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ON TRILATERAL AND TRILINEAR GENERATING FUNCTIONS

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Abstract: In this paper, we prove a general theorem on trilateral generating functions involving Laguerre, Jacobi and the two-parameter Srivastava polynomials of one variable. Some applications of these theorems lead us to derive certain trilinear and trilateral generating functions involving Laguerre and Jacobi polynomials of one variable.

Keywords: generating functions; Srivastava polynomials; Laguerre polynomials; Jacobi polynomials; Lauricella's function.

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1. Introduction

In 1972, Srivastava [7] introduced the following family of polynomials:

$$S_n^N(x)=\sum_{k=0}^{[\frac{n}{N}]}\frac{(-n)_{Nk}}{k!}A_{n,k}\,x^k \quad (n\in\mathbb{N}_0=\mathbb{N}\cup\{0\};N\in\mathbb{N}), \tag{1.1}$$

where $\mathbb{N}$ is the set of positive integers, $\{A_{n,k}\}_{n,k=0}^{\infty}$ is a bounded double sequence of real or complex numbers, $[a]$ denotes the greatest integer in $a\in\mathbb{R}$ and $(\lambda)_n$ denotes the Pochhammer symbol defined by [6]

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \quad \lambda\neq 0,-1,-2,\cdots \tag{1.2}$$

Afterwards, Gonzalez *at al.* [1] extended the Srivastava polynomials $S_n^N(x)$ as follows:

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$$S_{n,m}^{N}(x)=\sum_{k=0}^{\left[\frac{n}{N}\right]}\frac{(-n)_{Nk}}{k!}A_{n+m,k}\,x^{k}\quad (n,m\in\mathbb{N}_0;\ N\in\mathbb{N}). \tag{1.3}$$

In 2013, Kaanoglu and Ozarslan [2] introduced the following family of two-parameter one-variable Srivastava polynomials:

$$S_{n}^{p,q}(x)=\sum_{k=0}^{n}\frac{(-n)_{k}}{k!}A_{p+q+n,q+k}\,x^{k}\quad (p,q,n,k\in\mathbb{N}_0), \tag{1.4}$$

where $\{A_{n,k}\}$ is a bounded double sequence of real or complex numbers.

Also the following remarks are given in [2]:

Remark 1.1 Choosing $A_{m,n}=(-\alpha-m)_n\quad (m,n\in\mathbb{N}_0)$ in (1.4), we get

$$S_{n}^{p,q}\left(\tfrac{-1}{x}\right)=(-1)^{q}(\alpha+p+n+1)_{q}\frac{n!}{(-x)^{n}}L_{n}^{(\alpha+p)}(x), \tag{1.5}$$

where $L_n^{(\alpha)}(x)$ is the classical Laguerre polynomials defined by [8]

$$L_{n}^{(\alpha)}(x)=\frac{(-x)^{n}}{n!}\,{}_2F_0\left[-n,-\alpha-n;-;\frac{-1}{x}\right] \tag{1.6}$$

Remark 1.2 Choosing $A_{m,n}=\dfrac{(\alpha+\beta+1)_{2m}(-\beta-m)_n}{(\alpha+\beta+1)_m(-\alpha-\beta-2m)_n}\quad (m,n\in\mathbb{N}_0)$ in (1.4), we get

$$\begin{aligned}S_{n}^{p,q}\left(\tfrac{2}{1+x}\right)&=\frac{(\alpha+\beta+1)_{2p+2q+2n}(-\beta-p-q-n)_{q}(1+\alpha+\beta+2p+q)_{n}}{(\alpha+\beta+1)_{p+q+n}(-\alpha-\beta-2p-2q-2n)_{q}(1+\alpha+\beta+2p+q)_{2n}}\\&\quad\times n!\left(\tfrac{2}{1+x}\right)^{n}P_{n}^{(\alpha+p+q,\beta+p)}(x)\\&=\frac{(\lambda+\mu+1+p+q+n)_{n+p+q}(-\mu-p-q-n)_{q}}{(1+\lambda+\mu+2p+q+n)_{n}(-\lambda-\mu-2p-2q-2n)_{q}}\\&\quad\times n!\left(\tfrac{2}{1+x}\right)^{n}P_{n}^{(\alpha+p+q,\beta+p)}(x),\end{aligned} \tag{1.7}$$

where $P_n^{(\alpha,\beta)}(x)$ is the classical Jacobi polynomials defined by [4]

$$P_{n}^{(\alpha,\beta)}(x)=\binom{\alpha+\beta+1}{n}\left(\frac{(1+x)}{2}\right)^{n}{}_2F_1\left[-n,-\beta-n;-\alpha-\beta-2n;\frac{2}{1+x}\right]. \tag{1.8}$$

The general triple hypergeometric series $F^{(3)}[x,y,z]$ is defined as follows [8] :

$$F^{(3)}[x,y,z]=F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{matrix}x,y,z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \Lambda(m,n,p) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{1.9}$$

where

$$\Lambda(m,n,p) = \frac{\prod_{j=1}^{A}(a_j)_{m+n+p} \prod_{j=1}^{B}(b_j)_{m+n} \prod_{j=1}^{B'}(b'_j)_{n+p} \prod_{j=1}^{B''}(b''_j)_{p+m}}{\prod_{j=1}^{E}(e_j)_{m+n+p} \prod_{j=1}^{G}(g_j)_{m+n} \prod_{j=1}^{G'}(g'_j)_{n+p} \prod_{j=1}^{G''}(g''_j)_{p+m}}$$

$$\frac{\prod_{j=1}^{C}(c_j)_m \prod_{j=1}^{C'}(c'_j)_n \prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{H}(h_j)_m \prod_{j=1}^{H'}(h'_j)_n \prod_{j=1}^{H''}(h''_j)_p} \tag{1.10}$$

(a) abbreviates the array of A parameters $a_1, \cdots, a_A$ with similar interpretation for $(b), (b'), (b'')$ et cetera.

The Lauricella's function $F_C^{(3)}$ is defined as follows [8]

$$F_C^{(3)}(a, b; c_1, c_2, c_3; x_1, x_2, x_3)$$

$$= \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a)_{m_1+m_2+m_3} (b)_{m_1+m_2+m_3}}{(c_1)_{m_1} (c_2)_{m_2} (c_3)_{m_3}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} \frac{x_3^{m_3}}{m_3!} \tag{1.11}$$

$$|x_1|^{1/2} + |x_2|^{1/2} + |x_3|^{1/2} < 1 \ .$$

2. Main Results

In this section, we prove the following two theorems on trilateral generating functions involving Laguerre, Jacobi and the two-parameter Srivastava polynomials of one variable:

Theorem 2.1 The following family of trilateral generating functions holds true:

$$\sum_{n,p,q=0}^{\infty} \frac{[(n+p+q)!]^2}{(\alpha+1)_{n+p+q}(\beta+1)_{n+p+q}} L_{n+p+q}^{(\alpha)}(x) L_{n+p+q}^{(\beta)}(y)\, S_n^{p,q}(z) \frac{(-t)^n}{n!} \frac{u^p}{p!} \frac{v^q}{q!}$$

$$= \sum_{p,q=0}^{\infty} \frac{[(p+q)!]^2}{(\alpha+1)_{p+q}(\beta+1)_{p+q}} L_{p+q}^{(\alpha)}(x) L_{p+q}^{(\beta)}(y) A_{p+q,q} \frac{(u-t)^p}{p!} \frac{(v+zt)^q}{q!}. \tag{2.1}$$

Theorem 2.2 The following family of trilateral generating functions holds true:

$$\sum_{n,p,q=0}^{\infty} \frac{[(n+p+q)!]^2}{(\gamma+1)_{n+p+q}(\delta+1)_{n+p+q}} P_{n+p+q}^{(\alpha,\beta)}(x) P_{n+p+q}^{(\gamma,\delta)}(y)\, S_n^{p,q}(z) \frac{(-t)^n}{n!} \frac{u^p}{p!} \frac{v^q}{q!}$$

$$= \sum_{p,q=0}^{\infty} \frac{[(p+q)!]^2}{(\gamma+1)_{p+q}(\delta+1)_{p+q}} P_{p+q}^{(\alpha,\beta)}(x) P_{p+q}^{(\gamma,\delta)}(y) A_{p+q,q} \frac{(u-t)^p}{p!} \frac{(v+zt)^q}{q!} . \tag{2.2}$$

Proof of (2.1): Denoting the left hand side of (2.1) by S, expressing $S_n^{p.q}(z)$ as in (1.4) and using the following identity [8]

$$(-n)_k = \frac{(-1)^k n!}{(n-k)!}, \ 0 \le k \le n, \tag{2.3}$$

we obtain

$$S = \sum_{n,p,q=0}^{\infty} \frac{[(n+p+q)!]^2}{(\alpha+1)_{n+p+q}(\beta+1)_{n+p+q}} L_{n+p+q}^{(\alpha)}(x) L_{n+p+q}^{(\beta)}(y)$$

$$\times \sum_{k=0}^{n} \frac{(-z)^k}{k!} A_{n+p+q,q+k} \frac{(-t^n)}{(n-k)!} \frac{u^p}{p!} \frac{v^q}{q!} \tag{2.4}$$

Using the following result [8]:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+k), \tag{2.5}$$

we get

$$S = \sum_{n,p,q,k=0}^{\infty} \frac{[(n+p+q+k)!]^2}{(\alpha+1)_{n+p+q+k}(\beta+1)_{n+p+q+k}} L_{n+p+q+k}^{(\alpha)}(x) L_{n+p+q+k}^{(\beta)}(y)$$

$$\times A_{n+p+q+k,q+k} \frac{(-t)^n}{n!} \frac{u^p}{p!} \frac{v^q}{q!} \frac{(zt)^k}{k!} \tag{2.6}$$

Now, in (2.6) using the following results [8]:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k) , \tag{2.7}$$

we have

$$S = \sum_{p,q,k=0}^{\infty} \frac{[(p+q+k)!]^2}{(\alpha+1)_{p+q+k}(\beta+1)_{p+q+k}} L_{p+q+k}^{(\alpha)}(x) L_{p+q+k}^{(\beta)}(y)$$

$$A_{p+q+k,q+k} \frac{u^p}{p!} \frac{v^q}{q!} \frac{(zt)^k}{k!} \sum_{n=0}^{p} \frac{(-p)_n}{n!} \left(\frac{t}{u}\right)^n \tag{2.8}$$

Using the result [8]

$$\sum_{n=0}^{\infty} (\lambda)_n \frac{x^n}{n!} = (1-x)^{-\lambda}, \tag{2.9}$$

we have

$$S = \sum_{p,q,k=0}^{\infty} \frac{[(p+q+k)!]^2}{(\alpha+1)_{p+q+k}(\beta+1)_{p+q+k}} L_{p+q+k}^{(\alpha)}(x) L_{p+q+k}^{(\beta)}(y) A_{p+q+k,q+k} \frac{(u-t)^p}{p!} \frac{v^q}{q!} \frac{(zt)^k}{k!} \tag{2.10}$$

Using the result (2.7) again, we have

$$= \sum_{p,q=0}^{\infty} \frac{[(p+q)!]^2}{(\alpha+1)_{p+q}(\beta+1)_{p+q}} L_{p+q}^{(\alpha)}(x) L_{p+q}^{(\beta)}(y) A_{p+q,q} \frac{(u-t)^p}{p!} \frac{v^q}{q!} \sum_{k=0}^{q} \frac{(-q)_k (-zt/v)^k}{k!} \tag{2.11}$$

Finally, using (2.9), we easily arrive at the right-hand side of (2.1). This completes the proof of Theorem 2.1. By similar manner as in proof of Theorem 2.1, we can prove the Theorem 2.2.

Remark 2.1 On taking $u = t$ in (2.1) and (2.2), we deduce the following interesting corollaries:

Corollary 2.1.

$$\sum_{n,p,q=0}^{\infty} \frac{[(n+p+q)!]^2}{(\alpha+1)_{n+p+q}(\beta+1)_{n+p+q}} L_{n+p+q}^{(\alpha)}(x) L_{n+p+q}^{(\beta)}(y) S_n^{p,q}(z) \frac{(-t)^n}{n!} \frac{t^p}{p!} \frac{v^q}{q!}$$

$$= \sum_{q=0}^{\infty} \frac{q!}{(\alpha+1)_q(\beta+1)_q} A_{q,q} L_q^{(\alpha)}(x) L_q^{(\beta)}(y) (v+zt)^q \tag{2.12}$$

Corollary 2.2.

$$\sum_{n,p,q=0}^{\infty} \frac{[(n+p+q)!]^2}{(\gamma+1)_{n+p+q}(\delta+1)_{n+p+q}} P_{n+p+q}^{(\alpha,\beta)}(x) P_{n+p+q}^{(\gamma,\delta)}(y) S_n^{p,q}(z) \frac{(-t)^n}{n!} \frac{t^p}{p!} \frac{v^q}{q!}$$

$$= \sum_{q=0}^{\infty} \frac{q!}{(\gamma+1)_q(\delta+1)_q} A_{q,q} P_q^{(\alpha,\beta)}(x) P_q^{(\gamma,\delta)}(y) (v+zt)^q \tag{2.13}$$

Remark 2.2 On taking $v = 0$ in (2.12) and (2.13), we deduce the following trilateral generating functions involving the extended Srivastava polynomials $S_{n,m}^{1}(z)$:

Corollary 2.3

$$\sum_{n,p=0}^{\infty} \frac{[(n+p)!]^2}{(\alpha+1)_{n+p}(\beta+1)_{n+p}} L_{n+p}^{(\alpha)}(x) L_{n+p}^{(\beta)}(y) S_{n,p}^{1}(z) \frac{(-t)^n}{n!} \frac{t^p}{p!}$$

$$= \sum_{q=0}^{\infty} \frac{q!}{(\alpha+1)_q(\beta+1)_q} A_{q,q} L_q^{(\alpha)}(x) L_q^{(\beta)}(y) (zt)^q \tag{2.14}$$

Corollary 2.4

$$\sum_{n,p=0}^{\infty} \frac{[(n+p)!]^2}{(\gamma+1)_{n+p}(\delta+1)_{n+p}} P_{n+p}^{(\alpha,\beta)}(x) P_{n+p}^{(\gamma,\delta)}(y)\, S_{n,p}^{1}(z) \frac{(-t)^n}{n!} \frac{t^p}{p!}$$

$$= \sum_{q=0}^{\infty} \frac{q!}{(\delta+1)_q(\delta+1)_q} A_{q,q}\, P_q^{(\alpha,\beta)}(x) P_q^{(\gamma,\delta)}(y)(zt)^q \tag{2.15}$$

3. Applications

I. In (2.12) choosing $A_{m,n} = (-\mu-m)_n$, $A_{m,n} = \dfrac{(\alpha+\mu+1)_{2m}(-\mu-m)_n}{(\alpha+\mu+1)_m(-\alpha-\mu-2m)_n}$, using (1.5) and (1.7) respectively, we get :

$$\sum_{n,p,q=0}^{\infty} \frac{[(n+p+q)!]^2(1+\mu+p+n)_q}{(\alpha+1)_{n+p+q}(\beta+1)_{n+p+q}} L_{n+p+q}^{(\alpha)}(x) L_{n+p+q}^{(\beta)}(y)\, L_n^{(\mu+p)}(z) \left(\frac{t}{z}\right)^n \frac{t^p}{p!} \frac{v^q}{q!}$$

$$= \sum_{q=0}^{\infty} \frac{(\mu+1)_q\, q!}{(\alpha+1)_q(\beta+1)_q} L_q^{(\alpha)}(x) L_q^{(\beta)}(y)\, (v+t/z)^q \tag{3.1}$$

and

$$\sum_{n,p,q=0}^{\infty} \frac{[(n+p+q)!]^2}{(\alpha+1)_{n+p+q}(\beta+1)_{n+p+q}} \frac{(\alpha+\mu+1+p+q+n)_{n+p+q}(-\mu-p-q-n)_q}{(1+\alpha+\mu+2p+q+n)_n(-\alpha-\mu-2p-2q-2n)_q}$$

$$\cdot L_{n+p+q}^{(\alpha)}(x) L_{n+p+q}^{(\beta)}(y)\, P_n^{(\alpha+p+q,\mu+p)}(z) \left(\frac{2t}{1+z}\right)^n \frac{(-t)^p}{p!} \frac{v^q}{q!}$$

$$= \sum_{q=0}^{\infty} \frac{(\mu+1)_q\, q!}{(\alpha+1)_q(\beta+1)_q} L_q^{(\alpha)}(x) L_q^{(\beta)}(y)\, (v-2t/(1+z))^q \tag{3.2}$$

Now, in (3.1) and (3.2) using the generating function [6] (see also [8])

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n\, n!}{(\alpha+1)_n(\beta+1)_n} L_n^{(\alpha)}(x) L_n^{(\beta)}(y)\, t^n$$

$$= (1-t)^{-\lambda} \exp\left(\frac{xt}{t-1}\right) F^{(3)} \left[\begin{matrix} -::-;\ \lambda\ ;\ -\ :\alpha-\lambda+1;\ -\ ;-; \\ -::-;\beta+1;\alpha+1:\ \ -\ \ ;-\ ;-; \end{matrix}\ \frac{xt}{1-t}, \frac{yt}{t-1}, \frac{xyt}{(1-t)^2}\right], \tag{3.3}$$

we obtain respectively the following trilinear and trilateral generating functions:

$$\sum_{n,p,q=0}^{\infty} \frac{[(n+p+q)!]^2(1+\mu+p+n)_q}{(\alpha+1)_{n+p+q}(\beta+1)_{n+p+q}} L_{n+p+q}^{(\alpha)}(x) L_{n+p+q}^{(\beta)}(y)\, L_n^{(\mu+p)}(z) \left(\frac{t}{z}\right)^n \frac{t^p}{p!} \frac{v^q}{q!}$$

$$= (1-u)^{-\mu-1} \exp\left(\frac{xu}{u-1}\right) F^{(3)}\left[\begin{matrix} -::-;\mu+1; & - & :\alpha-\mu;-;-; \\ -::-;\beta+1;\alpha+1: & - & ;-\ ;-; \end{matrix} \frac{xu}{1-u}, \frac{yu}{u-1}, \frac{xyu}{(1-u)^2}\right], \quad (3.4)$$

and

$$\sum_{n,p,q=0}^{\infty} \frac{[(n+p+q)!]^2}{(\alpha+1)_{n+p+q}(\beta+1)_{n+p+q}} \frac{(\alpha+\mu+1+p+q+n)_{n+p+q}(-\mu-p-q-n)_q}{(\alpha+\mu+1+2p+q+n)_n(-\alpha-\mu-2p-2q-2n)_q}$$

$$\cdot L_{n+p+q}^{(\alpha)}(x) L_{n+p+q}^{(\beta)}(y)\, P_n^{(\alpha+p+q,\mu+p)}(z) \left(\frac{2t}{1+z}\right)^n \frac{(-t)^p}{p!} \frac{v^q}{q!}$$

$$= (1-w)^{-\gamma-1} \exp\left(\frac{xw}{w-1}\right) F^{(3)}\left[\begin{matrix} -::-;\mu+1; & - & :\alpha-\mu;-;-; \\ -::-;\beta+1;\alpha+1: & - & ;-\ ;-; \end{matrix} \frac{xw}{1-w}, \frac{yw}{w-1}, \frac{xyw}{(1-w)^2}\right], \quad (3.5)$$

where $u = v + \dfrac{t}{z}$ and $w = v - 2t/(1+z)$.

Further, if we take $v = 0$ in (3.4) and (3.5) , then we obtain

$$\sum_{n,p=0}^{\infty} \frac{[(n+p)!]^2}{(\alpha+1)_{n+p}(\beta+1)_{n+p}} L_{n+p}^{(\alpha)}(x) L_{n+p}^{(\beta)}(y)\, L_n^{(\mu+p)}(z) \left(\frac{t}{z}\right)^n \frac{t^p}{p!}$$

$$= (1-t/z)^{-\mu-1} \exp\left(\frac{xt}{t-z}\right) F^{(3)}\left[\begin{matrix} -::-;\mu+1; & - & :\alpha-\mu;-;-; \\ -::-;\beta+1;\alpha+1: & - & ;-\ ;-; \end{matrix} \frac{xt}{z-t}, \frac{yt}{t-z}, \frac{xyzt}{(z-t)^2}\right] \quad (3.6)$$

and

$$\sum_{n,p=0}^{\infty} \frac{[(n+p)!]^2 (\alpha+\beta+1+p+n)_p}{(\alpha+1)_{n+p}(\beta+1)_{n+p}} L_{n+p}^{(\alpha)}(x) L_{n+p}^{(\beta)}(y) P_n^{(\alpha+p,\mu+p)}(z) \left(\frac{2t}{1+z}\right)^n \frac{(-t)^p}{p!}$$

$$= \left(\frac{1+z+2t}{1+z}\right)^{-\mu-1} \exp\left(\frac{2xt}{1+z+2t}\right)$$

$$\cdot F^{(3)}\left[\begin{matrix} -::-;\mu+1; & - & :\alpha-\mu;-;-; \\ -::-;\beta+1;\alpha+1: & - & ;-\ ;-; \end{matrix} \frac{-2xt}{1+z+2t}, \frac{2yt}{1+z+2t} \frac{-2xyt(1+z)}{(1+z+2t)^2}\right] \quad (3.7)$$

Setting $\mu = \beta$ in (3.6) and (3.7), we get respectively the following trilinear and trilateral generating functions in the following form:

$$\sum_{n,p=0}^{\infty} \frac{[(n+p)!]^2}{(\alpha+1)_{n+p}(\beta+1)_{n+p}} L_{n+p}^{(\alpha)}(x) L_{n+p}^{(\beta)}(y)\, L_n^{(\beta+p)}(z) \left(\frac{t}{z}\right)^n \frac{t^p}{p!}$$

$$= (1-t/z)^{-\beta-1} \exp\left(\frac{(x+y)t}{t-z}\right) \Phi_3\left[\alpha-\beta; \alpha+1; \frac{xt}{z-t}, \frac{xyzt}{(z-t)^2}\right] \quad (3.8)$$

and

$$\sum_{n,p=0}^{\infty} \frac{[(n+p)!]^2 (\alpha+\beta+1+p+n)_p}{(\alpha+1)_{n+p}(\beta+1)_{n+p}} L_{n+p}^{(\alpha)}(x) L_{n+p}^{(\beta)}(y) P_n^{(\alpha+p,\beta+p)}(z) \left(\frac{2t}{1+z}\right)^n \frac{(-t)^p}{p!}$$

$$= \left(\frac{1+z+2t}{1+z}\right)^{-\beta-1} \exp\left(\frac{2(x+y)t}{1+z+2t}\right) \Phi_3\left[\alpha-\beta; \alpha+1; \frac{-2xt}{1+z+2t}, \frac{-2xyt(1+z)}{(1+z+2t)^2}\right], \quad (3.9)$$

where Φ_3 is Humbert's function of two variables defined by [8].

II. In (2.13) choosing $A_{m,n} = (-\alpha-\beta-m)_n$, $A_{m,n} = \dfrac{(2\alpha+\beta+1)_{2m}(-\alpha-\beta-m)_n}{(2\alpha+\beta+1)_m(-2\alpha-\beta-2m)_n}$ and using (1.5), (1.7) respectively, we get :

$$\sum_{n,p,q=0}^{\infty} \frac{[(n+p+q)!]^2 (1+\alpha+\beta+p+n)_q}{(\gamma+1)_{n+p+q}(\delta+1)_{n+p+q}} P_{n+p+q}^{(\alpha,\beta)}(x) P_{n+p+q}^{(\gamma,\delta)}(y) L_n^{(\alpha+\beta+p)}(z) \left(\frac{t}{z}\right)^n \frac{t^p}{p!} \frac{v^q}{q!}$$

$$= \sum_{q=0}^{\infty} \frac{(\alpha+\beta+1)_q q!}{(\gamma+1)_q(\delta+1)_q} P_q^{(\alpha,\beta)}(x) P_q^{(\gamma,\delta)}(y) (v+t/z)^q \quad (3.10)$$

and

$$\sum_{n,p,q=0}^{\infty} \frac{[(n+p+q)!]^2}{(\gamma+1)_{n+p+q}(\delta+1)_{n+p+q}} \frac{(2\alpha+\beta+1+p+q+n)_{n+p+q}(-\alpha-\beta-p-q-n)_q}{(2\alpha+\beta+1+2p+q+n)_n(-2\alpha-\beta-2p-2q-2n)_q}$$

$$\cdot P_{n+p+q}^{(\alpha,\beta)}(x) P_{n+p+q}^{(\gamma,\delta)}(y) P_n^{(\alpha+p+q,\alpha+\beta+p)}(z) \left(\frac{2t}{1+z}\right)^n \frac{(-t)^p}{p!} \frac{v^q}{q!}$$

$$= \sum_{q=0}^{\infty} \frac{(\alpha+\beta+1)_q q!}{(\gamma+1)_q(\delta+1)_q} P_q^{(\alpha,\beta)}(x) P_q^{(\gamma,\delta)}(y) (v-2t/(1+z))^q . \quad (3.11)$$

Now, in (3.10) and (3.11) using the generating function [8]

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n n!}{(\gamma+1)_n(\delta+1)_n} P_n^{(\alpha,\beta)}(x) P_n^{(\gamma,\delta)}(y) t^n$$

$$= \left(\frac{x+1}{2}\right)^{-\alpha-\beta-1} F_C^{(3)}\left[\alpha+\beta+1, \alpha+1; \alpha+1, \gamma+1, \delta+1; \frac{x-1}{x+1}, \frac{(y-1)t}{x+1} \frac{(y+1)t}{x+1}\right], \quad (3.12)$$

we obtain respectively the following trilateral and trilinear generating functions:

$$\sum_{n,p,q=0}^{\infty} \frac{[(n+p+q)!]^2 (1+\alpha+\beta+p+n)_q}{(\gamma+1)_{n+p+q}(\delta+1)_{n+p+q}} P_{n+p+q}^{(\alpha,\beta)}(x) P_{n+p+q}^{(\gamma,\delta)}(y) L_n^{(\alpha+\beta+p)}(z) \left(\frac{t}{z}\right)^n \frac{t^p}{p!} \frac{v^q}{q!}$$

$$=\left(\frac{x+1}{2}\right)^{-\alpha-\beta-1} F_C^{(3)}\left[\alpha+\beta+1,\alpha+1;\alpha+1,\gamma+1,\delta+1;\ \frac{x-1}{x+1},\frac{(y-1)u}{x+1}\frac{(y+1)u}{x+1}\right] \quad (3.13)$$

and

$$\sum_{n,p,q=0}^{\infty}\frac{[(n+p+q)!]^2}{(\gamma+1)_{n+p+q}(\delta+1)_{n+p+q}}\frac{(2\alpha+\beta+1+p+q+n)_{n+p+q}(-\alpha-\beta-p-q-n)_q}{(2\alpha+\beta+1+2p+q+n)_n(-2\alpha-\beta-2p-2q-2n)_q}$$

$$\cdot P_{n+p+q}^{(\alpha,\beta)}(x)P_{n+p+q}^{(\gamma,\delta)}(y)\,P_n^{(\alpha+p+q,\alpha+\beta+p)}(z)\left(\frac{2t}{1+z}\right)^n\frac{(-t)^p}{p!}\frac{v^q}{q!}$$

$$=\left(\frac{x+1}{2}\right)^{-\alpha-\beta-1} F_C^{(3)}\left[\alpha+\beta+1,\alpha+1;\alpha+1,\gamma+1,\delta+1;\ \frac{x-1}{x+1},\frac{(y-1)w}{x+1}\frac{(y+1)w}{x+1}\right], \quad (3.14)$$

where $u=v+t/z$ and $w=v-2t/(1+z)$.

Further, if we take $v=0$ in (3.13) and (3.14) , then we obtain

$$\sum_{n,p=0}^{\infty}\frac{[(n+p)!]^2}{(\gamma+1)_{n+p}(\delta+1)_{n+p}}P_{n+p}^{(\alpha,\beta)}(x)P_{n+p}^{(\gamma,\delta)}(y)\,L_n^{(\alpha+\beta+p)}(z)\left(\frac{t}{z}\right)^n\frac{t^p}{p!}$$

$$=\left(\frac{x+1}{2}\right)^{-\alpha-\beta-1} F_C^{(3)}\left[\alpha+\beta+1,\alpha+1;\alpha+1,\gamma+1,\delta+1;\ \frac{x-1}{x+1},\frac{(y-1)t}{(x+1)z}\frac{(y+1)t}{(x+1)z}\right], \quad (3.15)$$

and

$$\sum_{n,p=0}^{\infty}\frac{[(n+p)!]^2(2\alpha+\beta+1+p+n)_p}{(\gamma+1)_{n+p}(\delta+1)_{n+p}}P_{n+p}^{(\alpha,\beta)}(x)P_{n+p}^{(\gamma,\delta)}(y)\,P_n^{(\alpha+p,\alpha+\beta+p)}(z)\left(\frac{2t}{1+z}\right)^n\frac{(-t)^p}{p!}$$

$$=\left(\frac{x+1}{2}\right)^{-\alpha-\beta-1} F_C^{(3)}\left[\alpha+\beta+1,\alpha+1;\alpha+1,\gamma+1,\delta+1;\ \frac{x-1}{x+1},\frac{-2(y-1)t}{(x+1)(z+1)}\frac{-2(y+1)t}{(x+1)(z+1)}\right] \quad (3.16)$$

respectively.

Setting $\gamma=\alpha$, $\delta=\beta$ in (3.15) and (3.16) and using the hypergeometric transformation [5] ,(see also [3])

$$F_C^{(3)}[\alpha+\beta+1,\beta+1;\alpha+1,\beta+1,\beta+1;\ x,y,z]=(1+x-y-z)^{-\alpha-\beta-1}$$

$$\cdot F_4\left[\tfrac{1}{2}(\alpha+\beta+1),\tfrac{1}{2}(\alpha+\beta+2);\alpha+1,\beta+1;\ \frac{4x}{(1+x-y-z)^2},\frac{4yz}{(1+x-y-z)^2}\right], \quad (3.17)$$

we get respectively the following trilateral and trilinear generating functions :

$$\sum_{n,p=0}^{\infty} \frac{[(n+p)!]^2}{(\alpha+1)_{n+p}(\beta+1)_{n+p}} P_{n+p}^{(\alpha,\beta)}(x) P_{n+p}^{(\alpha,\beta)}(y) L_n^{(\alpha+\beta+p)}(z) \left(\frac{t}{z}\right)^n \frac{t^p}{p!} = (1+t/z)^{-\alpha-\beta-1}$$

$$\cdot F_4\left[\tfrac{1}{2}(\alpha+\beta+1), \tfrac{1}{2}(\alpha+\beta+2); \alpha+1, \beta+1; \frac{(1-x)(1-y)t}{z(1+z/t)^2}, \frac{(1+x)(1+y)t}{z(1+z/t)^2}\right] \quad (3.18)$$

and

$$\sum_{n,p=0}^{\infty} \frac{[(n+p)!]^2 (2\alpha+\beta+1+p+n)_p}{(\alpha+1)_{n+p}(\beta+1)_{n+p}} P_{n+p}^{(\alpha,\beta)}(x) P_{n+p}^{(\alpha,\beta)}(y) P_n^{(\alpha+p,\alpha+\beta+p)}(z) \left(\frac{2t}{1+z}\right)^n \frac{(-t)^p}{p!}$$

$$= \left(\frac{1+z-2t}{1+z}\right)^{-\alpha-\beta-1} F_4\left[\tfrac{1}{2}(\alpha+\beta+1), \tfrac{1}{2}(\alpha+\beta+2); \alpha+1, \beta+1; X, Y\right], \quad (3.19)$$

where F_4 is Appell double hypergeometric function defined by [8] and

$$X = \frac{-2t(1-x)(1-y)(1+z)}{(1+z-2t)^2}, \quad Y = \frac{-2t(1+x)(1+y)(1+z)}{(1+z-2t)^2} .$$

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