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INTEGRAL REPRESENTATIONS OF SEVERAL (p,q)-BERNSTEIN POLYNOMIALS AND THEIR APPLICATIONS

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Abstract. The aim of this work is to present some new results of multiplications of (p,q)-Bernstein polynomials and to obtain some new relations of those polynomials with related to (p,q)-Gamma and (p,q)-Beta functions.

Keywords: (p,q)-calculus; (p,q)-integral; (p,q)-Bernstein polynomials; (p,q)-Gamma function; (p,q)-Beta function.

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1. Introduction

Bernstein polynomials [4] have been one of the useful tools for many mathematicians since last century. Getting these tools as afflatus, we have recently seen a lot of important and interesting research studies. We exclusively advert to some of them as follows:

Relations between some special polynomials (Bernoulli and Frobenius-Euler polynomials etc.) and Bernstein polynomials were investigated, very interesting properties were obtained

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from these researchers [2,5]. Generating function of Bernstein polynomials was obtained. Also recurrence relations and derivative formula for Bernstein polynomials were proved with the aid of generating function [6].

Consider the q- and (p,q)- calculus to obtain many new results. Some new properties were obtained for q- analogue of some special polynomials[20]. Generalized Bernstein polynomials, so called q-Bernstein polynomials, were defined in [16]. By the motivation from [16], approximation properties for generalize Bernstein polynomials were studied and some estimates on the rate of convergence were given for q-Bernstein polynomials [15]. Results for q-Bernstein polynomials were obtained with regard to some special polynomials [3]. A new generating function for the generalize Bernstein type polynomials was constructed and some basic properties were established by using this new generating function [19]. Acar [1] constructed new modifications of Szász–Mirakyan operators based on (p,q)-integers and derived its new properties. In addition, a new analogue for Bernstein polynomials which is called (p,q)-Bernstein polynomials was defined and approximation properties were studied for these polynomials [14].

This paper is organised as follows:

We will give, respectively, basic definitions and notations which are important and useful for integral representations of Bernstein polynomials in (p,q)-calculus, a definition of (p,q)-Bernstein polynomials introduced by Mursaleen et al.[14], and then extend some well known properties from Bernstein polynomials to (p,q)-Bernstein polynomials and integral representations of (p,q)-Bernstein polynomials and also we will show relations between special functions and these polynomials by using integral representations of (p,q)-Bernstein polynomials. Finally, we will present the conclusions in this paper.

2. Basic Definitions and Notations

In this part, we mention the following definitions and notations that enable us to obtain some results for sequel of this paper.

Definition 2.1. [8] The (p,q)-numbers that are called twin basic number is defined as

$$[n]_{p,q}=\frac{p^n-q^n}{p-q}, 0<q<p\leq 1. \tag{2.1}$$

If we take $p=1$, (p,q) analog of n reduces to q analogue of n. Some (p,q)-numbers are determined as follows:

$$[1]_{p,q}=1$$
$$[2]_{p,q}=p+q$$
$$[3]_{p,q}=p^2+pq+q^2$$
$$\ldots$$

Definition 2.2. [9] The (p,q)-Binomial Formula is defined as

$$(a-b)^n_{p,q}=\sum_{k=0}^{n}\binom{n}{k}_{p,q}p^{(n-k)(n-k-1)/2}q^{k(k-1)/2}(-1)^k a^{n-k}b^k, \tag{2.2}$$

where $a,b\in\mathbb{R}$.

Definition 2.3. [9] The (p,q)-Binom coefficients are defined as below:

$$\begin{pmatrix} n \\ k \end{pmatrix}_{p,q}=\frac{[n]_{p,q}!}{[k]_{p,q}!\,[n-k]_{p,q}!}, \tag{2.3}$$

where

$$[n]_{p,q}!=\begin{cases} 1 & ,n=0 \\ [n]_{p,q}\cdot[n-1]_{p,q}\cdot\ldots\cdot[1]_{p,q} & ,n\neq 0 \end{cases}.$$

Definition 2.4. [17] $(p-q)^\infty_{p,q}$ is expressed multiplication of infinite powers as follows:

$$(p-q)^\infty_{p,q}=\prod_{k=0}^{n}\left(p^{n+1}-q^{n+1}\right). \tag{2.4}$$

Definition 2.5. [18] The (p,q)-derivative operator is determined as (2.5)

$$D_{p,q}\left[f(x)\right]=\frac{f(px)-f(qx)}{(p-q)x} \tag{2.5}$$

Definition 2.6. [9] Let f and g be arbitrary function. The (p,q)- analogue of derivative of product for two functions is defined as

$$D_{p,q}\left[f(x)g(x)\right]=f(px)D_{p,q}g(x)+g(qx)D_{p,q}f(x)=g(px)D_{p,q}f(x)+f(qx)D_{p,q}g(x). \tag{2.6}$$

where $f:\mathbb{R}\to\mathbb{R}$ and $x\in\mathbb{R}$.

The definite (p,q)-integral is known as below:

Definition 2.7. [9,18] Suppose that f be an arbitrary function and a be a real number, we construct

$$\int_0^a f(x)d_{p,q}x=(p-q)a\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f(\frac{q^k}{p^{k+1}}a),\ \left|\frac{q}{p}\right|<1. \tag{2.7}$$

When $a=1$, we see that

$$\int_0^1 f(x)d_{p,q}x=(p-q)\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f(\frac{q^k}{p^{k+1}}).$$

For example, taking $f(x)=x^n$ gives

$$\int_0^1 x^n d_{p,q}x=\frac{1}{[n+1]_{p,q}}.$$

where $0\le a\le b\le\infty$.

For more information about the applications of (p,q)-integral, see [18].

Now, we present definition of Gamma and Beta functions in (p,q)-calculus which is called post q-calculus. In post q-calculus, these special functions are very important and useful in mathematics, physics and engineering as ordinary sense. Firstly, we begin with definition of (p,q)-Gamma function.

Definition 2.8. [13,17] The (p,q) analogue of Gamma function is determined as (2.9)

$$\Gamma_{p,q}(x)=\frac{(p-q)_{p,q}^{\infty}}{(p^x-q^x)_{p,q}^{\infty}}(p-q)^{1-x},0<q<p\le 1, \tag{2.9}$$

in which (p,q)-Gamma function satisfies the following conditions:

$$\circ\Gamma_{p,q}(x+1)=[x]_{p,q}\Gamma_{p,q}(x)$$

$$\circ\Gamma_{p,q}(n+1)=[n]_{p,q}!$$

where n is a nonnegative integer.

Definition 2.9. [13] Let m and $n\in\mathbb{N}$. The (p,q)-Beta function is given as

$$B_{p,q}(m,n)=\int_0^1 x^{m-1}(1-qx)_{p,q}^{n-1}d_{p,q}x. \tag{2.10}$$

Remark 2.1. [13] (p,q)-Beta function which is defined by (2.10) is not commutatitve. Commutative (p,q)-Beta function is determined as below :

$$\widetilde{B}_{p,q}(m,n)=\int_0^1 p^{\frac{m(m-1)}{2}}x^{m-1}(1-qx)_{p,q}^{n-1}d_{p,q}x.$$

In post q-calculus, we have important relationships between Gamma and two type Beta functions just as ordinary calculus. These relations are shown, respectively,

$$B_{p,q}(m,n)=p^{\frac{(n-m)\cdot(2m+n-2)}{2}}\frac{\Gamma_{p,q}(m)\cdot\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}$$
$$\widetilde{B}_{p,q}(m,n)=p^{\frac{2mn+m^2+n^2-3m-3n+2}{2}}\frac{\Gamma_{p,q}(m)\cdot\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}.$$

3. Main results

In this part, we start by giving the definition of (p,q)-Bernstein operator which is constructed by Mursaleen et. al. in [14].

Definition 3.1. The (p,q)-Bernstein operator is defined by (3.1)

$$B_{n,p,q}(f;x)=\frac{1}{p^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)_{p,q}p^{\frac{k(k-1)}{2}}x^k\prod_{s=0}^{n-k-1}(p^s-q^sx)\,f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right). \tag{3.1}$$

Definition 3.2. Let k and n be arbitrary positive integers. The (p,q)-Bernstein polynomial of degree n is given by

$$B_{k,n}(x;p,q)=p^{\binom{k}{2}-\binom{n}{2}}\binom{n}{k}_{p,q}x^k(1-x)_{p,q}^{n-k},\ k\leq n, \tag{3.2}$$

where $\frac{n(n-1)}{2}$ is shown by $\binom{n}{2}$.

We now give some new corollaries listed below without giving proof because they can be derived by means of the definition of (p,q)-Bernstein polynomials.

Corollary 3.1. (A recursive definition) For $0\leq x\leq 1$,we have an equality as below:

$$B_{k,n}(x;p,q)=\left(1-x\left(\frac{q}{p}\right)^{n-k-1}\right)B_{k,n-1}(x;p,q)+xq^{n-k}B_{k-1,n-1}(x;p,q). \tag{3.3}$$

Corollary 3.2. For $0 \leq x \leq 1$ and $0 < q < p \leq 1$,we obtain an equality as follow:

$$\frac{p^{n-1}\left(p^{n-k-1}-x\left(q^{n-k-1}+p^{-k}\right)\right)}{\binom{n-1}{k}_{p,q}} B_{k.n-1}(x;p,q) = \frac{B_{k.n}(x;p,q)}{\binom{n}{k}_{p,q}} + \frac{B_{k+1.n}(x;p,q)}{\binom{n}{k+1}_{p,q}}. \tag{3.4}$$

Above equation shows that the (p,q)-Bernstein polynomials of degree $n-1$ is generated by means of (p,q)-Bernstein polynomials of degree n.

Corollary 3.3. (Derivative of (p,q)-Bernstein polynomials) For $0 \leq x \leq 1$ and $0 < q < p \leq 1$,we obtain an equality as follows:

$$\begin{aligned} \frac{d}{d_{p,q}x} B_{k,n}(x;p,q) &= \frac{[n]_{p,q}}{q^k p^{n-k}} \left(q^k B_{k-1,n-1}(qx;p,q) - pB_{k,n-1}(qx;p,q)\right) \\ &= \binom{n}{k}_{p,q} p^{\binom{k}{2}-\binom{n}{2}} \sum_{j=0}^{n-k} \binom{n-k}{j}_{p,q} p^{\binom{n-k-j}{2}} q^{\binom{j}{2}} (-1)^j [k+j]_{p,q} x^{k+j-1}. \end{aligned} \tag{3.5}$$

By aid of the corollary 3.3, we construct some new corollaries for different type (p,q)-Bernstein polynomials under (p,q)-derivative operator as below:

Corollary 3.4. For $0 \leq x \leq 1$, $0 < q < p \leq 1$ and $k_1, k_2, ..., k_s, n \in \mathbb{N}$ we obtain an equality as follows:

$$\begin{aligned} &\frac{d}{d_{p,q}x}\left(B^{p,q}_{k_1,n}(x)\prod_{j=2}^{s} B^{p,q}_{k_j,n}(\left(\frac{q}{p}\right)^{(j-1)n-\sum_{l=1}^{j-1}k_l} x)\right) \\ &= \left(\prod_{y=1}^{s}\binom{n}{k_y}_{p,q}\right) p^{\left(\sum_{y=1}^{s}\binom{k_y}{2}-s\binom{n}{2}\right)+\left(\sum_{m=2}^{s}\left((m-1)n-\sum_{i=1}^{s-1}k_i\right)(n-2k_s)\right)} \\ &\times q^{\left(\sum_{m=2}^{s}\left((m-1)n-\sum_{i=1}^{s-1}k_i\right)k_s\right)} \times \sum_{j=0}^{sn-(k_1+...+k_s)} \binom{sn-(k_1+...+k_s)}{j}_{p,q} \\ &\times p^{\binom{sn-(k_1+...+k_s)-j}{2}} q^{\binom{j}{2}} (-1)^j [k_1+...+k_s+j]_{p,q} x^{k_1+...+k_s+j-1}. \end{aligned}$$

Corollary 3.5. For $0 \leq x \leq 1$, $0 < q < p \leq 1$ and $k_i, n_i \in \mathbb{N}$, $i = 1, ..., s$, we have

$$\begin{aligned} &\frac{d}{d_{p,q}x}\left(B^{p,q}_{k_1,n_1}(x)\prod_{j=2}^{s} B^{p,q}_{k_j,n_j}(\left(\frac{q}{p}\right)^{\sum_{y=1}^{s-1}(n_y-k_y)} x)\right) \\ = &\left(\prod_{y=1}^{s}\binom{n_y}{k_y}_{p,q}\right) p^{\left(\sum_{y=1}^{s}\binom{k_y}{2}-\binom{n_y}{2}\right)+\left(\sum_{j=2}^{s}\left(\sum_{i=1}^{j-1}n_i-(s-1)k\right)(n_s-k)\right)-\left(\left(\sum_{m=2}^{s}n_{m-1}\right)-(s-1)k\right)k} \\ &\times q^{\left(\left(\sum_{m=2}^{s}n_{m-1}\right)-(s-1)k\right)k} \sum_{j=0}^{(n_1+...+n_s)-(k_1+...+k_s)} \binom{(n_1+...+n_s)-(k_1+...+k_s)}{j}_{p,q} \\ &\times p^{\binom{(n_1+...+n_s)-(k_1+...+k_s)-j}{2}} q^{\binom{j}{2}} (-1)^j \times [k_1+...+k_s+j]_{p,q} x^{k_1+...+k_s+j-1}. \end{aligned}$$

We are now in a position to state the integral representations of (p,q)-Bernstein polynomials under definite (p,q)-integral over the interval $[0,1]$. Let us start with the following theorem.

Theorem 3.1. *For $0<q<p\leq 1$ and $n-k>0$,*

$$\begin{aligned}\frac{\int_0^1 B_{k,n}(qx;p,q)d_{p,q}x}{p^{\binom{k}{2}-\binom{n}{2}}q^k} &= \binom{n}{k}_{p,q}\sum_{r=0}^{n-k}\binom{n-k}{r}_{p,q}(-1)^r q^{\binom{n-k-r}{2}+r}p^{\binom{r}{2}}\frac{1}{[k+r+1]_{p,q}} \\ &= p^{\frac{(n-k)(n+k+1)}{2}}\binom{n}{k}_{p,q}[k]_{p,q}[n-k]_{p,q}\frac{\Gamma_{p,q}(k)\Gamma_{p,q}(n-k)}{\Gamma_{p,q}(n+2)}\end{aligned}$$

Proof. Firstly, we consider

$$\int_0^1 B_{k,n}(qx;p,q)d_{p,q}x=\int_0^1 p^{\binom{k}{2}-\binom{n}{2}}\binom{n}{k}_{p,q}(qx)^k(1-qx)_{p,q}^{n-k}d_{p,q}x. \tag{3.6}$$

By using (p,q)-Binomial formula and re-arranging coefficients on right hand side of (3.6), we have

$$\begin{aligned}\frac{\int_0^1 B_{k,n}(qx;p,q)d_{p,q}x}{p^{\binom{k}{2}-\binom{n}{2}}q^k} &= \binom{n}{k}_{p,q}\int_0^1 x^k(1-qx)_{p,q}^{n-k}d_{p,q}x \\ &= \binom{n}{k}_{p,q}\int_0^1\sum_{r=0}^{n-k}\binom{n-k}{r}_{p,q}(-1)^r q^{\binom{n-k-r}{2}+r}p^{\binom{r}{2}}x^{k+r}d_{p,q}x \\ &= \binom{n}{k}_{p,q}\sum_{r=0}^{n-k}\binom{n-k}{r}_{p,q}(-1)^r q^{\binom{n-k-r}{2}+r}p^{\binom{r}{2}}\int_0^1 x^{k+r}d_{p,q}x \\ &= \binom{n}{k}_{p,q}\sum_{r=0}^{n-k}\binom{n-k}{r}_{p,q}(-1)^r q^{\binom{n-k-r}{2}+r}p^{\binom{r}{2}}\frac{1}{[k+r+1]_{p,q}}.\end{aligned}$$

On the other hand, we heuristically know that first equality of above equation has relation both Gamma and Beta functions as follows:

$$\begin{aligned}\frac{\int_0^1 B_{k,n}(qx;p,q)d_{p,q}x}{p^{\binom{k}{2}-\binom{n}{2}}q^k} &= \binom{n}{k}_{p,q}\int_0^1 x^k(1-qx)_{p,q}^{n-k}d_{p,q}x \\ &= \binom{n}{k}_{p,q}B_{p,q}(k+1,n-k+1) \\ &= \binom{n}{k}_{p,q}p^{\frac{(n-k)(n+k+1)}{2}}\frac{\Gamma_{p,q}(k+1)\cdot\Gamma_{p,q}(n-k+1)}{\Gamma_{p,q}(n+2)} \\ &= \binom{n}{k}_{p,q}p^{\frac{(n-k)(n+k+1)}{2}}[k]_{p,q}[n-k]_{p,q}\frac{\Gamma_{p,q}(k)\cdot\Gamma_{p,q}(n-k)}{\Gamma_{p,q}(n+2)}.\end{aligned}$$

The proof is completed.

Some of the integral representations of (p,q)-Bernstein polynomials for special values are as follows:

$$\begin{aligned}
\int_0^1 B_{0,1}(qx,p,q)d_{p,q}x &= \frac{p}{p+q}\\
\int_0^1 B_{1,1}(qx,p,q)d_{p,q}x &= \frac{p^2q}{p+q}\\
\int_0^1 B_{0,2}(qx,p,q)d_{p,q}x &= \frac{(p^{-1}q)^2}{p^2+pq+q^2}\\
\int_0^1 B_{1,2}(qx,p,q)d_{p,q}x &= \frac{pq^2}{p^2+pq+q^2}\\
\int_0^1 B_{2,2}(qx,p,q)d_{p,q}x &= q^2.
\end{aligned}$$

If we take $p=1$ and q approaches to 1^-, we obtain results in sense of ordinary calculus.

We now consider integral representation of multiplication of two (p,q)-Bernstein polynomials as below:

$$\int_0^1 B_{k,n}(qx;p,q).B_{k,m}\left(\left(\frac{q}{p}\right)^{n-k}qx;p,q\right)d_{p,q}x. \tag{3.7}$$

According to (3.7), we obtain

Theorem 3.2. For $0<q<p\leq 1$ and $n+m-2k+1>0$,

$$\begin{aligned}
&\frac{\int_0^1 B_{k,n}(qx;p,q).B_{k,m}\left(\left(\frac{q}{p}\right)^{n-k}qx;p,q\right)d_{p,q}x}{p^{2\binom{k}{2}-\binom{n}{2}-\binom{m}{2}-(n-k)(m-k)}q^{2k}\left(\frac{q}{p}\right)^{nk-k^2}}\\
&= \sum_{r=0}^{2k}\frac{\binom{2k}{r}_{p,q}p^{\binom{r}{2}}q^{\binom{2k-r}{2}+r}(-1)^r}{[n+m+l-2k+1]_{p,q}}\\
&= p^{\frac{2k(2n+2m-2k+1)}{2}}\frac{\Gamma_{p,q}(n+m-2k+1)\Gamma_{p,q}(2k+1)}{\Gamma_{p,q}(n+m+2)}.
\end{aligned}$$

Proof. Thanks to applying the definition of (p,q)-Bernstein polynomials in (3.7), we get

$$\frac{\int_0^1 B_{k,n}(qx;p,q).B_{k,m}\left(\left(\frac{q}{p}\right)^{n-k}qx;p,q\right)d_{p,q}x.}{p^{2\binom{k}{2}-\binom{n}{2}-\binom{m}{2}-(n-k)(m-k)}q^{2k}\left(\frac{q}{p}\right)^{nk-k^2}}$$
$$=\binom{n}{k}_{p,q}\binom{m}{k}_{p,q}\int_0^1 x^{2k}(1-qx)_{p,q}^{n-k}(1-\left(\frac{q}{p}\right)^{n-k}qx)_{p,q}^{m-k}d_{p,q}x.$$
$$=\binom{n}{k}_{p,q}\binom{m}{k}_{p,q}\int_0^1 x^{2k}(1-qx)_{p,q}^{n+m-2k}d_{p,q}x$$
$$=\binom{n}{k}_{p,q}\binom{m}{k}_{p,q}\int_0^1 x^{n+m-2k}(1-qx)_{p,q}^{2k}d_{p,q}x.$$

If we use the definition of (p,q)-Binomial formula on right hand side of above equation, we construct

$$\frac{\int_0^1 B_{k,n}(qx;p,q).B_{k,m}(\left(\frac{q}{p}\right)^{n-k}qx;p,q)d_{p,q}x.}{p^{2\binom{k}{2}-\binom{n}{2}-\binom{m}{2}-(n-k)(m-k)}q^{2k}\left(\frac{q}{p}\right)^{nk-k^2}}$$
$$=\binom{n}{k}_{p,q}\binom{m}{k}_{p,q}\sum_{r=0}^{2k}\binom{2k}{r}_{p,q}p^{\binom{r}{2}}q^{\binom{2k-r}{2}+r}(-1)^r\int_0^1 x^{n+m+r-2k}d_{p,q}x$$
$$=\binom{n}{k}_{p,q}\binom{m}{k}_{p,q}\sum_{r=0}^{2k}\frac{(-1)^r\binom{2k}{r}_{p,q}p^{\binom{r}{2}}q^{\binom{2k-r}{2}+r}}{[n+m+r-2k+1]_{p,q}}$$

If we again deal with previous equality, we generate an identity as below:

$$\frac{\int_0^1 B_{k,n}(qx;p,q).B_{k,m}\left(\left(\frac{q}{p}\right)^{n-k}qx;p,q\right)d_{p,q}x}{p^{2\binom{k}{2}-\binom{n}{2}-\binom{m}{2}-(n-k)(m-k)}q^{2k}\left(\frac{q}{p}\right)^{nk-k^2}}$$
$$=\binom{n}{k}_{p,q}\binom{m}{k}_{p,q}B_{p,q}(n+m-2k+1,2k+1)$$
$$=\binom{n}{k}_{p,q}\binom{m}{k}_{p,q}p^{\frac{(n+m-2k)(n+m+2k+1)}{2}}\frac{\Gamma_{p,q}(n+m-2k+1)\Gamma_{p,q}(2k+1)}{\Gamma_{p,q}(n+m+2)}.$$

Thus, the proof is completed.

Now, we extend the integral representation concept to product of three (p,q)-Bernstein polynomials and show an equality with related to special functions with the following corollary:

Theorem 3.3. For $0<q<p\le 1$ and $n+m+s-3k+1>0$,

$$\frac{\int_0^1 B_{k,n}(qx;p,q).B_{k,m}\left(\left(\frac{q}{p}\right)^{n-k}qx;p,q\right).B_{k,s}\left(\left(\frac{q}{p}\right)^{n+m-2k}qx;p,q\right)d_{p,q}x}{p^{3\binom{k}{2}-\left(\binom{n}{2}+\binom{m}{2}+\binom{s}{2}\right)-[(n-k)(m-k)+(n+m-2k)(s-k)]}q^{3k}\left(\frac{q}{p}\right)^{2nk+mk-3k^2}}$$
$$=p^{\frac{(n+m+s-3k)(n+m+s+3k+1)}{2}}\frac{\Gamma_{p,q}(n+m+s-3k+1)\Gamma_{p,q}(3k+1)}{\Gamma_{p,q}(n+m+s+2)}$$
$$=\sum_{r=0}^{3k}\frac{\binom{3k}{r}_{p,q}p^{\binom{r}{2}}q^{\binom{3k-r}{2}+r}(-1)^r}{[n+m+s+r-3k+1]_{p,q}}.$$

Proof. We see that

$$\int_0^1\left[B_{k,n}(qx;p,q)\times B_{k,m}\left(\left(\frac{q}{p}\right)^{n-k+1}x;p,q\right)\times B_{k,s}\left(\left(\left(\frac{q}{p}\right)^{n+m-2k+1}x;p,q\right)\right)\right]d_{p,q}x$$
$$=\int_0^1\left[p^{\binom{k}{2}-\binom{n}{2}}\binom{n}{k}_{p,q}(qx)^k(1-qx)_{p,q}^{n-k}p^{\binom{k}{2}-\binom{m}{2}}\binom{m}{k}_{p,q}\left(\left(\frac{q}{p}\right)^{n-k+1}x\right)^k(1-\left(\frac{q}{p}\right)^{n-k+1}x)_{p,q}^{m-k}\right]\times$$
$$p^{\binom{k}{2}-\binom{s}{2}}\binom{s}{k}_{p,q}\left(\left(\frac{q}{p}\right)^{n+m-2k+1}x\right)^k(1-\left(\frac{q}{p}\right)^{n+m-2k+1}x)_{p,q}^{s-k}.d_{p,q}x$$

after some basic operations, we obtain:

$$\frac{\int\limits_0^1\left[B_{k,n}(qx;p,q)\times B_{k,m}\left(\left(\frac{q}{p}\right)^{n-k+1}x;p,q\right)\times B_{k,s}\left(\left(\frac{q}{p}\right)^{n+m-2k+1}x;p,q\right)\right]d_{p,q}x}{p^{3\binom{k}{2}-\left(\binom{n}{2}+\binom{m}{2}+\binom{s}{2}\right)-[(n-k)(m-k)+(n+m-2k)(s-k)]}q^{3k}\left(\frac{q}{p}\right)^{2nk+mk-3k^2}}$$
$$=\binom{n}{k}_{p,q}\binom{m}{k}_{p,q}\binom{s}{k}_{p,q}\int_0^1 x^{3k}(1-qx)_{p,q}^{n+m+s-3k}d_{p,q}x$$
$$=\binom{n}{k}_{p,q}\binom{m}{k}_{p,q}\binom{s}{k}_{p,q}\int_0^1 x^{n+m+s-3k}(1-qx)_{p,q}^{3k}d_{p,q}x.$$

On the other hand, if we consider (p,q) analogs of Beta and Gamma functions, we have

$$\binom{n}{k}_{p,q}\binom{m}{k}_{p,q}\binom{s}{k}_{p,q}\int_0^1 x^{n+m+s-3k}(1-qx)_{p,q}^{3k}d_{p,q}x=\binom{n}{k}_{p,q}\binom{m}{k}_{p,q}\binom{s}{k}_{p,q}B_{p,q}(n+m+s-3k+1,3k+1)$$

and

$$B_{p,q}(n+m+s-3k+1,3k+1)=p^{\frac{(n+m+s-3k)(n+m+s+3k+1)}{2}}\frac{\Gamma_{p,q}(n+m+s-3k+1)\Gamma_{p,q}(3k+1)}{\Gamma_{p,q}(n+m+s+2)}.$$

By using the definition of (p,q)-Binomial formula, we obtain

$$\binom{n}{k}_{p,q}\binom{m}{k}_{p,q}\binom{s}{k}_{p,q}\int_0^1 x^{n+m+s-3k}(1-qx)^{3k}_{p,q}d_{p,q}x$$
$$=\binom{n}{k}_{p,q}\binom{m}{k}_{p,q}\binom{s}{k}_{p,q}\sum_{r=0}^{3k}\binom{3k}{r}_{p,q}p^{\binom{r}{2}}q^{\binom{3k-r}{2}+r}$$
$$\times(-1)^r\int_0^1 x^{n+m+s-3k+l}d_{p,q}x$$
$$=\binom{n}{k}_{p,q}\binom{m}{k}_{p,q}\binom{s}{k}_{p,q}\sum_{r=0}^{3k}\frac{\binom{3k}{r}_{p,q}p^{\binom{r}{2}}q^{\binom{3k-r}{2}+r}(-1)^r}{[n+m+s+r-3k+1]_{p,q}}.$$

Thus, the desired result is obtained.

The following corollary is actually a consequence of a more general previous theorems which we stated and proved in this part.

Corollary 3.6. For $0<q<p\leq 1, s\geq 2$

$$\frac{\int_0^1 B_{k,n}(qx;p,q)\prod_{i=1}^{s-1}B_{k,n_{i+1}}\left(\left(\frac{q}{p}\right)^{\sum_{l=1}^{i}n_l-ik}qx;p,q\right)d_{p,q}x}{p^{s\binom{k}{2}-\sum_{i=1}^{s}\binom{n_i}{2}-\left(\sum_{i=1}^{s-1}\left(\sum_{r=1}^{i}(n_r-ik)(n_{i+1}-k)\right)\right)}q^{sk}\left(\frac{q}{p}\right)^{k\sum_{i=1}^{s-1}(in_{s-1})-k^2\binom{s}{2}}}$$
$$=\binom{n_1}{k}_{p,q}\binom{n_2}{k}_{p,q}\cdots\binom{n_s}{k}_{p,q}\sum_{r=0}^{sk}\frac{\binom{sk}{r}_{p,q}p^{\binom{r}{2}}q^{\binom{sk-r}{2}+r}(-1)^r}{[n_1+n_2+\ldots+n_s+r-sk+1]_{p,q}}$$

and

$$\frac{\int_0^1 B_{k,n}(qx;p,q)\prod_{i=1}^{s-1}B_{k,n_{i+1}}\left(\left(\frac{q}{p}\right)^{\sum_{l=1}^{i}n_l-ik}qx;p,q\right)d_{p,q}x}{p^{s\binom{k}{2}-\sum_{i=1}^{s}\binom{n_i}{2}-\left(\sum_{i=1}^{s-1}\left(\sum_{r=1}^{i}(n_r-ik)(n_{i+1}-k)\right)\right)}q^{sk}\left(\frac{q}{p}\right)^{k\sum\limits_{i=1}^{s-1}(in_{s-1})-k^2\binom{s}{2}}}$$
$$=\binom{n_1}{k}_{p,q}\binom{n_2}{k}_{p,q}\cdots\binom{n_s}{k}_{p,q}B_{p,q}(sk+1,n_1+n_2+\ldots+n_s-sk+1)$$
$$=p^{\frac{[(n_1+n_2+\ldots+n_s-sk)(n_1+n_2+\ldots+n_s+sk+1)]}{2}}\frac{\Gamma_{p,q}(n_1+n_2+\ldots+n_s-sk+1)\Gamma_{p,q}(sk+1)}{\Gamma_{p,q}(n_1+n_2+\ldots+n_s+2)}.$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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