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SOME EXPLICIT CONSTRUCTIONS OF TERNARY NON-FULL-RANK TILINGS OF ABELIAN GROUPS

KHALID AMIN

University of Bahrain, College of Science, Department of Mathematics, PO Box 32038, Bahrain

Abstract. A tiling of a finite abelian group G is a pair (A,B) of subsets of G, such that both A and B contain the identity element e of G and every $g \in G$ can be uniquely written in the form $g = ab$, where $a \in A$ and $b \in B$. A tiling (A,B) of G is called *full-rank* if $\langle A \rangle = \langle B >= G$, Otherwise, it is called a *non-full rank* tiling. In this paper, we show some explicit constructions of *non-full rank* tilings of $3-$groups of order 3^4.

Keywords:factorization of Abelian groups; q-ary tilings.

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1. Introduction

A tiling of a finite abelian group G is a pair (A,B) of subsets of G containing the identity e of G and every $g \in G$ can be uniquely written in the form $g = ab$, where $a \in A$ and $b \in B$. Tilings are a special case of normalized factorizations of a finite abelian group G, where by a normalized factorization of G is meant a collection of subsets A_1, A_2,...,A_n of G, such that $e \in A_i$ for each $i = 1,2,...,n$ and every $g \in G$ can be uniquely written in the form $g = a_1a_2...a_n$,

E-mail address: kamin35@hotmail.com

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$a_i \in A_i$. The notion of factorization of an abelian group into subsets was introduced by G. Hajos [1], when he found the answer to a conjecture by H. Minkowski [4], about lattice tiling of $\mathbb{R}^n$ by unit cubes or clusters of unit cubes. Hajos first translated Minkowski's conjecture into a question about finite abelian groups and then he solved the question.

The group-theoretic version of Minkowski's conjecture reads as follows:

If G is a finite abelian group and $G = A_1 \ldots A_{i\ldots}\ A_k$ is a normalized factorization of G, where each of the subsets A_i is of the form $\{e, a, a^2, \ldots, a^k\}$, where $k <| a |$; (here $| a |$denotes order of a).then at least one of the subsets A_i is a subgroup of G.

2. Preliminaries

Hajos made use of the integral group ring $\mathbb{Z}(G)$. Corresponding to each subset A of G, we hav element $\overline{A}$ of G, where $\overline{A} = \sum_{a \in A} a$. If $B = \sum n_i g_i$, $n_i \in \mathbb{Z}$, $g_i \in G$ is an element of $\mathbb{Z}(G)$, then by $< b >$ is meant the subgroup of G generated by the support of b; viz. those elements g_i such that $n_i \neq 0$. We will also, use $\langle A \rangle$ to mean the subgroup generated by a subset A of G and $\langle b_1, b_2, \ldots, b_m \rangle$ will denote the subgroup generated by the support of $b_i \in \mathbb{Z}(G)$, $1 \leq i \leq m$.

Redei [4] made use of group characters; viz homomorphisms χ from G to the multiplicative group of complex numbers $\mathbb{C}$. These extend to ring homomorphisms χ from $\mathbb{Z}(G)$ to the multiplicative group of complex numbers $\mathbb{C}$, where $\chi\left(\sum n_i g_i\right) = \sum n_i \varkappa(g_i)$. He also defined the *annihilator* of the subset A of G, $Ann(A) = \{\varkappa : \varkappa(\overline{A}) = 0\}$ and observed that $A = B$ if and only if $\varkappa\left(\overline{A}\right) = \varkappa\left(\overline{B}\right)$.

In particular $G = A_1 \ldots A_{i\ldots}\ A_k$ is a*factorization of* G if and only if $| G |=| A_1 | \ldots | A_{i|\ldots} |A_k |$ and for each non-identity character $\varkappa$ there exists A_i such that $\varkappa(\overline{A_I}) = 0$...................................$(*)$

We will use$(*)$ to show that our constructions constitute factorizations of a given group G.

3. Main results

M. Dinitz [1], showed that if $p \geqslant 5$, then groups of order p^n admit full-rank tiling and left the case $p = 3$, as an open question. We answer this question by showing some explicit constructions of *non-full rank* tiliings of groups of order 3^4.

We recall that a finite abelian group G is said to be of type $(p_1^{\alpha_1}, p_2^{\alpha_2}, ..., p_r^{\alpha_r})$ if it is a product of cyclic groups of orders $p_1^{\alpha_1}, p_2^{\alpha_2}, ..., p_r^{\alpha_r}$. If $p_i = p$, for each i, G is called a $p-$group. We construct non-full-rank tilings of $3-$groups of order 3^4 of the following types: $\left(3^3,3\right), \left(3^2,3^2\right), \left(3^2,3,3\right)$ and $(3,3,3,3)$.

Construction 1

A non-full rank tiling of $3-$groups of type $\left(3^3,3\right)$. Let $G = \langle x\rangle \times \langle y\rangle$, where $|\, x \,| = 27$ and $|\, y \,| = 3$. Let $A = \left\langle x^9 y\right\rangle \cup x\left\langle x^9 y\right\rangle \cup x^2\left\langle x^9 y\right\rangle$ and $B = \left\langle x^9\right\rangle \cup x^3\langle y\rangle \cup x^6\langle y\rangle$.

We will use $(*)$ to show that AB is a tiling of G. First note that the possible orders of $\varkappa(x)$ are $1,3,9$ and 27 and the possible orders of $\varkappa(y)$ are 1 and 3.

So, altogether we have 8 different cases to consider. The result is summarized below:

Case	Order of $\varkappa(x)$	Order of $\varkappa(y)$	$\varkappa(A)$	$\varkappa(B)$
1	1	1	.	.
2	1	3	0	
3	3	1	0	
4	3	3	0	
5	9	1		0
6	9	3	0	
7	27	1	0	
8	27	3		0

We observe that no element of B has order greater than 9.

Therefore, $\langle B\rangle \subseteq \langle x^3, y\rangle$. Thus, $\langle B\rangle \neq G$.

Construction 2

A non-full rank tiling of $3-$groups of type $\left(3^2,3^2\right)$.

Let $G = \langle x\rangle \times \langle y\rangle$, where $|\, x \,| = |\, y \,| = 9$.

Let $A = (\langle x\rangle - \left\{x^5, x^8\right\}) \cup \left\{x^5 y^3, x^8 y^6\right\}$ and

$B == (\langle y\rangle - \left\{y^5, y^8\right\}) \cup \left\{x^6 y^5, x^3 y^8\right\}$.

Note that the possible orders of $\varkappa(x)$ are $1,3$ and 9 and similarly for orders of $\varkappa(y)$.

So, altogether we have 9 different cases to consider. The result is summarized below:

Case	Order of $\varkappa(x)$	Order of $\varkappa(y)$	$\varkappa(A)$	$\varkappa(B)$
1	1	1	.	.
2	1	3		0
3	1	9		0
4	3	1	0	
5	3	3	0	0
6	3	9		0
7	9	1	0	
8	9	3	0	
9	9	9	0^*	0^*

(*) All the other cases, except this one, which we will detail.

Let $\varkappa(x) = \xi$ and $\varkappa(y) = \eta$, where ξ and η are primitive $9-$th roots of unity. Then

$\varkappa(A) = 1 + \xi + \xi^2 + \xi^3 + \xi^4 + \xi^6 + \xi^7 + \xi^5\eta^3 + \xi^8\eta^6$

$= (1 + \xi^3 + \xi^6) + \xi(1 + \xi^3 + \xi^6) + \xi^2(1 + +\xi^3\eta^3 + \xi^6\eta^6)$

$= \xi^2(1 + +\xi^3\eta^3 + \xi^6\eta^6)$.

$\varkappa(B) = 1 + \eta + \eta^2 + \eta^3 + \eta^4 + \eta^6 + \eta^7 + \xi^6\eta^5 + \xi^3\eta^8$

$= (1 + \eta^3 + \eta^6) + \eta(1 + \eta^3 + \eta^6) + \eta^2(1 + \xi^3\eta^6 + \xi^6\eta^3)$

$= \eta^2(1 + \xi^3\eta^5 + \xi^6\eta^3)$.

Now, ξ and η are both primitive $9-$th root of unity. Hence:

$\eta = \xi, \xi^2, \xi^4, \xi^5, \xi^7$ or ξ^8. Easy calculations will show that when $\eta = \xi, \xi^4$ or ξ^7, we obtain $\varkappa(A) = 0$. In the remaining cases, we get that $\varkappa(B) = 0$. In this case, by construction, we get that, $\langle B \rangle \neq G$.

Construction 3

A non-full rank tiling of $3-$groups of type $(3^2, 3, 3)$.

Let $G = \langle x \rangle \times \langle y \rangle \times \langle z \rangle$, where $|x| = 9, |y| = 3$ and $|z| = 3$.

Let $A = \langle x^3 \rangle \cup x\langle y \rangle \cup x^2\langle y \rangle$ and $B = \langle x^3 y \rangle \cup z\langle x^3 y^2 \rangle \cup z^2\langle x^3 y \rangle$.

Note that the possible orders of $\varkappa(x)$ are $1,3$ and 9, while the possible orders of $\varkappa(y)$ and $\varkappa(z)$ are 1 and 3 only. So, altogether we have 12 different cases to consider. The result is summarized below:

Case	Order of $\chi(x)$	Order of $\chi(y)$	Order of $\chi(z)$	$\chi(A)$	$\chi(B)$
1	1	1	1	.	
2	1	1	3		0
3	1	3	1		0
4	1	3	3		0
5	3	1	1	0	
6	3	1	3		0
7	3	3	1	0	
8	3	3	3	0	
9	9	1	1		0
10	9	1	3		0
11	9	3	1	0^{**}	
12	9	3	3	0**	

(**) All the other cases, except these, in which case, we get the result by using a similar argument as in the previous case.

In this case, by construction, we get that, $\langle A\rangle \neq G$.

Construction 4

A non-full rank tiling of $3-$groups of type $(3,3,3,3)$.

Let $G=(\langle x\rangle \times \langle y\rangle \times \langle u\rangle \times \langle v\rangle$, where $|x|=|y|=|u|=|v|=3$.

Let $A=(\langle x,y\rangle - \left\{x^2y,x^2y^2\right\})\cup \left\{x^2yv,x^2y^2v^2\right\}$ and

$B==(\langle u,v\rangle - \left\{u^2v,u^2v^2\right\})\cup \left\{yu^2v^2,y^2u^2v\right\}$.

Note that the possible orders of $\varkappa(x)$ are $1,3$ only. Similarly with $\varkappa(y)$, $\varkappa(u)$ and $\varkappa(v)$. So, altogether we have 16 different cases to consider.

Let $\chi(x)=\alpha$

$\chi(y)=\beta$

$\chi(u)=\gamma$

$\chi(v) = \delta$

where α, β, γ,and δ are primitive 3rd roots of unity. Then

$\chi(A) = \alpha^2\beta\delta + \alpha^2\beta^2\delta^2 - \alpha^2\beta - \alpha^2\beta^2$

$= \alpha^2\beta(\delta + \beta\delta^2 - 1 - \beta)$.

$\chi(B) = \beta\gamma^2\delta^2 + \beta^2\gamma^2\delta - \gamma^2\delta - \gamma^2\delta^2$

$= \gamma^2\delta\left(\beta\delta + \beta^2 - 1 - \delta\right)$.

Now, if $\beta = 1$, then $\chi(B) = 0$. This takes care of 4 cases.

If $\delta = 1$, then $\chi(A) = A0$. This takes care of 8 cases.

Otherwise, $\beta \neq 1$ and $\delta \neq 1$. Then either $\beta = \delta$ or $\beta = \delta^2$.

If $\beta = \delta$, then $\chi(A) = 0$. This takes care of 4 more cases.

If $\beta = \delta^2$, then $\chi(B) = 0$. This takes care of the remaining 4 cases.

The result is summarized below.

Case	Order of $\chi(x)$	Order of $\chi(y)$	Order of $\chi(u)$	Order of $\chi(v)$	$\chi(A)$	$\chi(B)$
1	1	1	1	1	.	.
2	1	1	1	3		0
3	1	1	3	1	0	0
4	1	1	3	3		0
5	1	3	1	1	0	
6	1	3	1	3		0
7	1	3	3	1	0	
8	1	3	3	3		0
9	3	1	1	1		0
10	3	1	1	3	0	0
11	3	1	3	1		0
12	3	1	3	3		0
13	3	3	1	1	0	
14	3	3	1	3	0	
15	3	3	3	1	0	
16	3	3	3	3	0	

In this case, by construction, we get that in fact, neither $\langle A\rangle \neq G$

nor $\langle B\rangle \neq G$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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