification of Laplace Adomian decomposition method for VFIDEs. The main advantage of this method is the fact that it gives the analytical solution. Also, this method is combining of two powerful methods for obtaining exact solutions of nonlinear Volterra-Fredholm integro differential. Also, we proved the convergence of LADM applied to the Volterra-Fredholm integro differential equations.

Conflict of Interests

The authors declare that there is no conflict of interests.

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