Available online at http://scik.org
J. Math. Comput. Sci. 7 (2017), No. 3, 495-521
ISSN: 1927-5307

STABILITY AND HOPF BIFURCATION IN A DELAYED PREDATOR-PREY SYSTEM WITH PARENTAL CARE FOR PREDATORS

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Abstract. In this paper a stage-structured predator-prey model (stage structure on predators) with two discrete time delays has been discussed. It is assumed that immature predators are raised by their parents in the sense that they cannot catch the prey and their foods are provided by parents. We suppose that the growth is of logistic type. The two discrete time delays occur due to gestation delay and maturation delay. Linear stability analysis for both non delays and as well as with delays reveals that certain thresholds have to be maintained for coexistence. We analyzed the global stability of the interior equilibrium and the boundary equilibrium points by using a suitable Lyapunov function. In addition, the normal form of the Hopf bifurcation arising in the system is determined to investigate the direction and the stability of periodic solutions bifurcating from these Hopf bifurcations. We present some numerical examples to illustrate our analytical works.

Keywords: stage structure; maturation delay; gestation delay; stability; hopf bifurcation.

2010 AMS Subject Classification: 92D25, 37G15.

1. Introduction

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Received March 6, 2017; Published May 1, 2017

Differential equation models for interactions between species are one of the classical application of Mathematics to Biology. In the natural world, there are many species whose individuals have a life history that take them through two stages, immature and mature, where immature predators are raised by their parents and the rate they attack at prey and the reproductive rate can be ignored. Stage structured models have recieved much attention in recent years [1]-[7]. Recently Wang, Takeuchi, Saito, Nakaoka [8] studied the following predator prey system with parental care for predators.

$$\begin{aligned}
\dot{x}(t) &= xg(x) - \beta x y_2 \\
\dot{y_1}(t) &= k_1 \beta x y_2 \frac{y_2}{w y_1 + y_2} - d_1 y_1 - k_2 \beta x y_2 \frac{w y_1}{w y_1 + y_2} \\
\dot{y_2}(t) &= k_2 \beta x y_2 \frac{w y_1}{w y_1 + y_2} - d_2 y_2
\end{aligned} \tag{1.1}$$

where x represents the prey, y_1 and y_2 represents immature and mature predators respectively, k_1 and k_2 are conversion coefficient and proportionality constant respectively, d_1 and d_2 are death rate of immature and mature predators respectively, and $g(x)$ is the per capita birth rate of the prey.

$$\begin{aligned}
\dot{x}(t) &= xg(x) - \beta x y_2(t-\tau_1) \\
\dot{y_1}(t) &= k_1 \beta x y_2 \frac{y_2}{w y_1 + y_2} - d_1 y_1 - k_2 \beta x y_2 \frac{w y_1}{w y_1 + y_2} \\
\dot{y_2}(t) &= k_2 \beta x(t-\tau_2) y_2 \frac{w y_1}{w y_1 + y_2} - d_2 y_2
\end{aligned} \tag{1.2}$$

Mathematical models with time delay are much more realistic, as in reality time delays occur in almost every biological problem and assumed to be one of the reasons of regular fluctuations in population density [9]-[13]. On the other hand, the growth of species often has its development process, while in each stage of its development, it always shows different characteristic. For instance, the mature species have preying capacity, while the immature species are not able to prey. The age to maturity is represented by a time delay. Also reproduction of predator after consuming prey is not instantaneous, but mediated by some time lag required for gestation. Therefore to make a predator prey model biologically more realistic, one has to consider this gestation and maturation delays in the model system.

Motivated by the work of Sandip Banerjee, Mukopadhyay and Bhattacharya [14], in the present paper we incorporate two discrete time delays in system (1.1) to make the model more realistic as follows. where $\tau_1 \geq 0$ is called the gestation delay and $\tau_2 \geq 0$ is the delay in the predator maturation.

The initial conditions for the system takes the form

$$x(\theta) = \phi(\theta) \geq 0,\ y_1(\theta) = \psi_1(\theta) \geq 0,\ y_2(\theta) = \psi_2(\theta) \geq 0,\ \phi(0) > 0,\ \psi_1(0) > 0,\ \psi_2(0) > 0 \tag{1.3}$$

where $\tau = max\{\tau_1, \tau_2\}$, $(\phi(\theta), \psi_1(\theta), \psi_2(\theta)) \in C([-\tau, 0], R^3_{+0})$, the banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^3_{+0} , where

$$R^3_{+0} = \{(x_1, x_2, x_3) : x_i \geq 0, i = 1, 2, 3\}$$

as the interior of R^3_{+0}.

The organisation of this paper is organised as follows. In the next section results on positivity and boundedness of the system are presented. Section 3 shows the inherent logistic growth of prey. We try to interpret our results by numerical simulation in Section 4. Section 5 with discussions completes the paper .

2. Positivity and Boundedness

In this section, we discuss the positivity and boundedness of the solutions of the system (1.2). Positivity means that the species is persistent and boundedness implies a natural restriction, that is, our model (1.2) is reasonable in part.

Theorem 2.1 *Every solution of system* (1.2) *with initial conditions* (1.3) *is bounded for all* $t \geq 0$ *and all of these solutions are ultimately bounded.*

Proof:

From (1.2)

$$\dot{x}(t) \leq xg(x),$$

$$\lim_{t\to\infty} sup\, x(t) \leq g(x).$$

There exists a $t_0 > 0$ such that $x(t) < g(x) + \varepsilon = M_1$, $t \geq t_0$, $\varepsilon > 0$ is sufficiently small.

Choosing function

$$\begin{aligned} \rho(t) &= y_1 + y_2, \\ \dot{\rho}(t) &= \dot{y_1} + \dot{y_2}, \\ \dot{\rho}(t) &\leq \frac{k_1 \beta M_1 y_2^2}{w y_1 + y_2} - d_1 y_1 - d_2 y_2, t \geq t_0 + \tau_2 \end{aligned} \tag{2.1}$$

Then

$$\dot{\rho}(t) + \sigma\rho(t) \leq \frac{k_1 \beta M_1 y_2^2}{w y_1 + y_2} - y_1(d_1 - \sigma) - y_2(d_2 - \sigma), t \geq t_0 + \tau_2. \tag{2.2}$$

where σ is a positive constant. Thus there exists a positive constant c such that $\dot{\rho}(t) + \sigma\rho(t) \leq c$.

Then

$$\rho(t) < \frac{c}{\sigma} + \left(\rho(t^*) - \frac{c}{\sigma}\right) e^{-\sigma(t-t^*)}. \tag{2.3}$$

Choose a positive constant $M_2 > \frac{c}{\sigma}$ sufficiently close to $\frac{c}{\sigma}$ and let

$$\Omega_1 = \left\{(x, y_1, y_2) \in {R_+}^3 / x(t) \leq M_1,\ y_1(t) \leq M_2,\ y_2(t) \leq M_2\right\} \tag{2.4}$$

Definition 2.1 *A system is said to be permanent if there exists a compact region $\Omega_0 \in int\ R_+^3$ such that every solution of system with initial conditions will eventually enter and remain in region Ω_0.*

Theorem 2.2 *System* (1.2) *is permanent provided that $g(x) - \beta M_2 > 0$.*

Proof:

From (1.2), $\dot{x}(t) \geq x(g(x) - \beta M_2)$, for $t > T$

According to $g(x) - \beta M_2 > 0$, we can choose $\varepsilon > 0$ sufficiently small such that $g(x) - \beta M_2 - \varepsilon > 0$. Therefore there exists $t_1 > T$ such that $x(t) > r - \beta M_2 - \varepsilon = m_1$

From (1.2) we have for $t > t_1 + T_2$,

$$\begin{aligned} \dot{y_2}(t) &\geq \frac{k_2 \beta m_1 y_2 w y_1}{w y_1 + y_2} - d_2 y_2 \\ &= y_2 \left[\frac{k_2 \beta m_1 w y_1}{w y_1 + y_2} - d_2\right] \end{aligned} \tag{2.5}$$

Consider the comparison equation ,

$$\begin{aligned} \dot{u}(t) &= u(t)\left[\frac{k_2\beta m_1 w v(t)}{wy_1+y_2} - d_2\right] \\ \dot{v}(t) &= k_1\beta m_1(u(t)^2) - d_1 v(t) - \frac{k_2\beta m_1 u(t) w v(t)}{wy_1+y_2} \end{aligned} \tag{2.6}$$

Let $(u(t), v(t))$ be the solution of the system (2.6) with initial conditions $(u(0), v(0))$, $0 < u(0) < \psi_1(0)$, $0 < v(0) < \psi_2(0)$. According to comparison theorem, $u(t) < y_2(t)$, $v(t) < y_1(t)$ for $t > t_2 + \tau_2$ and

$$\lim_{t\to\infty} u(t) = b, \lim_{t\to\infty} v(t) = \frac{-bw + b\sqrt{w^2+4kd}}{2d}, \tag{2.7}$$

Hence there exists $t_3 > t_2 + \tau_2$ such that

$$y_1(t) > \frac{-bw + b\sqrt{w^2+4kd}}{2d} - \varepsilon = m_2,\ y_2(t) > b - \varepsilon = m_3 \tag{2.8}$$

Therefore

$$\Omega_0 = \left\{(x, y_1, y_2) \in {R_+}^3 / m_1 \le x(t) \le M_1,\ m_2 \le y_1(t) \le M_2,\ m_3 \le y_2(t) \le M_3\right\} \tag{2.9}$$

3. Inherent logistic growth of prey

In this section, we study the dynamical behaviors of (1.2) under the assumption that the demographic structure of the prey is governed by a Logistic growth. The model to be governed is

$$\begin{aligned} \dot{x}(t) &= x(r - \alpha x - \beta y_2(t-\tau_1)) \\ \dot{y_1}(t) &= k_1\beta x y_2 \frac{y_2}{wy_1+y_2} - d_1 y_1 - k_2\beta x y_2 \frac{wy_1}{wy_1+y_2} \\ \dot{y_2}(t) &= k_2\beta x(t-\tau_2) y_2 \frac{wy_1}{wy_1+y_2} - d_2 y_2 \end{aligned} \tag{3.1}$$

where α is the density dependent coefficient of the prey.

3.1 Equilibria analysis By introducing scaling variables $u = \frac{x\beta k_2}{d_2}$, $v_1 = y_1 w$, $v_2 = y_2$, $\theta = d_2 t$, and then still using old variables for simplicity in notations, we obtain

$$\begin{aligned} \dot{x}(t) &= x(b - ax - cy_2(t-\tau_1)) \\ \dot{y_1}(t) &= \frac{kxy_2^2}{y_1+y_2} - dy_1 - w\frac{xy_1y_2}{y_1+y_2} \\ \dot{y_2}(t) &= \frac{x(t-\tau_2)y_1y_2}{y_1+y_2} - y_2 \end{aligned} \tag{3.2}$$

where $b = \frac{r}{d_2}$, $a = \frac{\alpha}{\beta k_2}$, $c = \frac{\beta}{d_2}$, $k = \frac{wk_1}{k_2}$, $d = \frac{d_1}{d_2}$.

System admits a unique positive equilibrium $E^* = (x^*, y_1{}^*, y_2{}^*)$, where $x^* = \frac{y_1+y_2}{y_1}$, $y_2^* = \frac{(b-a)y_1}{a+cy_1}$ and y_1^* is the unique positive solution of the equation

$$dc^2y_1^2 + c[(b-a)w + 2ad]y_1 - k(b-a)^2 + a[w(b-a) + ad] = 0 \tag{3.3}$$

Theorem 3.1 *System* (3.2) *admits a unique positive equilibrium if and only if* $b > a$ *and* $k(b-a)^2 > a[w(b-a)+ad]$ *holds.*

The Jacobian matrix of (3.2) at E^* is

$$\begin{vmatrix} -ax^* - \lambda & 0 & -cx^*e^{-\lambda\tau_1} \\ \frac{y_2^*(ky_2^* - wy_1^*)}{y_1^*+y_2^*} & \frac{-(kx^*y_2^{*2} + dy_1^{*2} + 2dy_1^*y_2^* + dy_2^{*2} + x^*y_2^{*2}w)}{(y_1^*+y_2^*)^2} - \lambda & \frac{x^*(2ky_2^*y_1^* + ky_2^{*2} - wy_1^{*2})}{(y_1^*+y_2^*)^2} \\ \frac{y_1^*y_2^*e^{-\lambda\tau_2}}{(y_1^*+y_2^*)} & \frac{x^*y_2^{*2}}{(y_1^*+y_2^*)^2} & \frac{-x^*y_1^*y_2^*}{(y_1^*+y_2^*)^2} - \lambda \end{vmatrix} = 0, \tag{3.4}$$

The characteristic equation of the Jacobian matrix is

$$\lambda^3 + A\lambda^2 + B\lambda + C + (D_1\lambda + E_1)e^{-\lambda\tau_1} + (D_2\lambda + E_2)e^{-\lambda\tau_2} + (F\lambda + G)e^{-\lambda(\tau_1+\tau_2)} = 0. \tag{3.5}$$

where $A = ax^* + \frac{(kx^*y_2^{*2} + dy_1^{*2} + 2dy_1^*y_2^* + dy_2^{*2} + x^*y_2^{*2}w + x^*y_1^*y_2^*)}{(y_1^*+y_2^*)^2}$,

$B = ax^*\frac{(kx^*y_2^{*2} + dy_1^{*2} + 2dy_1^*y_2^* + dy_2^{*2} + x^*y_2^{*2}w)}{(y_1^*+y_2^*)^2} + \frac{ax^{*2}y_1^*y_2^*}{(y_1^*+y_2^*)^2} + \frac{xy_1^*y_2^*(kx^*y_2^{*2} + dy_1^{*2} + 2dy_1^*y_2^* + dy_2^{*2} + x^*y_2^{*2}w)}{(y_1^*+y_2^*)^4}$
$- \frac{x^{*2}y_2^{*2}(2ky_1^*y_2^* + ky_2^{*2} - wy_1^{*2})}{y_1^*+y_2^*)^4}$,

$C = ax^{*2}y_1^*y_2^* + \frac{(kx^*y_2^{*2} + dy_1^{*2} + 2dy_1^*y_2^* + dy_2^{*2} + x^*y_2^{*2}w)}{(y_1^*+y_2^*)^4} - \frac{ax^{*3}y_2^{*2}(2ky_1^*y_2^* + ky_2^{*2} - wy_1^{*2})}{y_1^*+y_2^*)^4}$, $D_1 = 0$,

$E_1 = \frac{cx^{*2}y_2^{*3}(ky_2^* - wy_1^*)}{(y_1^*+y_2^*)^3}$, $D_2 = 0$, $E_2 = 0$, $F = \frac{cx^*y_2^*y_1^*}{y_1^*+y_2^*}$, $G = \frac{cx^*y_1^*y_2^*(kx^*y_2^{*2} + dy_1^{*2} + 2dy_1^*y_2^* + dy_2^{*2} + x^*y_2^{*2}w)}{(y_1^*+y_2^*)^3}$.

Case 1: $\tau_1 = 0, \tau_2 = 0$

In this case the characteristic equation (3.5) reduces to

$$\lambda^3 + A\lambda^2 + (B + D_1)\lambda + (C + E_1) + ((D_2 + F)\lambda + E_2 + G)e^{-\lambda\tau_2} = 0 \tag{3.6}$$

Assume that $(H_1) : (B+D_1+D_2+F) > 0, (C+E_1+E_2+G) > 0$. Thus the stability of E^* is determined by the sign $H = A(B+D_1+D_2+F)-(C+E_1+E_2+G)$. By [13], $H = (h_1 c+h_0)x^*/(y_1^*+y_2^*)^4$, where $h = (y_1^*+y_2^*)x^* y_1^* y_2^* a(y_1^*+y_2^*)^2+(1-d)y_1^* y_2^*$,
$h_0 = a(d+ax^*)(y_1^*+y_2^*)^2+x^* y_2^*((k+w)y_2^*+y_1^*)\times d(y_1^*+y_2^*)^2+x^* y_1^* y_2^*+(k+w)x^*(y_2^*)^2$. Note that $H > 0$ if $d < 1$.

From this we observe that [13], the system (3.2) without delay is locally asymptotically stable if $d < 1$ around $E^* = (x^*, y_1^*, y_2^*)$.

Case 2: $\tau_1 = 0, \tau_2 > 0$

In this case the characteristic equation (3.5) reduces to

$$\lambda^3+A\lambda^2+(B+D_1)\lambda+(C+E_1)+((D_2+F)\lambda+E_2+G)e^{-\lambda\tau_2} = 0 \tag{3.7}$$

Let $i\omega(\omega > 0)$ be a root of the equation (3.7), then

$$-i\omega^3-A\omega^2+(B+D_1)i\omega+(C+E_1)+((D_2+F)i\omega+(E_2+G))e^{-i\omega\tau_2} = 0 \tag{3.8}$$

Equating real and imaginary parts, we obtain

$$\begin{aligned} &(E_2+G)cos\omega\tau_2+(D_2+F)\omega sin\omega\tau_2 = A\omega^2-(C+E_1) \\ &(D_2+F)\omega cos\omega\tau_2-(E_2+G)sin\omega\tau_2 = \omega^3-(B+D_1)\omega \end{aligned} \tag{3.9}$$

which implies that

$$\omega^6+\omega^4(A^2-2(B+D_1))+\omega^2[(B+D_1)^2-2A(C+E_1)-(D_2+F)^2]+[(C+E_1)^2-(E_2+G)^2] = 0 \tag{3.10}$$

If $(H_2): A^2-2(B+D_1) > 0, (B+D_1)^2-(D_2+F)^2-2A(C+E_1) > 0, (C+E_1)^2-(E_2+G)^2 > 0$ hold then (3.10) has no positive roots. Hence all the roots of (3.10) have negative real parts when $\tau_2 \in [0,\infty)$ under (H_1) and (H_2).
If (H_1) and $(H_3) : (C+E_1)^2-(E_2+G)^2 < 0$ hold, then (3.10) has a unique positive root ω_0^2. Substituting ω_0^2 into (3.10), we have

$$\tau_{2n} = \frac{1}{\omega_0}cos^{-1}\left[\frac{\omega_0{}^4(D_2+F)+\omega_0{}^2[A(E_2+G)-((D_2+F)(B+D_1))]-[(E_2+G)(C+E_1)]}{[(E_2+G)^2+(D_2+F)^2\omega_0{}^2]}\right]+\frac{2n\pi}{\omega_0}, \tag{3.11}$$

where n = 0, 1, 2...

If (H_1) and $(H_4) : 2(B+D_1)-A^2 > 0, (D_2+F)^2-(B+D_1)^2+2A(C+E_1) > 0$, $(C+E_1)^2-(E_2+G)^2 > 0$ and $[(D_2+F)^2-(B+D_1)^2+2A(C+E_1)]^2 > 4[(C+E_1)^2-(E_2+G)^2]$ hold then (3.10) has two positive roots ω_+^2 and ω_-^2. Substituting $\omega_\pm^2$ into (3.10) gives

$$\tau_{2k}^{\pm} = \frac{1}{\omega_\pm} cos^{-1}\left[\frac{\omega_\pm{}^4(D_2+F)+\omega_\pm{}^2[A(E_2+G)-((D_2+F)(B+D_1))]-[(E_2+G)(C+E_1)]}{[(E_2+G)^2+(D_2+F)^2\omega_\pm{}^2]}\right]+\frac{2n\pi}{\omega_\pm}, \tag{3.12}$$

Let $\lambda(\tau_2)$ be the root of (3.7) satisfying $Re\, \lambda(\tau_{2n}) = 0(rep.\, Re\, \lambda(\tau_{2k}^{\pm}) = 0)$ and $Im\lambda(\tau_{2n}) = \omega_0$ $(rep.\, Im\, \lambda(\tau_{2k}^{\pm}) = \omega_\pm)$ Then

$$\left[\frac{d}{d\tau_2}Re(\lambda)\right]_{\tau_2=\tau_{20},\omega=\omega_0} > 0, \ \left[\frac{d}{d\tau_2}Re(\lambda)\right]_{\tau_2=\tau_{2k}^{+},\omega=\omega_+} > 0, \ \left[\frac{d}{d\tau_2}Re(\lambda)\right]_{\tau_2=\tau_{2k}^{-},\omega=\omega_-} > 0 \tag{3.13}$$

From corollary 2.4 in Ruan and Wei [15], we have the following conclusion.

Lemma 3.1

For $\tau_1 = 0$, assume that (H_1) is satisfied. Then the following conclusion holds.

1. If (H_2) holds, then equilibrium (x^*, y_1^*, y_2^*) is asymptotically stable for all $\tau_2 \geq 0$.

2. If (H_3) holds, then equilibrium (x^*, y_1^*, y_2^*) is asymptotically stable for $\tau_2 < \tau_{20}$, and unstable for $\tau_2 > \tau_{20}$. Furthermore, system undergoes a Hopf bifurcation at $(x^*, y_1^*, y_2^*)$when $\tau_2 = \tau_{20}$.

3. If (H_4) holds, then there exists a positive integer m such that the equilibrium is stable when $\tau_2 \in [0, \tau_{20}^{+}) \cup (\tau_{20}^{-}, \tau_{21}^{+}) \cup ..... \cup (\tau_{2m-1}^{-}, \tau_{2m}^{+})$ and unstable when $\tau_2 \in [\tau_{20}^{+}, \tau_{20}^{-}) \cup (\tau_{21}^{+}, \tau_{21}^{-}) \cup ... \cup (\tau_{2m}^{+}, \tau_{2m}^{-}) \cup (\tau_{2m}^{+}, \infty)$

Furthermore system undergoes a Hopf bifurcation at (x^*, y_1^*, y_2^*) when $\tau_2 = \tau_{2k}^{\pm}$, k = 0, 1, 2,...

Case 3: $\tau_1 > 0, \tau_2 = 0$

In this case the characteristic equation (3.5) becomes

$$\lambda^3 + A\lambda^2 + (B+D_2)\lambda + (C+E_2) + ((D_1+F)\lambda + (E_1+G))e^{-\lambda\tau_1} = 0 \tag{3.14}$$

Let $i\omega(\omega > 0)$ be a root of the equation (3.14) , then we have

$$\begin{aligned} (E_1+G)cos\omega\tau_1 + (D_1+F)\omega sin\omega\tau_1 &= A\omega^2 - (C+E_2) \\ (D_1+F)\omega cos\omega\tau_1 - (E_1+G)sin\omega\tau_1 &= \omega^3 - (B+D_2)\omega \end{aligned} \tag{3.15}$$

which implies that

$$cos\omega\tau_1 = \left[\frac{(D_1+F)\omega^4 + (A(E_1+G)-(D_1+F)(B+D_2))\omega^2 - ((E_1+G)(C+E_2))}{(E_1+G)^2+\omega^2(D_1+F)^2}\right] \tag{3.16}$$

$$sin\omega\tau_1 = \left[\frac{[A(D_2+F)-(E_2+G)]\omega^3+((E_2+G)(B+D_1)-(D_2+F)(C+E_1))\omega}{(D_2+F)^2\omega^2+(E_2+G)^2}\right] \tag{3.17}$$

Squaring and adding we get

$$\omega^6+\omega^4(A^2-2(B+D_2))+[(B+D_2)^2+2A(C+E_2)-(D_1+F)^2]\omega^2+(C+E_2)^2-(E_1+G)^2=0 \tag{3.18}$$

Let

$$\psi(W)\equiv W^3+W^2(A^2-2(B+D_2))+[(B+D_2)^2+2A(C+E_2)-(D_1+F)^2]W+(C+E_2)^2-(E_1+G)^2=0 \tag{3.19}$$

where $W=\omega^2$.

The function ψ has positive roots iff

$$(C+E_2)^2-(E_1+G)^2<0,$$

Without loss of generality, let W_p be the positive roots of $\psi=0$ and let $\omega_p=\sqrt{W_p}$. The unique solution of $\theta=[0,2\pi]$ of (3.16) and (3.17) is

$$\theta = cos^{-1}\left[\frac{(D_1+F)\omega^4+(A(E_1+G)-(D_1+F)(B+D_2))\omega^2-((E_1+G)(C+E_2))}{(E_1+G)^2+\omega^2(D_1+F)^2}\right] \tag{3.20}$$

if $sin(\theta)>0$, that is, if $(A(D_2+F)-(E_2+G))\omega^2+(E_2+G)(B+D_1)-(D_2+F)(C+E_1)>0$ and

$$\theta = 2\pi - cos^{-1}\left[\frac{(D_1+F)\omega^4+(A(E_1+G)-(D_1+F)(B+D_2))\omega^2-((E_1+G)(C+E_2))}{(E_1+G)^2+\omega^2(D_1+F)^2}\right] \tag{3.21}$$

if $(A(D_2+F)-(E_2+G))\omega^2+(E_2+G)(B+D_1)-(D_2+F)(C+E_1)\leq 0$.

Define,

$$\tau_{1,p}^{1,i}=\frac{1}{\omega_p}\left[cos^{-1}\left[\frac{(D_1+F)\omega^4+(A(E_1+G)-(D_1+F)(B+D_2))\omega^2-((E_1+G)(C+E_2))}{(E_1+G)^2+\omega^2(D_1+F)^2}\right]+2i\pi\right]$$

$$\tau_{1,p}^{2,i}=\frac{1}{\omega_p}\left[2\pi-cos^{-1}\left[\frac{(D_1+F)\omega^4+(A(E_1+G)-(D_1+F)(B+D_2))\omega^2-((E_1+G)(C+E_2))}{(E_1+G)^2+\omega^2(D_1+F)^2}\right]+2i\pi\right]$$

Theorem 3.2 *Let $\tau_{1,p}^*=\tau_{1,p}^{1,i}$ or $\tau_{1,p}^*=\tau_{1,p}^{2,i}$, that is $\tau_{1,p}^*$ represents an element either of the sequence $\tau_{1,p}^{1,i}$ or $\tau_{1,p}^{2,i}$ associated with ω_p. Then the equation $\lambda^3+A\lambda^2+(B+D_2)\lambda+(C+$*

$E_2)+((D_1+F)\lambda+(E_1+G))e^{-\lambda\tau_1}=0$ *has a pair of simple conjugate roots* $\pm i\omega_p$ *for* $\tau_2=\tau^*_{1,p}$ *which satisfies*

$$sign\left\{\frac{dRe\lambda}{d\tau_1}|_{\tau=\tau^*_{1,p}}\right\}=sign\ \acute{\psi}(\omega^2 p) \tag{3.22}$$

Denoting $\tau_1^*=min_{i\in N}\left\{\tau^{1,i}_{1,p},\tau^{2,i}_{1,p}\right\}$, *it is concluded that the steady state* (x^*,y_1^*,y_2^*) *is locally asymptotically stable if* $\tau_1=\tau_1^*$ *iff* $\acute{\psi}(\omega^2 p)>0$.

Proof:

Let $\pm i\omega_p$ be a pair of purely imaginary roots of (3.14) and let $\lambda(\tau_1)=\phi(\tau_1)+i\omega(\tau_1)$ be a branch of roots of (3.14) with $\phi(\tau_1^*,p)=0$ and $\omega(\tau_1^*,p)=\omega_p$. We assume that $\lambda(\tau_1^*,p)$ is not a simple root of (3.14), then both (3.14) and derivatives of (3.14) share the same root, which implies

$$\lambda^3+A\lambda^2+(B+D_2)\lambda+(C+E_2)+((D_1+F)\lambda+(E_1+G))e^{-\lambda\tau_1}=0 \tag{3.23}$$

$$(3\lambda^2+2A\lambda+(B+D_2)+(D_1+F)-\tau_1((D_1+F)\lambda+(E_1+G))e^{-\lambda\tau_1})\frac{d\lambda}{d\tau_1}-\lambda((D_1+F)\lambda+(E_1+G))e^{-\lambda\tau_1}=0 \tag{3.24}$$

at $\lambda=\lambda(\tau_1^*,p)$. Put $\lambda=\lambda(\tau_1^*,p)=\omega(\tau_1^*,p)=\omega_p$ and by seperating real and imaginary parts, we get respectively

$$\begin{aligned}(E_1+G)\omega_p cos(\omega_p\tau^*_{1,p})+(D_1+F)\omega_p^2 sin(\omega_p\tau^*_{1,p})=0\\(D_1+F)\omega_p^2 cos(\omega_p\tau^*_{1,p})-(E_1+G)\omega_p sin(\omega_p\tau^*_{1,p})=0\end{aligned} \tag{3.25}$$

$$\begin{aligned}(E_1+G)cos(\omega_p\tau^*_{1,p})+(D_1+F)\omega_p sin(\omega_p\tau^*_{1,p})=A\omega_p^2-(C+E_2)\\(D_1+F)\omega_p cos(\omega_p\tau^*_{1,p})-(E_1+G)sin(\omega_p\tau^*_{1,p})=\omega_p^3-(B+D_2)\omega_p\end{aligned} \tag{3.26}$$

Let us consider $\omega_p>0$. From (3.25) and (3.26) , we obtain $A(B+D_2)=C+E_2$,we arrive at a contradiction.

Hence $\pm i\omega_p$ are simple roots of (3.14). From (3.14) and (3.26) we get

$$e^{\lambda\tau_1}=-\frac{(D_1+F)\lambda+(E_1+G)}{\lambda^3+A\lambda^2+(B+D_2)\lambda+(C+E_2)} \tag{3.27}$$

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1}=\frac{(3\lambda^2+2A\lambda+(B+D_2))e^{\lambda\tau_1}+(D_1+F)}{\lambda((D_1+F)\lambda+(E_1+G))}-\frac{\tau_1}{\lambda} \tag{3.28}$$

By removing $e^{\lambda\tau_1}$, we get

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1} = -\frac{3\lambda^2+2A\lambda+(B+D_2)}{\lambda(\lambda^3+A\lambda^2+(B+D_2)\lambda+(C+E_2))} + \frac{D_1+F}{\lambda((D_1+F)\lambda+(E_1+G))} - \frac{\tau_1}{\lambda} \tag{3.29}$$

Then

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1}_{\tau_1=\tau^*_{1,p}} = -\frac{-3\omega_p^2+2Ai\omega_p+(B+D_2)}{i\omega_p(C+E_2)-i\omega_p^3A+\omega_p^4-\omega_p^2(B+D_2)} + \frac{D_1+F}{i\omega_p(E_1+G)-\omega_p(D_1+F)} - \frac{\tau_1}{i\omega_p} \tag{3.30}$$

Consequently

$$Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}_{\tau_1=\tau^*_{1,p}} = \frac{3\omega_p^2+2\omega_p(A^2-2(B+D_2))+(B+D_2)^2-2A(C+E_2)}{((C+E_2)-A\omega_p^2)^2+\omega_p^2(\omega_p^2-(B+D_2))^2} - \frac{(D_1+F)^2}{(D_1+F)^2\omega_p^2+(E_1+G)^2} \tag{3.31}$$

Now

$$\begin{aligned}(D_1+F)^2\omega_p^2+(E_1+G)^2 &= \omega_p^6+\omega_p^4(A^2-2(B+D_2))+\omega_p^2((B+D_2)^2-2(C+E_2)A)+(C+E_2)^2\\ &= ((C+E_2)-A\omega_p^2)^2+(\omega_p^2-(B+D_2)^2)\omega_p^2\end{aligned} \tag{3.32}$$

$$\begin{aligned}Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}_{\tau_1=\tau^*_{1,p}} &= \frac{3\omega_p^2+2\omega_p(A^2-2(B+D_2))+(B+D_2)^2-2A(C+E_2)-(D_1+F)^2}{((C+E_2)-A\omega_p^2)^2+\omega_p^2(\omega_p^2-(B+D_2))^2}\\ &= \frac{\psi(\omega_p^2)}{((C+E_2)-A\omega_p^2)^2+\omega_p^2(\omega_p^2-(B+D_2))^2}\end{aligned} \tag{3.33}$$

Since $sign\left\{Re(\frac{d\lambda}{d\tau_1})^{-1}_{\tau_1=\tau^*_{1,p}}\right\} = sign\left\{\frac{dRe\lambda}{d\tau_1}|_{\tau=\tau^*_{1,p}}\right\}$ we get

$$sign\left\{\frac{dRe\lambda}{d\tau_1}|_{\tau=\tau^*_{1,p}}\right\} = sign\ \psi(\omega^2 p)$$

If $\psi(\omega_p^2)>0$, then $sign\left\{\frac{dRe\lambda}{d\tau_1}|_{\tau=\tau^*_{1,p}}\right\}>0$. Hence the system will be locally asymptotically stable when $\tau_1=\tau^*_{1,p}$ and a Hopf bifurcation occurs at (x^*,y_1^*,y_2^*) at $\tau_1=\tau^*_{2,p}$ iff $\psi(\omega_p^2)>0$

Case 4: $\tau_1>0,\tau_2>0$

Proposition 3.1 *If all the roots of the equation* (3.5) *have negative real parts for some* $\tau_1>0$, *then there exists a* $\tau_2^*(\tau_1)>0$ *such that all the roots of equation* (3.5) *(i.e with* $\tau_2>0$*) have negative real parts when* $\tau_2<\tau_2^*(\tau_1)$.

Considering the above proposition we can now state the following theorem.

Theorem 3.3 *If we assume that the proposition 3.1 hold, then for any* $\tau_1\in[0,\tau_1^*)$, *(*τ_1^* *having the same definition as in theorem 3.2) there exists a* $\tau_2^*(\tau_1)>0$ *such that the positive steady state* (x^*,y_1^*,y_2^*) *of the system is locally asymptotically stable when* $\tau_1\in[0,\tau_1^*)$.

Proof:

Using the above proposition, we can say that all the roots of (3.5) have negative real parts when $\tau_1 \in [0, \tau_1^*)$ and by proposition we can conclude that there exists a $\tau_2^*(\tau_1) > 0$ such that all the roots of equation (3.5) have negative real parts when $\tau_2 < \tau_2^*(\tau_1)$. Hence the steady state (x^*, y_1^*, y_2^*) of system (3.2) is locally asymptotically stable when $\tau_1 \in [0, \tau_1^*)$.

3.2 Global Stability

Lemma 3.2 *Consider the following system*

$$\dot{u}(t) = u(t)\left[\frac{(b/a+\varepsilon)v(t)}{u(t)+v(t)} - 1\right] \approx P(u,v)$$
$$\dot{v}(t) = \left[k\frac{(b/a+\varepsilon)u(t)^2}{u(t)+v(t)} - dv(t) - w\frac{(b/a+\varepsilon)u(t)v(t)}{u(t)+v(t)}\right] \approx Q(u,v)$$

where $\varepsilon > 0$ is sufficiently small, we have

i) the unique equilibrium (0,0) of system (3.2) *is globally asymptotically stable.*

ii) the positive equilibrium (u^*, v^*) is globally asymptotically stable.

Proof:

(0,0) is the unique non negative asymptotically stable equilibrium of system (3.2). From the proof of Theorem 2.1, we can conclude that all solutions of system(3.2) are uniformly bounded. Since $\frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v} < 0$, hence according to Bendixson- Dulac theorem, the unique equilibrium is globally asymptotically stable. Hence the proof. In a similar way, the positive equilibrium (u^*, v^*) is globally asymptotically stable.

Theorem 3.4 *The boundary equilibrium $E_1 = (b/a, 0, 0)$ is globally asymptotically stable.*

Proof:

According to Theorem 2.1, for simplicity we assume that $x(t) < (b/a+\varepsilon)$, for $t > 0$ ($\varepsilon > 0$ is sufficiently small).

The third equation of system (3.2) yields,

$$\dot{y_2}(t) \leq y_2\left[\frac{(b/a+\varepsilon)y_1}{y_1+y_2} - 1\right]$$

Consider the comparison system

$$\dot{u}(t) = u(t)\left[\frac{(b/a+\varepsilon)v(t)}{u(t)+v(t)} - 1\right]$$

$$\dot{v}(t) = \left[k\frac{(b/a+\varepsilon)u(t)^2}{u(t)+v(t)} - dv(t) - w\frac{(b/a+\varepsilon)u(t)v(t)}{u(t)+v(t)} \right]$$

By comparison theorem in differential equations, we have $u(t) \geq y_2(t)$, $v(t) \geq y_1(t)$ for $t > 0$. From Lemma 3.2,

$$\lim_{t\to\infty} u(t) = 0, \ \lim_{t\to\infty} v(t) = 0, \tag{3.34}$$

Incorporating into the positivity of $y_1(t)$ and $y_2(t)$, we have

$$\lim_{t\to\infty} y_2(t) = 0, \ \lim_{t\to\infty} y_1(t) = 0, \tag{3.35}$$

Therefore there exists a $T_1 > 0$ such that $y_2(t) < \varepsilon$ for $T > T_1 - \tau_1$.

The first equation of system (3.2) yields,

$$\dot{x}(t) \geq x(t)(b - ax - \varepsilon)$$

Thus it is easy to obtain

$$\lim_{t\to\infty} inf\ x(t) \geq b/a - \varepsilon$$

From Theorem 2.1, we can deduce

$$\lim_{t\to\infty} x(t) = b/a \tag{3.36}$$

From (3.34) and (3.36), the boundary equilibrium $E_{10} = (b/a, 0, 0)$ is globally asymptotically stable.

Theorem 3.5 *The boundary equilibrium $E_2 = (0, \tilde{y_1}, \tilde{y_2})$ is globally asymptotically stable.*

Proof:

According to the positivity of the solutions of system (3.2), we have

$$\dot{y_2}(t) \geq -y_2(t)$$

Consider the comparison equations,

$$\dot{u}(t) = -u(t)$$

$$\dot{v}(t) = -dv(t) \tag{3.37}$$

$(\tilde{y_2}, \tilde{y_1})$ is the unique positive equilibrium of system (3.2) which is globally asymptotically stable.

Let (u(t), v(t)) be the solution of (3.2) with initial value (u(0), v(0)) and $u(0) \leq \psi_1(0)$, $v(0) \leq \psi_2(0)$. In view of comparison theorem we have $u(t) \leq y_2(t)$, $v(t) \leq y_1(t)$ for $t > 0$ and hence

$$\lim_{t\to\infty} inf\ y_2(t) \geq \tilde{y_2}$$

We can choose $\varepsilon_1 > 0$ sufficiently small such that

$$\frac{b}{a} < c\ (\tilde{y_2} - \varepsilon_1) \tag{3.38}$$

Let $T_1 > 0$ be large enough such that

$$\dot{y_2}(t) > \tilde{y_2} - \varepsilon_1\ for\ t > T_1$$

Then we have for $t > T_1 + \tau_1$

$$\dot{x(t)} < x(t)(b - ax - (\tilde{y_2} - \varepsilon_1))$$

From (3.38) and comparison theorem we have

$$\lim_{t\to\infty} sup\ x(t) \leq 0$$

From Lemma 3.2,

$$\lim_{t\to\infty} x(t) = 0$$

Let $\xi > 0$ be sufficiently small and in view of $\lim_{t\to\infty} x(t) = 0$, we obtain that there is a $T_2 > T_1 + \tau_1$ such that $-\xi < x(t) < \xi$ for $t > T_2 + \tau_2$. For the third equation of system (3.2), we have

$$\dot{y_2}(t) > y_2(t)\left(\frac{-\xi y_1(t)}{y_1(t) + y_2(t)} - 1\right),\ for\ t > T_2 + \tau_2,$$

and

$$\dot{y_2}(t) < y_2(t)\left(\frac{\xi y_1(t)}{y_1(t) + y_2(t)} - 1\right), for\ t > T_2 + \tau_2$$

Consider the comparison equations

$$\dot{u_1}(t) = u(t)\left(\frac{-\xi w v_1(t)}{w v_1(t) + u_1(t)} - 1\right),$$
$$\dot{v_1}(t) = -\frac{k\xi u_1(t)^2}{v_1(t) + u_1(t)} - d v_1(t) + \frac{\xi u_1(t) v_1(t)}{v_1(t) + u_1(t)}$$

and

$$\dot{u_2}(t) = u(t)\left(\frac{\xi w v_2(t)}{w v_2(t)+u_2(t)} - 1\right),$$
$$\dot{v_2}(t) = \frac{k\xi u_2(t)^2}{v_2(t)+u_2(t)} - d v_2(t) - \frac{\xi u_2(t) v_2(t)}{v_2(t)+u_2(t)}$$

Let $(u_1(t), v_1(t))$ be the solution of system (3.39) with initial value $(u_1(0), v_1(0))$ and $0 < u_1(0) < \psi_1(0), 0 < v_1(0) < \psi_2(0)$. By comparison theorem we have $u_1(t) < y_2(t)$, $v_1(t) < y_1(t)$ for $t > T_3 + \tau_2$ and note that ξ is sufficiently small hence

$$\lim_{t\to\infty} inf\ y_2(t) \geq \tilde{y_2}, \ \lim_{t\to\infty} inf\ y_1(t) \geq \tilde{y_1}$$

Similarly

$$\lim_{t\to\infty} sup\ y_2(t) \leq \tilde{y_2}, \ \lim_{t\to\infty} sup\ y_1(t) \leq \tilde{y_1}$$

Hence

$$\lim_{t\to\infty} inf\ y_2(t) = \tilde{y_2}, \ \lim_{t\to\infty} sup\ y_1(t) = \tilde{y_1}$$

The boundary equilibrium $E_{20} = (0, \tilde{y_1}, \tilde{y_2})$ is globally asymptotically stable.

Theorem 3.6 *The positive equilibrium $E^* = (x^*, {y_1}^*, {y_2}^*)$ of system* (3.2) *is globally asymptotically stable.*

Proof:

By the transformation $X(t) = x(t) - x^*$, $Y(t) = y_2(t) - y_2^*$, $Z(t) = y_1(t) - y_1^*$, system (3.2), is reduced to

$$\begin{aligned}
\dot{X}(t) &= (X(t)+x^*)(-aX(t) - cY(t-\tau_1)) \\
\dot{Y}(t) &= (Y(t)+y_2^*)\frac{X(t-\tau_2)Y(t)Z(t)}{Y(t)+Z(t)} \\
\dot{Z}(t) &= \frac{kX(t)Y(t)^2}{Y(t)+Z(t)} - (Z(t)+y_1^*)\left(\frac{wX(t)Y(t)}{Y(t)+Z(t)}\right)
\end{aligned} \tag{3.39}$$

Consider the following Lyapunov function

$$V(X(t),Y(t),Z(t)) = \left[X(t) - x^* ln\left(1+\frac{X(t)}{x^*}\right)\right] + \left[Y(t) - y_2^* ln\left(1+\frac{Y(t)}{y_2^*}\right)\right] + \frac{1}{2}\int_{-\tau_2}^{0} X^2(t+S)ds + \frac{1}{2}\int_{-\tau_1}^{0} Y^2(t+S)ds + lZ^2(t)$$

Calculate and estimate the derivative of V(t) along the solutions of system (3.39)

$$\begin{aligned}
\frac{dV(t)}{dt} &= X(t)(-aX(t)-cY(t-\tau_1))+Y(t)\left(\frac{X(t-\tau_2)Z(t)}{Y(t)+Z(t)}\right)+\frac{1}{2}(X^2(t)-X^2(t-\tau_2)) \\
&+\frac{1}{2}(Y^2(t)-Y^2(t-\tau_1))+2lZ(t)\left(-Z(t)\left(\frac{wX(t)Y(t)}{Y(t)+Z(t)}\right)+\frac{kX(t)(Y(t))^2}{Y(t)+Z(t)}\right) \\
&\leq \frac{-1}{2}(1-c^2)X^2(t)-\frac{1}{2}Y^2(t)-wl\frac{X(t)Y(t)(Z(t))^2}{Y(t)+Z(t)}-\left(\sqrt{lw}Z(t)-\sqrt{\frac{l}{w}}kY(t)\right)^2 \\
&\frac{X(t)Y(t)}{X(t)+Y(t)}+\frac{lk^2X(t)(Y(t))^3}{w(X(t)+Y(t))} \\
&\leq \frac{-1}{2}(1-c^2)X^2(t)-\frac{1}{2}Y^2(t)-\frac{X(t)Y(t)}{Y(t)+Z(t)}(-wl(Z(t))^2-\left(\sqrt{lw}Z(t)-\sqrt{\frac{l}{w}}kY(t)\right)^2 \\
&+\frac{lk^2}{w}(Y(t))^2) \\
&\leq \frac{-1}{2}(1-c^2)X^2(t)-\frac{1}{2}(1-\frac{2lk^2}{w})(Y(t))^2-wl(Z(t))^2.
\end{aligned} \tag{3.40}$$

Now we can choose $l \in \frac{w}{2k^2}$ such that $1-\frac{2lk^2}{w}>0$. Thus $\frac{dV}{dt}\leq 0$ and $\frac{dV}{dt}=0$ if and only if X = Y = Z = 0. Hence the equilibrium (0, 0, 0) of system (3.2), that is the positive equilibrium E^* of system (3.39) is globally asymptotically stable. This completes the proof.

3.3 Direction and Stability of Hopf bifurcation In this section we shall study the direction of the Hopf bifurcations and stability of bifurcating periodic solutions by applying the normal theory and the center manifold theorem introduced by Hassard et al. [16]. Throughout this section, we always assume that the system undergoes a hopf bifurcation at the positive equilibrium $E(x^*,y_1^*,y_2^*)$ for $\tau_1=\tau_{1_0}$, and then $\pm i\omega$ denotes the corresponding purely imaginary roots of the characteristic equation at the positive equilibrium $E(x^*,y_1^*,y_2^*)$.

Without loss of generality, we assume that $\tau_2^*<\tau_{1_0}$ where $\tau_2^*\in(0,\tau_{2_0}^*)$ and $\tau_1=\tau_{10}+\mu$. Let $x_{11}=x-x^*$, $x_{21}=y_1-y_1^*$, $x_{31}=y_2-y_2^*$, $\bar{x_{i1}}=\mu_i(\tau t)$, i=1,2,3... Here $\mu=0$ is the bifurcation parameter and dropping the bars, the system becomes a functional differential equation in $C=C([-1,0],R^3)$ as

$$\frac{dX}{dt}=L_\mu(X_t)+f(\mu,X_t) \tag{3.41}$$

where $x(t) = (x_{11}, x_{21}, x_{31}) \in R^3$ and $L_\mu : C \to R^3$, $f : R \times C \to R^3$ are respectively given by

$$L_\mu(\phi) = (\tau_{1_0} + \mu)B \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} + (\tau_{1_0} + \mu)C \begin{pmatrix} \phi_1(\frac{-\tau_2^*}{\tau_1}) \\ \phi_2(\frac{-\tau_2^*}{\tau_1}) \\ \phi_3(\frac{-\tau_2^*}{\tau_1}) \end{pmatrix} + (\tau_{1_0} + \mu)D \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix} \tag{3.42}$$

and

$$f(\mu, \phi) = (\tau_{1_0} + \mu)Q \tag{3.43}$$

where Q= $\begin{pmatrix} -a\phi_1^2(0) - c\phi_1(0)\phi_3(-1) \\ \frac{k\phi_1(0)\phi_3^2(0)}{\phi_2(0)+\phi_3(0)} - \frac{w\phi_1(0)\phi_2(0)\phi_3(0)}{\phi_2(0)+\phi_3(0)} \\ \frac{\phi_1(\frac{-\tau_2*}{\tau_1})\phi_2(0)\phi_3(0)}{\phi_2(0)+\phi_3(0)} \end{pmatrix}$, respectively where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C$,

$$B = \begin{pmatrix} -ax_1^* & 0 & 0 \\ \frac{y_2^*(ky_2^* - wy_1^*)}{y_1 * + y_2^*} & \frac{-(kx^*y_2^{*2} + dy_1^{*2} + 2dy_1^*y_2^* + dy_2^{*2} + x^*y_2^{*2}w)}{(y_1^* + y_2^*)^2} & \frac{x^*(2ky_2^*y_1^* + ky_2^{*2} - wy_1^{*2})}{(y_1^* + y_2^*)^2} \\ 0 & \frac{x^*y_2^{*2}}{(y_1 * + y_2^*)^2} & \frac{-x^*y_1^*y_2^*}{(y_1^* + y_2^*)^2} \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{y_1^*y_2^*}{y_1^* + y_2^*} & 0 & 0 \end{pmatrix}, \; D = \begin{pmatrix} 0 & 0 & -cx^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By the Riesz representation theorem, we claim about the existence of a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0)$ such that

$$L_\mu(\phi) = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta) \quad for\ \phi \in C \tag{3.44}$$

Now let us choose ,

$$\eta(\theta, \mu) = \begin{cases} (\tau_{1_0} + \mu)(B + C + D), & \theta = 0 \\ (\tau_{1_0} + \mu)(C + D), & \theta \in [\frac{-\tau_2^*}{\tau_1}, 0) \\ (\tau_{1_0} + \mu)(D), & \theta \in (-1, \frac{-\tau_2^*}{\tau_1}) \\ 0, & \theta = -1. \end{cases}$$

For $\phi \in C([-1,0],R^3)$, we define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1,0) \\ \\ \int_{-1}^{0} d\eta(s,\mu)\phi(s), & \theta = 0 \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1,0) \\ \\ f(\mu,\phi), & \theta = 0 \end{cases}$$

Then the system is equivalent to

$$\frac{dX}{dt} = A(\mu)X_t + R(\mu)X_t, \tag{3.45}$$

where $X_t(\theta) = X(t+\theta)$ for $\theta \in [-1,0]$.

Now for $\psi \in \acute{C}([-1,0],(R^3)^*)$, we define

$$A^*\psi(s) = \begin{cases} \frac{-d\psi(s)}{ds}, & s \in (0,1] \\ \\ \int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0 \end{cases}$$

Further we define a bilinear inner product

$$< \psi(s),\ \phi(0) > = \ \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\zeta=0}^{\theta} \bar{\psi}(\zeta-\theta)d\eta(\theta)\phi(\zeta)d\zeta. \tag{3.46}$$

where $\eta(\theta) = \eta(\theta,0)$. We know that $\pm i\omega_0\tau_{1_0}$ are eigenvalues of $A(0)$. Thus they are also eigenvalues of A^*. To determine the poincare normal form of the operator A, we need to calculate the eigen vector q of A belonging to the eigenvalue $i\omega_0\tau_{1_0}$ and the eigen vector q^* of A^* belonging to the eigenvalue $-i\omega_0\tau_{1_0}$.

Let $q(\theta) = (1\ \alpha\ \beta)^T e^{i\omega_0\tau_{1_0}\theta}$ be the eigen vector of $A(0)$ corresponding to $i\omega_0\tau_{1_0}$ where
$\alpha = \frac{y_2^*(y_1^*+y_2^*)(ky_2^*-wy_1^*)-\frac{(ax^*+i\omega_0)}{cx^*}e^{i\omega_0\tau_{1_0}}x^*(2ky_2^*y_1^*+ky_2^{*2}-wy_1^{*2})}{kx^*y_2^{*2}+dy_1^{*2}+2dy_1^*y_2^*+dy_2^{*2}+x^*y_2^{*2}w+i\omega(y_1^*+y_2^*)^2}$, $\beta = \frac{-(ax^*+i\omega_0)e^{i\omega\tau_{1_0}}}{cx^*}$

Similarly if $q^*(s) = M(1\ \alpha^*\ \beta^*)e^{i\omega_0\tau_{1_0}s}$ be the eigen vector of A^* where

$$\alpha^* = \frac{(ax^*-i\omega)x^*y_2^{*2}(y_1^*+y_2^*)}{x^*y_2^{*3}(ky_2^*-wy_1^*)+y_1^*y_2^*e^{i\omega\frac{\tau_2^*}{\tau_{1_0}}}(kx^*y_2^{*2}+dy_1^{*2}+2dy_1^*y_2^*+dy_2^{*2}+x^*y_2^{*2}w-i\omega(y_1^*+y_2^*)^2)},$$

$$\beta^* = \frac{(ax^* - i\omega_0)(y_1^* + y_2^*) - \alpha y_2^*(ky_2^* - wy_1^*)}{y_1^* y_2^* e^{i\omega_0 \frac{\tau_2^*}{\tau_{1_0}}}}$$

Then we have to determine M from $< q^*(s),\ q(\theta) > = 1$.

$$< q^*(s), q(\theta) > = \bar{M}(1\ \bar{\alpha}^*\ \bar{\beta}^*)(1\ \alpha\ \beta)^T - \int_{-1}^{0}\int_{0}^{\theta} \bar{M}(1\ \bar{\alpha}^*\ \bar{\beta}^*)e^{-i\omega_0\tau_{1_0}(\zeta-\theta)}d\eta(\theta)(1\ \alpha\ \beta)^T e^{i\omega_0\tau_{1_0}\zeta}d\zeta$$

$$= \bar{M}(1\ \bar{\alpha}^*\ \bar{\beta}^*)(1\ \alpha\ \beta)^T - \int_{-1}^{0} \bar{M}(1\ \bar{\alpha}^*\ \bar{\beta}^*)\theta e^{i\omega_0\tau_{1_0}\theta}d\eta(\theta)(1\ \alpha\ \beta)^T$$

$$= \bar{M}\left[1 + \alpha\bar{\alpha}^* + \beta\bar{\beta}^* + \tau_{1_0}\left(-\beta x^* c e^{-i\omega_0\tau_{1_0}} + \frac{\tau_2^*}{\tau_{1_0}}\left(\frac{\beta y_1^* y_2^*}{y_1^* + y_2^*}e^{-i\omega_0\frac{\tau_2^*}{\tau_{1_0}}}\right)\right)\right]$$

Thus we can take

$$\bar{M} = \frac{1}{\left[1 + \alpha\bar{\alpha}^* + \beta\bar{\beta}^* + \tau_{1_0}\left(-\beta x^* c e^{-i\omega_0\tau_{1_0}} + \frac{\tau_2^*}{\tau_{1_0}}\left(\frac{\beta y_1^* y_2^*}{y_1^* + y_2^*}e^{-i\omega_0\frac{\tau_2^*}{\tau_{1_0}}}\right)\right)\right]} \tag{3.47}$$

We first compute the coordinate to describe the center manifold C_0 at $\mu = 0$. Let X_t be the solution of the system (3.45) when $\mu = 0$. Define $z(t) = < q^*, X_t >$

$$W(t,\theta) = X_t(\theta) - 2Rez(t)q(\theta) \tag{3.48}$$

On the center manifold C_0, we have

$W(t,\theta) = W(z(t),\bar{z}(t),\theta)$ where

$$W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + ... \tag{3.49}$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$.

Note that W is real if X_t is real. We consider only real solutions. For solution $X_t \in C_0$ of equation (3.41), since $\mu = 0$ we have

$$\begin{aligned}\dot{z}(t) &= i\omega_0\tau_{1_0}z + \langle \bar{q}^*(0)f(0, W(z,\bar{z},0) + 2Rezq(\theta))\rangle \\ &\cong i\omega_0\tau_{1_0}z + \bar{q}^*(0)f_0(z,\bar{z}) \\ &= i\omega_0\tau_{1_0}z + g(z,\bar{z})\end{aligned} \tag{3.50}$$

where

$$\begin{aligned}g(z,\bar{z}) &= \bar{q}^*(0)f_0(z,\bar{z}) \\ &= g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{z^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + ...\end{aligned} \tag{3.51}$$

From (3.48) and (3.49), we get

$$\begin{aligned} X_t(\theta) &= W(t,\theta)+2Rez(t)q(\theta) \\ &= W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\bar{z}+W_{02}(\theta)\frac{\bar{z}^2}{2}+zq+\bar{z}\bar{q}+.... \qquad (3.52) \\ &= W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\bar{z}+W_{02}(\theta)\frac{\bar{z}^2}{2}+(1\alpha\beta)^T e^{i\omega_0\tau_{1_0}}z+(1\bar{\alpha}\bar{\beta})^T e^{i\omega_0\tau_{1_0}}\bar{z}+.... \end{aligned}$$

Hence we have

$$\begin{aligned} g(z,\bar{z}) &= \bar{q^*}(0)f_0(z,\bar{z}) \\ &= \bar{q^*}(0)f(0,X_t) \\ &= \tau_{1_0}\bar{M}(1\ \bar{\alpha^*}\ \bar{\beta^*})T \\ &= \tau_{1_0}\bar{M}(p_1z^2+2p_2z\bar{z}+p_3\bar{z}^2+p_4z^2\bar{z})+H.O.T \end{aligned} \qquad (3.53)$$

where $T=\begin{pmatrix} -ax_{1t}^2(0)-cx_{1t}(0)x_{3t}(-1) \\ \frac{kx_{1t}(0)x_{3t}^2(0)}{x_{2t}(0)+x_{3t}(0)}-\frac{wx_{1t}(0)x_{2t}(0)x_{3t}(0)}{x_{2t}(0)+x_{3t}(0)} \\ \frac{x_{1t}(\frac{-\tau_{2^*}}{\tau_1})x_{2t}(0)x_{3t}(0)}{x_{2t}(0)+x_{3t}(0)} \end{pmatrix}$

$p_1=-a-c\beta e^{-i\omega_0\tau_{1_0}}$, $p_2=-a-\frac{c}{2}\left[\beta e^{-i\omega_0\tau_{1_0}}+\bar{\beta}e^{i\omega_0\tau_{1_0}}\right]$, $p_3=-a-c\bar{\beta}e^{i\omega_0\tau_{1_0}}$,

$p_4=-aW_{20}^{(1)}(0)-2aW_{11}^{(1)}(0)-c\left(\frac{W_{20}^{(3)}(-1)}{2}+\frac{\bar{\beta}e^{i\omega_0\tau_{1_0}}W_{20}^{(1)}(0)}{2}+W_{11}^{(3)}(-1)+\beta e^{-i\omega_0\tau_{1_0}}W_{11}^{(1)}(0)\right)$.

Comparing (3.51) and (3.53)

$g_{20}=2\tau_{1_0}\bar{M}p_1$, $g_{11}=2\tau_{1_0}\bar{M}p_2$, $g_{02}=2\tau_{1_0}\bar{M}p_3$, $g_{21}=2\tau_{1_0}\bar{M}p_4$.

For unknown $W_{20}^{(i)}(\theta)$, $W_{11}^{(i)}(\theta)$, i=1,2 in g_{21}, we still have to compute them. From (3.45) and (3.48)

$$\begin{aligned} \dot{W} &= \dot{X}_t-\dot{z}q-\dot{\bar{z}}\bar{q} \\ &= \begin{cases} AW-2Re\{\bar{q^*}(0)f_0q(\theta)\} & ,-1\leq\theta\leq 0, \\ AW-2Re\{\bar{q^*}(0)f_0q(\theta)\}+f_0 & ,\theta=0, \end{cases} \qquad (3.54) \\ \acute{W} &= AW+H(z,\bar{z},\theta) \end{aligned}$$

where

$$H(z,\bar{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\bar{z}+H_{02}(\theta)\frac{\bar{z}^2}{2}+..... \qquad (3.55)$$

From (3.54) and (3.55)

$$\begin{aligned} W_{20}(\theta) &= -H_{20}(\theta) \\ A(0)W_{11}(\theta) &= -H_{11}(\theta) \end{aligned} \tag{3.56}$$

From (3.54) we have for $\theta \in [-1,0)$

$$H(z,\bar{z},\theta) = -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta) \tag{3.57}$$

Comparing (3.55) and (3.58)

$$\begin{aligned} H_{20}(\theta) &= -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta) \\ H_{11}(\theta) &= -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta) \end{aligned} \tag{3.58}$$

Using definitions of $A(\theta)$ and from the above equations

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_{1_0}}q(0)e^{i\omega_0\tau_{1_0}\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau_{1_0}}\bar{q}(0)e^{-i\omega_0\tau_{1_0}\theta} + E_1e^{2i\omega_0\tau_{1_0}\theta}. \tag{3.59}$$

and

$$W_{11}(\theta) = \frac{-ig_{11}}{\omega_0\tau_{1_0}}q(0)e^{i\omega_0\tau_{1_0}\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_{1_0}}\bar{q}(0)e^{-i\omega_0\tau_{1_0}\theta} + E_2. \tag{3.60}$$

where $q(\theta) = (1\ \alpha\ \beta)^T e^{i\omega_0\tau_{1_0}\theta}$, $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)}) \in R^3$ and $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)}) \in R^3$ are constant vectors. From (3.54) and (3.55)

$$\begin{aligned} H_{20}(0) &= -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_{1_0}(c_1\ c_2\ c_3)^T \\ H_{11}(0) &= -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\tau_{1_0}(d_1\ d_2\ d_3)^T \end{aligned} \tag{3.61}$$

where $(c_1\ c_2\ c_3)^T = C_1$, $(d_1\ d_2\ d_3)^T = D_1$ are respective coefficients of z^2 and $z\bar{z}$ of $f_0(z\ \bar{z})$ and they are

$$C_1 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -a - c\beta e^{-i\omega_0\tau_{1_0}} \\ 0 \\ 0 \end{pmatrix} \text{ and } D_1 = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} -2a - c\left(\beta e^{-i\omega_0\tau_{1_0}} + \bar{\beta}e^{i\omega_0\tau_{1_0}}\right) \\ 0 \\ 0 \end{pmatrix}$$

Finally we have $(2i\omega_0\tau_{1_0}I - \int_{-1}^{0} e^{2i\omega_0\tau_{1_0}\theta}d\eta(\theta))E_1 = 2\tau_{1_0}C_1$ or $C^*E_1 = 2C_1$ Where

$C^* =$

$$\begin{vmatrix} 2i\omega_0 + ax^* & 0 & cx^* e^{-2i\omega_0 \tau_{1_0}} \\ \frac{-y_2^*(ky_2^* - wy_1^*)}{y_1^* + y_2^*} & \frac{(kx^* y_2^{*2} + dy_1^{*2} + 2dy_1^* y_2^* + dy_2^{*2} + x^* y_2^{*2} w)}{(y_1^* + y_2^*)^2} + 2i\omega_0 & \frac{-x^*(2ky_2^* y_1^* + ky_2^{*2} - wy_1^{*2})}{(y_1^* + y_2^*)^2} \\ \frac{-y_2^* y_1^* e^{-2i\omega^* \tau_2}}{(y_1^* + y_2^*)} & \frac{-x^* y_2^{*2}}{(y_1^* + y_2^*)^2} & \frac{x^* y_1^* y_2^*}{(y_1^* + y_2^*)^2} + 2i\omega_0 \end{vmatrix}, \tag{3.62}$$

Thus $E_1^i = \frac{2\Delta i}{\Delta}$ where $\Delta = Det(C^*)$ and Δ_i be the value of the determinant U_i, where U_i formed by replacing i^{th} column vector of C^* by another column vector $(c_1\ c_2\ c_3)^T$, i =1, 2, 3.

Similarly $D^* E_2 = 2D_1$, where

$D^* =$

$$\begin{vmatrix} ax^* & 0 & cx^* \\ \frac{-y_2^*(ky_2^* - wy_1^*)}{y_1^* + y_2^*} & \frac{(kx^* y_2^{*2} + dy_1^{*2} + 2dy_1^* y_2^* + dy_2^{*2} + x^* y_2^{*2} w)}{(y_1^* + y_2^*)^2} & \frac{-x^*(2ky_2^* y_1^* + ky_2^{*2} - wy_1^{*2})}{(y_1^* + y_2^*)^2} \\ \frac{-y_2^* y_1^*}{(y_1^* + y_2^*)} & \frac{-x^* y_2^{*2}}{(y_1^* + y_2^*)^2} & \frac{x^* y_1^* y_2^*}{(y_1^* + y_2^*)^2} \end{vmatrix}, \tag{3.63}$$

Thus $E_2^i = \frac{2\bar{\Delta} i}{\bar{\Delta}}$ where $\bar{\Delta} = Det(D^*)$ and $\bar{\Delta}_i$ be the value of the determinant V_i, where V_i formed by replacing i^{th} column vector of D^* by another column vector $(d_1\ d_2\ d_3)^T$, i =1,2, 3. Thus we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (3.62) and (3.63). Furthermore using them we can compute g_{21} and derive the following values.

$C_1(0) = \frac{i}{2\omega_0 \tau_{1_0}}(g_{20} g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2}$,

$\mu_2 = \frac{-Re\{C_1(0)\}}{Re\left\{\frac{d\lambda(\tau_{1_0})}{d\tau}\right\}}$,

$\beta_2 = 2Re\{C_1(0)\}$,

$T_2 = \frac{-Im\left\{C_1(0) + \mu_2 Im\left\{\frac{d\lambda(\tau_{1_0})}{d\tau}\right\}\right\}}{\omega_0 \tau_{1_0}}$.

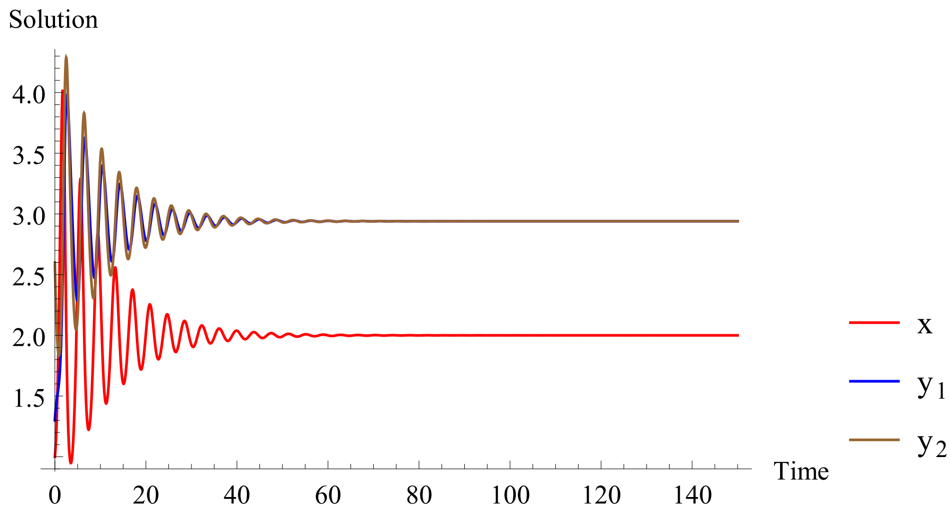
These formulas give a description of the Hopf bifurcation periodic solutions of (3.2) at $\tau = \tau_{1_0}$ on the center manifold. Hence we have the following result.

Theorem 3.7 *The periodic solutions is supercritical (resp. subcritical) if $\mu_2 > 0$ (resp. $\mu_2 < 0$). The bifurcating periodic solutions are orbitally asymptotically stable with an asymptotical phase (resp. unstable) if $\beta_2 < 0$ (resp. $\beta_2 > 0$). The period of bifurcating periodic solutions increases(resp. decreases) if $T_2 > 0$ (resp.$T_2 < 0$).*

4. Numerical Simulation

Fig 1: $\tau_1 = 0$ and $\tau_2 = 0$

$(x(\theta), y_1(\theta), y_2(\theta))^T = (1, 1.3, 2.6)^T$



In this section, we have investigated a class of predator prey system (3.2) with two delays. By means of analysis approach, we give the criteria for the boundedness, permanence and existence of positive periodic solutions. From section 3.3, we may determine the direction of a Hopf bifurcation and the stability of the bifurcation periodic solutions.

The parameters are chosen as follows. $r = 3$, $k_1 = 2$, $k_2 = 1$, $w = 0.5$, $d = 0.5$, $\beta = 1$, $\alpha = 0.03$. Then (3.2) becomes

$$\begin{aligned} \dot{x}(t) &= x(3 - 0.03x - y_2(t - \tau_1)) \\ \dot{y_1}(t) &= \frac{xy_2^2}{y_1 + y_2} - 0.5y_1 - 0.5\frac{xy_2y_1}{y_1 + y_2} \\ \dot{y_2}(t) &= \frac{x(t - \tau_2)y_2y_1}{y_1 + y_2} - y_2 \end{aligned} \tag{4.1}$$

which has a positive equilibrium $E^*(x, y_1, y_2) = (2, 2.94, 2.94)$. When $\tau_1 = 0, \tau_2 = 0$, the equilibrium E^* is asymptotically stable if $d < 1$ and is unstable if $d > 1$. Fig 1 shows that the positive solutions of () approach E^* is an oscillatory form if E^* is stable. Hence less mortality rate of juvenile predators relative to that of adult predators has a stabilizing effect and a larger one destabilizes the equilibrium and produces cycle. Fig 2, 3, 4 and 5 shows that the steady state is asymptotically stable, though damped oscillations can be observed. The time delays are $[\tau_1 = 0.06,\ \tau_2 = 0]$, $[\tau_2 = 0.05,\ \tau_1 = 0]$,$[\tau_1 = \tau_2$ i.e., $\tau_1 = 0.02,\ \tau_2 = 0.02]$ and $[\tau_1 \neq \tau_2$ i.e., $\tau_1 = 0.02,\ \tau_2 = 0.03]$ respectively.

Fig 2: $\tau_1 = 0.06$ and $\tau_2 = 0$

$(x(\theta), y_1(\theta), y_2(\theta))^T = m(1, 1.8, 2.9)^T$

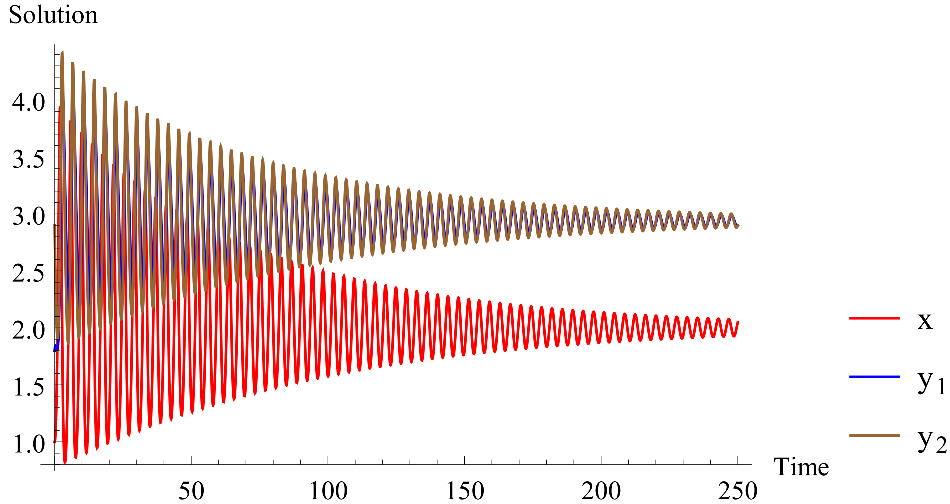
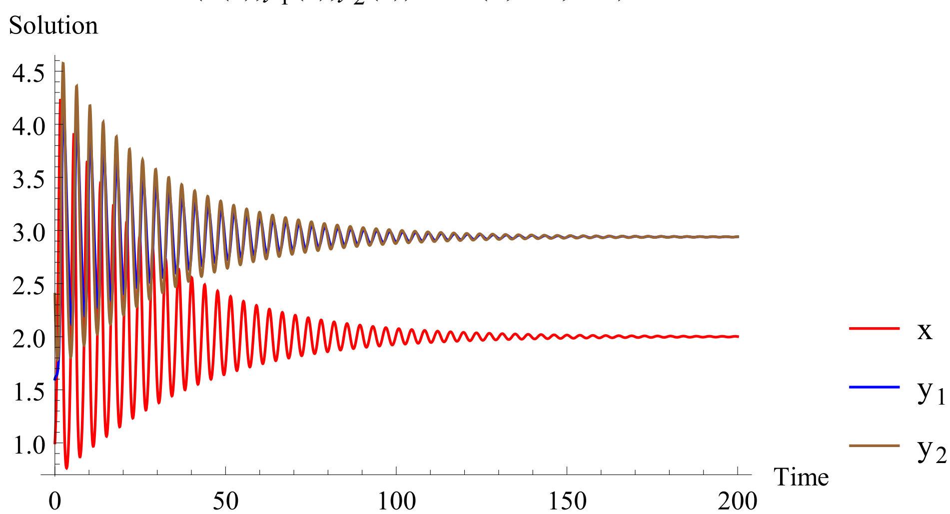


Fig 3: $\tau_1 = 0$ and $\tau_2 = 0.05$

$(x(\theta), y_1(\theta), y_2(\theta))^T = m(1, 1.8, 2.9)^T$



5. Discussion

In this paper, a stage structured predator prey system with two dicrete delays which is an extension of the ordinary differential equation model studied by [8]. For non delay case, if the prey grows in the form of the logistic type and the transition rate is the linear function of the nutrient availability to one immature predator in unit time, then the model has a periodic solution and a positive equilibrium of the model admits multiple stability switches as one of the parameters w and d_1 changes. Based on the system proposed in [14], we further incorporate

Fig 4: $\tau_1 = 0.02$ and $\tau_2 = 0.02$

$(x(\theta), y_1(\theta), y_2(\theta))^T = (1, 1.5, 2.5)^T$

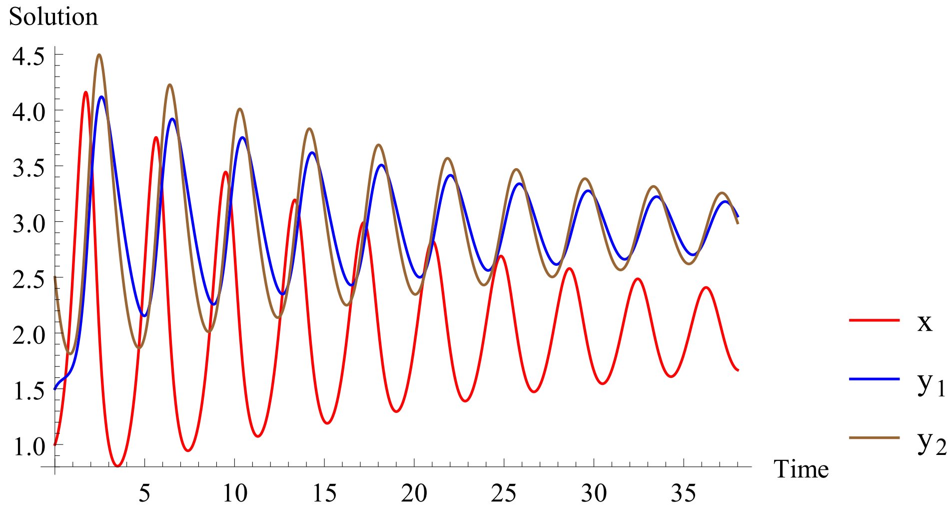
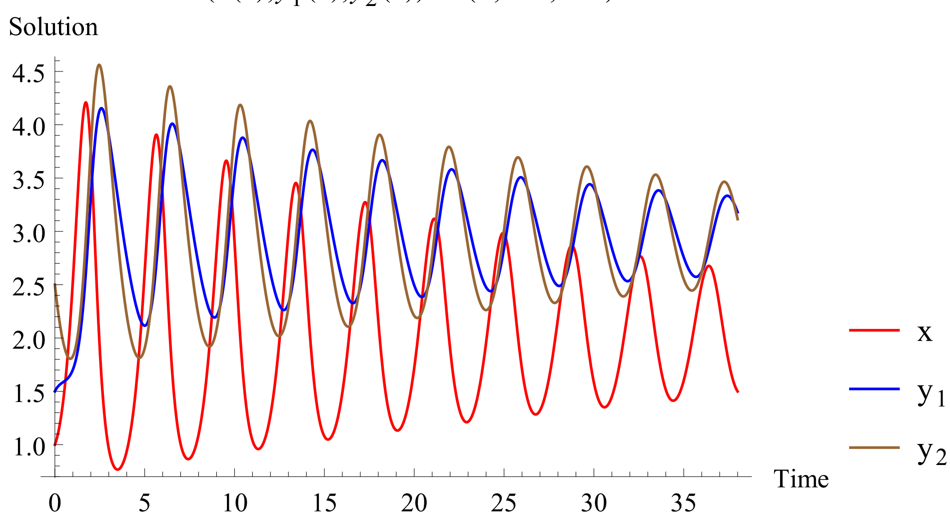


Fig 5: $\tau_1 = 0.02$ and $\tau_2 = 0.03$

$(x(\theta), y_1(\theta), y_2(\theta))^T = (1, 1.5, 2.5)^T$



time delays due to gestation and maturation. The main purpose of this paper is to investigate the effects of two delays on the system for logistic growth of prey. By choosing the possible combinations of the two delays as bifurcating parameters, sufficient conditions for local stability and existence of local Hopf bifurcation are obtained. When the time delay is below the corresponding critical value, we get that the system is local stable. Otherwise, a local Hopf bifurcation occurs at the positive equilibrium. We give the sharp threshold conditions which are both necessary and sufficient for the permanence of the system (1.2) and by theorem 3.6, we give the sufficient conditions for the global stability of the coexistence equilibrium. By theorem

2.2, we found that the system (1.2) is permanent if $g(x) - \beta M_2 > 0$ holds true. The properties of the bifurcated periodic solutions such as the direction and the stability are determined. And a numerical example is also given to support the theoretical results.We found that small sufficiently delays cannot change the stability of positive equilibrium solution and large delays cannot only destabilize the positive equilibrium solution but also cause an oscillation near the positive equilibrium solution. Hence we can see that the species in the system considered in this paper can coexist under some certain conditions. Further investigations of this problem is presently in progress.

Conflict of Interests

The authors declare that there is no conflict of interests.

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