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STABILITY OF QUARTIC MAPPINGS IN FUZZY BANACH SPACES

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Abstract. In this paper, we will consider the stability of quartic mapping in fuzzy normed spaces. First we define the concept of convergent sequence and Cauchy sequence. Also, we show that quartic mapping exist another fuzzy approximately quartic mapping. We discuss under what condition does the quartic mapping be a fuzzy continuity mapping.

Keywords: fuzzy normed spaces; stability of quartic mapping; Cauchy sequence.

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1. Introduction

Problems concerning stability of group homomorphisms was first posed by Ulam [13] in 1940, In the next year, Hyers [11] gave an affirmative answer to the question of Ulam in Banach spaces. The notion of fuzzy stability of the functional equations was given in [5,6]. Later, several various fuzzy versions of stability concerning Jensen, cubic and quadratic functional equations were investigated in [2,10,3,4].

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Studies on fuzzy normed spaces are relatively recent in the field of fuzzy functional analysis. In 1984, it was Katsaras [1] who first introduce the idea of fuzzy norm on a linear space. In 1994, Cheng and Mordeson [9] defined another type of fuzzy norm on a linear space whose associated fuzzy metric is of Kramosil and Michalek type [16]. Finally, in [8] Bag and Samanta redefined the concept of fuzzy norm given in [9].

In this paper, we consider the functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(x) \tag{1.1}$$

This functional equation is called quartic, since the quartic mapping $f(x)=cx^4(c\in R)$ is a solution of the functional equation. Every solution of the quartic functional equation is said to be a quartic mapping. In section 2, we give some basic definitions. In section 3, we use the basic methods to establish Hyers-Ulam-Rassias stability of quartic type functional equations (1.1) in fuzzy Banach spaces. In fact, we will show that, under some conditions, every almost quartic mapping from a linear space to a fuzzy normed space can be approximated by a unique quartic function. Finally, we will show that there exists a close relation between the continuity behavior of an almost quartic mapping and the exact quartic function which approximates it.

2. Preliminaries

In this section, we introduce some notations to be used in the sequel.

Definition 2.1. [7] *Let X be a real linear space. A fuzzy subset N of $X\times\mathbb{R}$ is called a* fuzzy norm *on X if and only if*

(N1) *For all $t\in\mathbb{R}$ with $t\le 0$, $N(x,t)=0$;*

(N2) *For all $t\in\mathbb{R}$ with $t>0$, $N(x,t)=1$ if and only if $x=0$;*

(N3) *For all $\lambda\in\mathbb{R}$ with $\lambda\neq 0$, $N(\lambda x,t)=N(x,t/|\lambda|)$;*

(N4) *For all $s,t\in\mathbb{R}$, $N(x+y,s+t)\ge\min\{(N(x,s),N(y,t)\}$;*

(N5) *$N(x,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$;*

(N6) *For $x\neq 0, N(x,\cdot)$ is (upper semi) continuous on $\mathbb{R}$.*

Then (X,N) is called a fuzzy normed linear space.

Example 2.2.[2] *Let $(X,\|\cdot\|)$ be a normed space. For every $x \in X$, we define*

$$N(x,t)=\begin{cases}\dfrac{t}{t+\|x\|}, & when\ t>0,\\[2ex] 0, & when\ t\leq 0.\end{cases}$$

Then (X,N) is a fuzzy normed linear space.

Definition 2.3.[12] *A sequence $\{x_n\}$ in a fuzzy normed linear space (X,N) is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x,t) = 1$ for all $t > 0$. In that case, x is called the fuzzy limit of the sequence $\{x_n\}$.*

Definition 2.4.[12] *A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n,t) > 1-\varepsilon$.*

Definition 2.5.[2] *Let $f : \mathbb{R} \to X$ be a function, X is a fuzzy normed space. If for all $\varepsilon > 0$ and all $0 < \alpha < 1$, there exist a positive real number δ, and for each $0 < |s-s_0| < \delta$, such that $N(f(sx)-f(s_0x),\varepsilon) \geq \alpha$. Then f is called fuzzy continuous at a point $s_0 \in \mathbb{R}$.*

3. fuzzy stability of quartic mapping

Lemma 3.1. *A mapping $f : X \to Y$ satisfies (1.1) if and only if the mapping $f : X \to Y$ is quartic.*

Theorem 3.2. *Let X be a linear space, (Y,N) and $(Z,N^{'})$ be a fuzzy Banach space and a fuzzy normed linear space respectively. Suppose that α is a constant satisfies $0 < \alpha < 16$, φ is a mapping from $X \times X \to Z$ such that*

$$N^{'}(\varphi(2x,0),t) \geq N^{'}(\alpha\varphi(x,0),t) \tag{3.1}$$

for all $x \in X, t > 0$, and

$$\lim_{k\to\infty} N^{'}(\varphi(2^k x, 2^k y), 16^k t) = 1$$

for all $x,y \in X, t > 0, k \geq 0$. If $f : X \to Y$ be a φ-approximately quartic mapping in the sense that

$$N(Df(x,y),t) \geq N^{'}(\varphi(x,y),t) \tag{3.2}$$

for all $x,y \in X, t > 0$. Then there exists a unique quartic mapping $C : X \to Y$ such that

$$N(C(x)-f(x),t) \geq N^{'}(\varphi(x,0),(16-\alpha)t)$$

or all $x \in X, t > 0$. Moreover,

$$C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{16^n}$$

for all $x \in X$

Proof. Put $y = 0$ in (3.2), we get

$$N(2f(2x) - 32f(x), t) = N(\frac{f(2x)}{16} - f(x), \frac{t}{32}) \geq N^{'}(\varphi(x,0), t) \tag{3.3}$$

for all $x \in X, t > 0$, replacing x by $2^n x$ in (3.3) and using $(N3)$,(3.1), we get

$$N(\frac{f(2^{n+1}x)}{16^{n+1}} - \frac{f(2^n x)}{16^n}, \frac{t}{32(16^n)}) \geq N^{'}(\varphi(2^n x, 0), t) \geq N^{'}(\varphi(x,0), t/\alpha^n)$$

for all $x \in X, t > 0, n \geq 0$. By replacing t by t/α^n, we can get that

$$N(\frac{f(2^{n+1}x)}{16^{n+1}} - \frac{f(2^n x)}{16^n}, \frac{\alpha^n t}{32(16^n)}) \geq N^{'}(\varphi(x,0), t) \tag{3.4}$$

for all $x \in X, t > 0, n \geq 0$. According to (3.4), we can get that

$$N(\frac{f(2^n x)}{16^n} - f(x), \sum_{k=0}^{n-1} \frac{\alpha^k t}{32(16^k)}) \geq \min \bigcup_{k=0}^{n-1} \{N(\frac{f(2^{k+1}x)}{16^{k+1}} - \frac{f(2^k x)}{16^k}, \frac{\alpha^k t}{32(16^k)}\} \geq N^{'}(\varphi(x,0), t) \tag{3.5}$$

for all $x \in X, t > 0, n > 0$. By replacing x with $2^m x$ in (3.5) and using (3.1), we get that

$$N(\frac{f(2^{n+m}x)}{16^{n+m}} - \frac{f(2^m x)}{16^m}, \sum_{k=m}^{n+m-1} \frac{\alpha^k t}{32(16^k)}) \geq N^{'}(\varphi(x,0), t) \tag{3.6}$$

for all $x \in X, t > 0, m > 0, n \geq 0$. By replacing $\sum_{k=m}^{n+m-1} \frac{\alpha^k t}{32(16^k)}$ with t in (3.6). We see that

$$N(\frac{f(2^{n+m}x)}{16^{n+m}} - \frac{f(2^m x)}{16^m}, t) \geq N^{'}(\varphi(x,0), \frac{t}{\sum_{k=m}^{n+m-1} \frac{\alpha^k}{32(16^k)}})$$

for all $x \in X, t > 0, m > 0, n \geq 0$. Since $0 < \alpha < 16$, We can obtain that $\sum_{k=m}^{n+m-1} \frac{\alpha^k}{32(16^k)} \to 0$, as $m, n \to \infty$.This show that $\{\frac{f(2^n x)}{16^n}\}$ is a Cauchy sequence in (Y,N). Since (Y,N) is a Banach space, $\{\frac{f(2^n x)}{16^n}\}$ is converges to $C(x)$ in (Y,N). Hence $C : X \to Y$ is a mapping which satisfies

$$C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{16^n}$$

According to (3.5) we get that

$$N(\frac{f(2^n x)}{16^n} - f(x), t) \geq N^{'}(\varphi(x,0), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{32(16^k)}}) \tag{3.7}$$

for all $x \in X, t > 0, n > 0$. Using (3.7) and (N4), we see that

$$\begin{aligned} N(f(x)-C(x),t) &= N(f(x)-\frac{f(2^n x)}{16^n}+\frac{f(2^n x)}{16^n}-C(x),t) \\ &\geq \min\{N(f(x)-\frac{f(2^n x)}{16^n},\frac{t}{2}),N(\frac{f(2^n x)}{16^n}-C(x),\frac{t}{2})\} \\ &\geq \min\{N'(\varphi(x,0),\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{16(16^k)}}),N(\frac{f(2^n x)}{16^n}-C(x),\frac{t}{2})\} \end{aligned}$$

for all $x \in X, t > 0$. Let $n \to \infty$, we can deduce that

$$N(f(x)-C(x),t) \geq N'(\varphi(x,0),\frac{t}{\sum_{k=0}^{\infty}\frac{\alpha^k}{16(16^k)}}) = N'(\varphi(x,0),(16-\alpha)t)$$

for all $x \in X, t > 0$.

According to $(N4)$, and $C(x) = \lim_{n\to\infty}\frac{f(2^n x)}{16^n}$, as $k \to \infty$, we have

$$N(DC(x,y)-\frac{1}{16^k}Df(2^k x,2^k y),t) \to 1$$

for all $x, y \in X, t > 0$. Hence we can deduce that

$$\begin{aligned} N(DC(x,y),t) &= N(DC(x,y)-\frac{1}{16^k}Df(2^k x,2^k y)+\frac{1}{16^k}Df(2^k x,2^k y),t) \\ &\geq \min\{N(DC(x,y)-\frac{1}{16^k}Df(2^k x,2^k y),\frac{t}{2}),N(Df(2^k x,2^k y),\frac{16^k t}{2}) \end{aligned}$$

for all $x, y \in X, t > 0$.Let $k \to \infty$, we get that $N(DC(x,y),t) \to 1$, for all $x, y \in X, t > 0$, from $(N2)$ we can see that $DC(x,y) = 0$, for all $x, y \in X$. We can conclude that C is quartic mapping. Now we prove the uniqueness of C. Set $T : X \to Y$ is a quartic mapping which satisfies $N(T(x)-f(x),t) \geq N'(\varphi(x,0),(16-\alpha)t)$, for all $x \in X, t > 0$, with $(N4)$ we have that

$$\begin{aligned} N(C(x)-T(x),t) &= N(C(x)-f(x)+f(x)-T(x),t) \\ &\geq \min\{N(C(x)-f(x),\frac{t}{2}),N(f(x)-T(x),\frac{t}{2}) \\ &\geq N'(\varphi(x,0),(16-\alpha)t) \end{aligned}$$

Hence C, T are quartic mapping, we see that $C(2x) = 16C(x), T(2x) = 16T(x)$, then we get

$$\begin{aligned} N(C(x)-T(x),t) &= N(C(2^k x)-T(2^k x),16^k t) \geq N'(\varphi(2^k x,0),16^k(16-\alpha)t) \\ &\geq N'(\alpha^k\varphi(x,0),16^k(16-\alpha)t) = N'(\varphi(x,0),\frac{16^k(16-\alpha)t}{\alpha^k}) \end{aligned}$$

for all $x \in X, t > 0$, and $k \in N$. Since $0 < \alpha < 16$, $N(C(x) - T(x), t) \to 1$, as $k \to \infty$. We can conclude that $C(x) = T(x)$.

Corollary 3.3. *Let $(X, \|\cdot\|)$ be a normed space, (Y,N) be a fuzzy Banach space and $(Z,N^{'})$ be a fuzzy normed space, p,q be non-negative real numbers satisfies $p,q < 4$. For some $u_1,u_2 \in Z$, If $f : X \to Y$ be a mapping satisfies that*

$$N(Df(x,y),t) \geq N^{'}(\|x\|^p u_1 + \|y\|^q u_2, t)$$

for all $x,y \in X, t > 0$. Then there exists a unique quartic mapping $C : X \to Y$ such that

$$N(f(x) - C(x), t) \geq N^{'}(\|x\|^p u_1, t(16 - 2^p))$$

.

Proof. We define $\varphi : X \times X \to Z$ by

$$\varphi(x,y) = \|x\|^p u_1 + \|y\|^q u_2$$

for all $x,y \in X$. With Theorem 3.2, we can conclude that

$$N(\varphi(2x,0)) \geq N^{'}(2^p \varphi(x,0), t)$$

and according to $(N4)$ we get

$$N^{'}(\|2^k x\|^p u_1 + \|2^k y\|^q u_2, 16^k t) \geq \min\{N^{'}(\|2^k x\|^p u_1, \frac{16^k t}{2}), N^{'}(\|2^k y\|^q u_2, \frac{16^k t}{2})\}$$

According to $(N5)$ and $0 < \alpha < 16$, we can conclude that

$$\lim_{k\to\infty} N^{'}(\|2^k x\|^p u_1, \frac{16^k t}{2}) = 1, \ \lim_{k\to\infty} N^{'}(\|2^k y\|^q u_2, \frac{16^k t}{2}) = 1$$

Hence

$$\lim_{k\to\infty} N^{'}(\|2^k x\|^p u_1 + \|2^k y\|^q u_2, 16^k t) = 1$$

It follows the conditions of Theorem 3.2, then completes the proof.

The case that $\alpha > 16$ has been proved in the next theorem.

Theorem 3.4. *Let X be a linear space, (Y,N) and $(Z,N^{'})$ be a fuzzy Banach space and a fuzzy normed linear space respectively. Suppose that α is a constant satisfies $\alpha > 16$, φ is a mapping* *from $X \times X \to Z$, such that*

$$N^{'}(\varphi(\frac{x}{2},0),t) \geq N^{'}(\varphi(x,0),\alpha t) \tag{3.8}$$

for all $x \in X, t > 0$, and

$$\lim_{k\to\infty} N^{'}(\varphi(2^k x, 2^k y), 16^k t) = 1$$

for all $x, y \in X, t > 0, k \geq 0$. If $f : X \to Y$ be a φ-approximately quartic mapping in the sense that

$$N(Df(x,y),t) \geq N^{'}(\varphi(x,y),t) \tag{3.9}$$

for all $x, y \in X, t > 0$. Then there exists a unique quartic mapping $C : X \to Y$ such that

$$N(C(x) - f(x),t) \geq N^{'}(\varphi(x,0),(\alpha - 16)t) \tag{3.10}$$

or all $x \in X, t > 0$. Moreover,

$$C(x) = \lim_{n\to\infty} 16^n f(\frac{x}{2^n})$$

for all $x \in X$

Proof. Replacing x by $\frac{x}{2^{n+1}}$ in (3.3) and using $(N3)$,(3.8), we get

$$N(16^n f(\frac{x}{2^n}) - 16^{n+1} f(\frac{x}{2^{n+1}}), \frac{16^n t}{32}) \geq N^{'}(\varphi(\frac{x}{2^{n+1}},0),t) \geq N^{'}(\varphi(x,0),\alpha^{n+1}t)$$

for all $x \in X, t > 0, n \geq 0$. By replacing t by $\alpha^{n+1}t$, we can get that

$$N(16^n f(\frac{x}{2^n}) - 16^{n+1} f(\frac{x}{2^{n+1}}), \frac{16^n t}{32\alpha^{n+1}}) \geq N^{'}(\varphi(x,0),t)$$

for all $x \in X, t > 0, n \geq 0$. We can get that

$$N(f(x) - 16^n f(\frac{x}{2^n}), \sum_{k=0}^{n-1}, \frac{16^k t}{32\alpha^{k+1}}) \geq \min \bigcup_{k=0}^{n-1} \{N(16^k f(\frac{x}{2^k}) - 16^{k+1} f(\frac{x}{2^{k+1}}), \frac{16^k t}{32\alpha^{k+1}})$$
$$\geq N^{'}(\varphi(x,0),t)$$

for all $x \in X, t > 0, n > 0$. It follows that

$$N(16^m f(\frac{x}{2^m}) - 16^{n+m} f(\frac{x}{2^{n+m}}), \sum_{k=m}^{n+m-1} \frac{16^k t}{32\alpha^{k+1}}) \geq N^{'}(\varphi(x,0),t) \tag{3.11}$$

for all $x \in X, t > 0, m > 0, n \geq 0$. By replacing $\sum_{k=m}^{n+m-1} \frac{16^k t}{32\alpha^{k+1}}$ with t in (3.11). We see that

$$N(16^m f(\frac{x}{2^m}) - 16^{n+m} f(\frac{x}{2^{n+m}}),t) \geq N^{'}(\varphi(x,0), \frac{t}{\sum_{k=m}^{n+m-1} \frac{16^k t}{32\alpha^{k+1}}})$$

for all $x \in X, t > 0, m > 0, n \geq 0$. Since $\alpha > 16$, We can obtain that $\sum_{k=m}^{n+m-1} \frac{16^k t}{32\alpha^{k+1}} \to 0$, as $m,n \to \infty$.This show that $16^n f(\frac{x}{2^n})$ is a Cauchy sequence in (Y,N). Since (Y,N) is a Banach space, $16^n f(\frac{x}{2^n})$ is converges to $C(x)$ in (Y,N). Hence $C : X \to Y$ is a mapping which satisfies

$$C(x) = \lim_{n\to\infty} 16^n f(\frac{x}{2^n})$$

for all $x \in X$.

Similar with the prove of Theorem 3.2, we can conclude that C is a unique quartic mapping satisfies (3.10).

Corollary 3.5. *Let $(X, \|\cdot\|)$ be a normed space, (Y,N) be a fuzzy Banach space and $(Z,N^{'})$ be a fuzzy normed space, p,q be non-negative real numbers satisfies $p,q > 4$. For some $u_1,u_2 \in Z$, If $f : X \to Y$ be a mapping satisfies that*

$$N(Df(x,y),t) \geq N^{'}(\|x\|^p u_1 + \|y\|^q u_2, t)$$

for all $x,y \in X, t > 0$. Then there exists a unique quartic mapping $C : X \to Y$ such that

$$N(f(x) - C(x),t) \geq N^{'}(\|x\|^p u_1, t(2^p - 16))$$

.

Proof. Similar with the proof of corollary 3.3.

4. fuzzy continuity of quartic mapping

In this section, we discuss fuzzy continuity of quartic mapping in Theorem 3.2 or Theorem 3.4. we will deal with the question that under what conditions does $C(x)$ is continuity in fuzzy normed space.

The following result gives an property of continuous approximately quartic mapping satisfies Theorem 3.2.

Theorem 4.1. *Let for each $x \in X$ and $s \in \mathbb{R}$, the function $s \to f(sx)$ is fuzzy continuous, then $s \to C(sx)$ is fuzzy continuous and $C(sx) = s^4 C(x)$.*

Proof. For fixed $x \in X$, $s \in \mathbb{R}$ and $0 < \beta < 1$. There exist $n_0 \in \mathbb{N}$, for all $n > n_0$ and $t > 0$ such that

$$N^{'}(\varphi(sx), 0), (\frac{16}{\alpha})^n (16-\alpha)\frac{t}{3}) > \beta$$

According to Theorem 3.2, we have

$$\begin{aligned}
N(C(sx) - 2^{-4n} f(2^n sx), \frac{t}{3}) &= N(2^{-4n} C(2^n sx) - 2^{-4n} f(2^n sx), \frac{t}{3}) \\
&= N(C(sx) - f(2^n sx), 2^{4n}\frac{t}{3}) \\
&\geq N^{'}(\varphi(2^n sx), 0), 2^{4n}(16-\alpha)\frac{t}{3}) \\
&\geq N^{'}(\alpha^n \varphi(sx), 0), 2^{4n}(16-\alpha)\frac{t}{3}) \\
&= N^{'}(\varphi(sx), 0), (\frac{16}{\alpha})^n (16-\alpha)\frac{t}{3}) > \beta
\end{aligned}$$

for each $x \in X, t > 0$, $s \in \mathbb{R}$ and $n \in \mathbb{N}$.

By the fuzzy continuous of the mapping $s \to f(sx)$,We can find some $\delta > 0$ with $0 < |s - s_0| < \delta$, such that

$$N(\frac{f(2^n sx)}{16^n}) - \frac{f(2^n s_0 x)}{16^n}, \frac{t}{3}) > \beta$$

It follows that

$$\begin{aligned}
N(C(sx) - C(s_0 x), t) &\geq \min\{N(C(sx) - \frac{f(2^n sx)}{16^n}, \frac{t}{3}), , N(\frac{f(2^n sx)}{16^n} - C(s_0 x), \frac{t}{3})\} \\
&\geq \min\{N(C(sx) - \frac{f(2^n sx)}{16^n}, \frac{t}{3}), N(\frac{f(2^n sx)}{16^n}) - \frac{f(2^n s_0 x)}{16^n}, \frac{t}{3}), \\
&N(\frac{f(2^n s_0 x)}{16^n} - C(s_0 x), \frac{t}{3})\} \\
&> \beta
\end{aligned}$$

This proves the fuzzy continuity of $s \to C(sx)$.

Next we use the fuzzy continuity of $s \to C(sx)$ to prove that $C(sx) = s^4 C(x)$, for all x and all rational numbers $s \in Q$. We can easily prove that $C(nx) = n^4 C(x)$ for all non-negative integer n and $C(-x) = -C(x)$. It follows that

$$C(\frac{m}{k}x) = m^4 C(\frac{1}{k}x) = (\frac{m}{k})^4 C(x) \ (m, k \in \mathbb{N})$$

It means that for all rational number r such that $C(rx) = r^4 C(x)$. Then there exists a sequence r_n of rational number, for every real number s such that $r_n \to s$. By the fuzzy continuity of $C(x)$, we can see that

$$C(sx) = \lim_{n\to\infty} C(r_n x) = \lim_{n\to\infty} r_n^4 C(x) = s^4 C(x)$$

This completes the proof.

Theorem 4.2. *If the condition of Theorem* 3.4 *hold, and the function $s \to f(sx)$ is fuzzy continuous, then $s \to C(sx)$ is fuzzy continuous and $C(sx) = s^4 C(x)$, for all $x \in X$ and $s \in \mathbb{R}$.*

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. K. Katsaras, Fuzzy topological vector spaces. II, Fuzzy Sets Syst. 12 (2) (1984), 143-154.

[2] A.K.Mirmostafaee and M.S.Moslehian, Fuzzy approximately cubic mappings, Inf. Sci. 178 (2008), 3791-3798.

[3] A.K. Mirmostafaee, M. Mirzavaziri and M.S. Moslehian, Fuzzy stability of the Jensen functional equation, Fuzzy Sets Syst. 159 (6) (2008), 730-738 .

[4] A.K. Mirmostafaee and M.S. Moslehian, Fuzzy almost quadratic functions, Results Math. 52 (2008), 161-177.

[5] A.K.Mirmostafaee, A fixed point approach to almost quartic mappings in quasi fuzzy normed spaces, Fuzzy Sets Syst. 160 (2009), 1653-1662.

[6] A.K.Mirmostafaee and M.S.Moslehian, Fuzzy versions of Hyers-Ulam-Rassias theorem, Fuzzy Sets Syst. 159 (2008), 720-729.

[7] T. Bag and S. K. Samanta, Finite dimensional fuzzy normed linear spaces, J. Fuzzy Math. 11 (3) (2003), 687-705.

[8] T.Bag and S. K.Samanta,: Finite dimensional fuzzy normed linear spaces. J. Fuzzy Math. 11(3) (2003), 687-705.

[9] S. C. Cheng and J. N. Mordeson, Fuzzy linear operators and fuzzy normed linear spaces, Bull. Calcutta Math. Soc. 86 (5) (1994), 429-436.

[10] D. Mihet, The fixed point method for fuzzy stability of the Jensen functional equation, Fuzzy Sets Syst. 160 (11) (2009), 1663-1667.

[11] D.H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA 27 (1941), 222-224.

[12] T.Z.Xu, On fuzzy approximately cubic type mapping in fuzzy Banach spaces, Inf. Sci. 278 (2014), 56-66.

[13] S.M. Ulam, Problems in Modern Mathematics, Science ed., John Wiley and Sons, New York, 1964.

[14] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan 2 (1950), 64-66.

[15] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978), 297-300.

[16] I. Kramosil and J. Michalek, Fuzzy metrics and statistical metric spaces Kybernetika, 11 (5) (1975), 326-334.