Available online at http://scik.org
J. Math. Comput. Sci. 7 (2017), No. 4, 642-657
ISSN: 1927-5307

TOPOLOGIES AND APPROXIMATION OPERATORS INDUCED BY BINARY RELATIONS

NURETTIN BAĞIRMAZ[1,*], A. FATIH ÖZCAN[2], HATICE TAŞBOZAN[3] AND İLHAN İÇEN[2]

[1]Mardin Artuklu University, Mardin, Turkey

[2]Department of Mathematics, Inonu University, Malatya, Turkey

[3]Department of Mathematics, Mustafa Kemal University, Hatay, Turkey

Abstract. Theory of rough sets is an extension of classic set theory, which is an important mathematical tool for dealing with uncertain or vague information. The core concepts of classical rough sets are lower and upper approximations based on equivalence relations. Topologies via relations have very important role in rough set theory. This paper studies some new topologies induced by a binary relation on universe with respect to neighborhood operators. Moreover, the relations among them are studied. In addition, lower and upper approximations of rough sets using the binary relation with respect to neighborhood operators are studied and examples are given.

Keywords: topological spaces; rough sets; binary relations; lower and upper approximations.

2010 AMS Subject Classification: 54A99, 68P01.

1. Introduction

---

*Corresponding author

E-mail address: nurettinbagirmaz@artuklu.edu.tr

Received March 7, 2017

The theory of rough sets was introduced by Pawlak as a mathematical tool to process information with uncertainty and vagueness [11].The theory of rough sets deals with the approximation of sets for classification of objects through equivalence relations. Important applications of the rough set theory have been applied in many fields, for example in medical science, data analysis, knowledge discovery in database [12, 13, 15, 20]

In the original theory of rough sets, an ordered pair (U,R) is called an approximation space, where U is a finite non- empty set called universe and R is an equivalent relation on U. But for practical use, it needs to some extensions on original rough set concept. This is to replace the equivalent relation by a general binary relation [4, 5, 6, 10, 16, 23, 24, 25, 26, 27]. We call the ordered pair (U,R) as a generalized approximation space when R is a binary relation on U.Topology is one of the most important subjects in mathematics, which to study rough sets. Many authors studied relationship between rough sets and topologies based on binary relations [1, 2, 7, 8, 9, 14, 17, 18, 19, 21, 22]

In this paper, we proposed and studied connections between topologies generated using successor, predecessor, successor-and-predecessor, and successor-or-predecessor neighborhood operators as a subbase by various binary relations on a universe, respectively. Let R be a binary relation on a given universe U. Sets

$$\mathscr{S}_i = \bigcup\{R_i(x)|x \in U\} \text{ ,where } i : s, p, s \wedge p \textit{ and } s \vee p$$

defined by successor, predecessor, successor-and-predecessor, and successor-or-predecessor neighborhood operators, respectively. If $\mathscr{S}_i$ *,where* $i : s,\ p,\ s \wedge p$ *and* $s \vee p$, forms a subbase for some topology on U, then the topology generated $\mathscr{S}_i$, denoted T_i. Z. Pei at al. in [14] used $\mathscr{S}_s$ as a subbase for some topology on U if and only if R is invers serial which for each $x \in U$, there exists $y \in U$ such that $(y,x) \in R$. A basic problem is: when does $\mathscr{S}_i$*,where* $i : s,\ p,\ s \wedge p$ *and* $s \vee p$, form a subbase for some topologies on U ? We solve this problem completely and the relations among this topologies are studied in section 3. In addition to this, we investigate connection between lower and upper approximation operators using successor, predecessor, successor-and-predecessor, and successor-or-predecessor neighborhood operators by various binary relations on a universe, respectively in section 4. Moreover, we give several examples for a better understanding of the subject.

2. Preliminaries

In this section, we shall briefly review basic concepts and relational propositions of the relation based rough sets and topology. For more details, we refer to [24, 25, 14].

2.1. Basic properties relation based rough approximations and neighborhood operators

In this paper, we always assume that U is a finite universe, i.e., a non-empty finite set of objects, R is a binary relation on U, i.e., a subset of $U^2 = U \times U$ [14].

R is serial if for each $x \in U$, there exists $y \in U$ such that $(x,y) \in R$; R is inverse serial if for each $x \in U$, there exists $y \in U$ such that $(y,x) \in R$; R is reflexive if for each $x \in U$, $(x,x) \in R$; R is symmetric if for all $x,y \in U$, $(x,y) \in R$ implies $(y,x) \in R$; R is transitive if for all $x,y,z \in U$, $(x,y) \in R$ and $(y,z) \in R$ imply $(x,z) \in R$ [9].

R is called a pre-order relation if R is both reflexive and transitive; R is called a similarity (or, tolerance) relation if R is both reflexive and symmetric; R is called an equivalence relation if R is reflexive, symmetric and transitive [14].

Given a universe U and a binary relation R on U, $x,y \in U$, the sets

$$R_s(x) = \{y \in U | (x,y) \in R\},$$

$$R_p(x) = \{y \in U | (y,x) \in R\},$$

$$R_{s\wedge p}(x) = \{y \in U | (x,y) \in R \textit{ and } (y,x) \in R\} = R_s(x) \cap R_p(x),$$

$$R_{s\vee p}(x) = \{y \in U | (x,y) \in R \textit{ or } (y,x) \in R\} = R_s(x) \cup R_p(x)$$

are called the successor, predecessor, successor-and-predecessor, and successor-or-predecessor neighborhood of x, respectively, and the following four set-valued operators from U to the power set $P(U)$

$$R_s : x \mapsto R_s(x),$$

$$R_p : x \mapsto R_p(x),$$

$$R_{s\wedge p} : x \mapsto R_{s\wedge p}(x),$$

$$R_{s\vee p} : x \mapsto R_{s\vee p}(x)$$

are called the successor, predecessor, successor-and-predecessor, and successor-or-predecessor neighborhood operators, respectively. Relationships between these neighborhood systems can be expressed as:

$$R_{s\wedge p}(x) \subseteq R_s(x) \subseteq R_{s\vee p}(x),$$

$$R_{s\wedge p}(x) \subseteq R_p(x) \subseteq R_{s\vee p}(x)$$

[25, 14].

Definition 2.1 [25] Let R be a binary relation on U. The ordered pair (U,R) is called a generalized approximation space based on the relation R. For $X \subseteq U$, the lower and upper approximations of X with respect to $R_s(x)$, $R_p(x)$, $R_{s\wedge p}(x)$, $R_{s\vee p}(x)$ are respectively defined as follows:

$$\underline{apr}_{R_s}(X) = \{x \in U | R_s(x) \subseteq X\},$$

$$\underline{apr}_{R_p}(X) = \left\{x \in U | R_p(x) \subseteq X\right\},$$

$$\underline{apr}_{R_{s\wedge p}}(X) = \left\{x \in U | R_{s\wedge p}(x) \subseteq X\right\},$$

$$\underline{apr}_{R_{s\vee p}}(X) = \left\{x \in U | R_{s\vee p}(x) \subseteq X\right\},$$

$$\overline{apr}_{R_s}(X) = \{x \in U | R_s(x) \cap X \neq \emptyset\},$$

$$\overline{apr}_{R_p}(X) = \left\{x \in U | R_p(x) \cap X \neq \emptyset\right\},$$

$$\overline{apr}_{R_{s\wedge p}}(X) = \left\{x \in U | R_{s\wedge p}(x) \cap X \neq \emptyset\right\},$$

$$\overline{apr}_{R_{s\vee p}}(X) = \left\{x \in U | R_{s\vee p}(x) \cap X \neq \emptyset\right\}.$$

In Pawlak's classical rough set theory for lower and upper approximations operators, an equivalence relation R is used. In this case, four neighborhood operators become the same, i.e., $R_s(x) = R_p(x) = R_{s\wedge p}(x) = R_{s\vee p}(x) = [x]_R$, where $[x]_R$ is the equivalence class containing x.

Proposition 2.2 [24] For an arbitrary neighborhood operator in an approximation space (U,R), the pair of approximation operators satisfy the following properties:

$(L0)$ $\underline{apr}(X) = (\overline{apr}(X^c))^c$,
$(U0)$ $\overline{apr}(X) = \left(\underline{apr}(X^c)\right)^c$,
$(L1)$ $\underline{apr}(U) = U$,
$(U1)$ $\underline{apr}(\emptyset) = \emptyset$,
$(L2)$ $\underline{apr}(X \cap Y) = \underline{apr}(X) \cap \underline{apr}(Y)$,
$(U2)$ $\overline{apr}(X \cup Y) = \overline{apr}(X) \cup \overline{apr}(Y)$.

where X^c is the complement of X with respect to U.

Moreover, if R is reflexive, then

$(L3)$ $\underline{apr}(X) \subseteq X$,
$(U3)$ $X \subseteq \overline{apr}(X)$.

If R is symmetric, then

$(L4)$ $X \subseteq \underline{apr}(\overline{apr}(X))$,
$(U4)$ $\overline{apr}(\underline{apr}(X)) \subseteq X$.

If R is transitive, then

$(L5)$ $\underline{apr}(X) \subseteq \underline{apr}(\underline{apr}(X))$,
$(U5)$ $\overline{apr}(\overline{apr}(X)) \subseteq \overline{apr}(X)$.

2.2. The concept of a topological space

In this section, we give some basic information about the topology [3, 14].

Definition 2.3 [3] A topological space is a pair (U,T) consisting of a set U and a set T of subsets of U (called "open sets"), such that the following axioms hold:

$(A1)$ Any union of open sets is open.

$(A2)$ The intersection of any two open sets is open.

$(A3)$ $\emptyset$ and U are open.

The pair (U,T) speaks simply of a topological space U.

Definition 2.4 [14] Let U be a topological space.

(1) $X \subseteq U$ is called closed when X^c is open.

(2) $X \subseteq U$ is called a neighborhood of $x \in X$ if there is an open set V with $x \in V \subseteq X$.

(3) A point x of a set X is an interior point of X if X is a neighborhood of x, and the set of all interior points of X is called the interior of X. The interior of X is denoted by $\overset{o}{X}$.

(4) The closure of a subset X of a topological space U is the intersection of the family of all closed sets containing X. The closure of X is denoted by $\overline{X}$.

In topological space U, the operator

$$^{o} : P(U) \to P(U),\ X \mapsto \overset{o}{X}$$

is an interior operator on U and for all $X,Y \subseteq U$ the following properties hold:

I1) $\overset{o}{U} = U$,

I2) $\overset{o}{X} \subseteq X$,

I3) $\overset{o}{\left(\overset{o}{X}\right)} = X$,

I4) $\left(X \overset{o}{\cap} Y\right) = \left(\overset{o}{X}\right) \cap \left(\overset{o}{Y}\right)$.

In topological space U, the operator

$$^{-} : P(U) \to P(U),\ X \mapsto \overline{X}$$

is a closure operator on U and for all $X,Y \subseteq U$ the following properties hold:

C1) $\overline{\emptyset} = \emptyset$,

C2) $X \subseteq \overline{X}$

C3) $\overline{\overline{X}} = X$,

C4) $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$.

In a topological space (U,T) a family $\mathscr{B} \subseteq T$ of sets is called a base for the topology T if for each point x of the space, and each neighborhood X of x, there is a member V of $\mathscr{B}$ such that $x \in V \subseteq X$. We know that a subfamily $\mathscr{B}$ of a topology T is a base for T if and only if each member of T is the union of members of $\mathscr{B}$. Moreover, $\mathscr{B} \subseteq P(U)$ forms a base for some topologies on U if and only if $\mathscr{B}$ satisfies the following conditions:

B1) $U = \bigcup \{B | B \in \mathscr{B}\}$,

B2) For every two members X and Y of $\mathscr{B}$ and each point $x \in X \cap Y$, there is $Z \in \mathscr{B}$ such that $x \in Z \subseteq X \cap Y$.

Also, a family $\mathscr{S} \subseteq T$ of sets is a subbase for the topology T if the family of all finite intersections of members of $\mathscr{S}$ is a base for T. Moreover, $\mathscr{S} \subseteq P(U)$ is a subbase for some topology on U if and only if $\mathscr{S}$ satisfies the following condition:

(S0) $U = \bigcup \{S | S \in \mathscr{S}\}$.

3. Generating topologies by relations

In this section, we show the subbase generated using successor, predecessor, successor-and-predecessor, and successor-or-predecessor neighborhood operators by various binary relations on a universe, respectively. Then, we introduce the topologies generated by these subbases and compare these topologies.

Let R be a binary relation on a given universe U. Sets

$$\mathscr{S}_i = \bigcup \{R_i(x) | x \in U\} \text{ ,where } i : s, p, s \wedge p \textit{ and } s \vee p$$

defined by successor, predecessor, successor-and-predecessor, and successor-or-predecessor neighborhood operators, respectively. If $\mathscr{S}_i$, forms a subbase for some topology on U, then the topology generated by $\mathscr{S}_i$, denoted by T_i, *where* $i : s$, p, $s \wedge p$ *and* $s \vee p$, respectively.

The following theorem states that a inverse serial is sufficient for the $\mathscr{S}_s$ forms a subbase for some topologies on U.

Theorem 3.1[14] If R is a binary relation on U, then $\mathscr{S}_s$ forms a subbase for some topologies on U if and only if R is inverse serial.

Remark 3.2 It is clear that if R is inverse serial on U, then

$$\forall x \in U, R_p(x) \neq \varnothing \textit{ and } U = \bigcup_{x \in U} R_s(x).$$

Thus the family $\mathscr{S}_s$ is covering of U.

Theorem 3.3 If R is a binary relation on U, then $\mathscr{S}_p$ forms a subbase for some topologies on U if and only if R is serial.

Proof. If R is serial, then

$$\forall x \in U, R_s(x) \neq \varnothing \textit{ and } U = \bigcup_{x \in U} R_p(x).$$

Thus the family S_p provides the condition (S0). Therefore $\mathscr{S}_p$ forms a subbase for some topologies on U. □

Example 3.4 Let $U = \{a,b,c,d\}$ and

$$R = \{(a,a),(a,c),(b,b),(b,c),(c,a),(d,c)\}.$$

Then R is a serial relation on U, and

$$R_s(a) = \{a,c\},\ R_s(b) = \{c\},\ R_s(c) = \{a\},\ R_s(d) = \{c\},$$

$$R_p(a) = \{a,c\},\ R_p(b) = \{b\},\ R_p(c) = \{b,d\},\ R_p(d) = \varnothing.$$

Thus

$$\mathscr{S}_p = \bigcup \left\{R_p(x) | x \in U\right\} = \{\{a,c\},\{b\},\{b,d\}\}.$$

Hence

$$T_p = \{\varnothing, U, \{b\},\{b,d\},\{a,c\},\{a,b,c\}\}.$$

Lemma 3.5 Let U be the universe and R is a symmetric relation. Then

$$R \textit{ is a serial relation} \Longleftrightarrow R \textit{ is an inverse serial relation}.$$

Proof. Suppose that R is a symmetric relation on U.

$$\begin{aligned} R\ is\ a\ serial\ relation &\Longleftrightarrow \forall x \exists y[(x,y) \in R] \\ &\Longleftrightarrow \forall x \exists y[(y,x) \in R] \\ &\Longleftrightarrow R\ is\ an\ inverse\ serial\ relation \end{aligned}$$

□

Theorem 3.6 If R is a binary relation on U, then $\mathscr{S}_{s\wedge p}$ forms a subbase for some topologies on U if and only if R is symmetric and serial or inverse serial.

Proof. From Lemma 3.5 and Theorem 3.3 we get

$$R_s(x) = R_p(x) \neq \varnothing.$$

Thus

$$\mathscr{S}_{s\wedge p} = \bigcup \left\{R_{s\wedge p}(x) | x \in U\right\} = \bigcup \{R_s(x) | x \in U\} = \bigcup \left\{R_p(x) | x \in U\right\} \neq \varnothing$$

, therefore $\mathscr{S}_{s\wedge p}$ forms a subbase for some topologies on U. □

Theorem 3.7 If R is a binary relation on U, then $\mathscr{S}_{s\vee p}$ forms a subbase for some topologies on U if and only if R is serial or inverse serial.

Proof. If R is serial or inverse serial on U, then

$$\forall x \in U, R_{s\vee p}(x) \neq \varnothing \ and\ U = \bigcup_{x\in U} R_{s\vee p}(x).$$

Thus the family $\mathscr{S}_{s\vee p}$ provides the condition (S0). Therefore $\mathscr{S}_{s\vee p}$ forms a subbase for some topologies on U. □

Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be covering of U. A partition $\mathscr{S}_1$ is finner than $\mathscr{S}_2$, or is coarser than $\mathscr{S}_1$, for each neighborhood operator in $\mathscr{S}_1$ produced by x, is subset the neighborhood operator in $\mathscr{S}_2$ by x. This relation is denoted as $\mathscr{S}_1 \preceq \mathscr{S}_2$.

$$\mathscr{S}_1 \preceq \mathscr{S}_2 \Longleftrightarrow \text{if every set of } \mathscr{S}_1 \text{ is contained in some sets of } \mathscr{S}_2, \text{ for all } x \in U.$$

Moreover, T_1 and T_2 are two topologies on U and $T_1 \subseteq T_2$, then we say that T_2 is finer than T_1.

In effect, the following proposition holds.

Proposition 3.8 Let U be the universe and R is general binary relation. Then

$$R_i(x) \subseteq R_j(x) \Leftrightarrow \mathscr{S}_i \preceq \mathscr{S}_j, for\ all\ x \in U, where\ i, j : s, p, s \wedge pands \vee p.$$

Proposition 3.9 Let U be the universe and R is a serial relation. Then the following conditions are provided:

(1) $\mathscr{S}_p \preceq \mathscr{S}_{s\vee p}$,

(2) $T_p \preceq T_{s\vee p}$.

Proof. Clearly, (2) is a direct corollary of (1) We only prove (1).

Since $R_p(x) \subseteq R_{s\vee p}(x)$ for all $x \in U$,then from Proposition 3.8 we get $\mathscr{S}_p \preceq \mathscr{S}_{s\vee p}$. □

Proposition 3.10 Let U be the universe and R is an inverse serial relation. Then the following conditions are provided:

(1) $\mathscr{S}_s \preceq \mathscr{S}_{s\vee p}$,

(2) $T_s \preceq T_{s\vee p}$.

Proof. Clearly, (2) is a direct corollary of (1) We only prove (1).

Since $R_s(x) \subseteq R_{s\vee p}(x)$ for all $x \in U$,then from Proposition 3.8 we get $\mathscr{S}_s \preceq \mathscr{S}_{s\vee p}$. □

The next proposition presents equality between $\mathscr{S}_{s\wedge p}, \mathscr{S}_s, \mathscr{S}_p$ and $\mathscr{S}_{s\vee p}$ subbases, and topologies generated by them that are equality each other.

Proposition 3.11 Let U be the universe and R is a symmetric and a serial (or inverse serial) relation. Then the following conditions are provided:

(1) $\mathscr{S}_{s\wedge p} = \mathscr{S}_s = \mathscr{S}_p = \mathscr{S}_{s\vee p}$,

(2) $T_{s\wedge p} = T_s = T_p = T_{s\vee p}$.

Proof. (1) If R is a symmetric and a serial (or inverse serial) relation on U, then $R_{s\wedge p}(x) = R_s(x) = R_p(x) = R_{s\vee p}(x)$. Therfore $\mathscr{S}_{s\wedge p} = \mathscr{S}_s = \mathscr{S}_p = \mathscr{S}_{s\vee p}$.

(2) is a direct corolary of (1). □

Corollary 3.12 Let U be the universe and R is a tolerance (symmetric and reflexive) relation. Then, the following conditions are provided:

(1) $\mathscr{S}_{s\wedge p} = \mathscr{S}_s = \mathscr{S}_p = \mathscr{S}_{s\vee p}$,

(2) $T_{s\wedge p} = T_s = T_p = T_{s\vee p}$.

Proposition 3.13 Let U be the universe and R is a reflexive relation. Then, the following conditions are provided:

(1) $\mathscr{S}_{s\wedge p} \preceq \mathscr{S}_s, \mathscr{S}_p \preceq \mathscr{S}_{s\vee p}$,

(2) $T_{s\wedge p} \preceq T_s, T_p \preceq T_{s\vee p}$.

Proof. (1) Let R is a reflexive relation on U. Then, R are serial and invers serial relation. Since $R_{s\wedge p}(x) = R_s(x) \cap R_p(x)$ and $R_{s\vee p}(x) = R_s(x) \cup R_p(x)$, then $R_{s\wedge p}(x) \subseteq R_s(x), R_p(x) \subseteq R_{s\vee p}(x)$.Thus $\mathscr{S}_{s\wedge p} \preceq \mathscr{S}_s, \mathscr{S}_p \preceq \mathscr{S}_{s\vee p}$.

(2) is a direct corollary of (1). $\square$

Remark 3.14 Notice that the reflexive relation R does not need to hold $R_s(x) = R_p(x)$ for each $x \in U$. This is evident by following example.

Example 3.15 Let $U = \{a,b,c,d\}$ and

$$R = \{(a,a),(a,c),(b,b),(b,c),(c,c),(d,d)\}.$$

Then R is a reflexive relation on U, and

$$R_s(a) = \{a,c\},\ R_s(b) = \{b,c\},\ R_s(c) = \{c\},\ R_s(d) = \{d\},$$

$$R_p(a) = \{a\}, R_p(b) = \{b\}, R_p(c) = \{b,c\}, R_p(d) = \{d\},$$

$$R_{p\wedge s}(a) = \{a\},\ R_{p\wedge s}(b) = \{b\},\ R_{p\wedge s}(c) = \{c\},\ R_{p\wedge s}(d) = \{d\}$$

$$R_{p\vee s}(a) = \{a,c\},\ R_{p\vee s}(b) = \{b,c\},\ R_{p\vee s}(c) = \{b,c\},\ R_{p\vee s}(d) = \{d\}.$$

Thus $R_s(a) \neq R_p(a)$, for $a \in U$. Let us note that $R_{p\wedge s}(a) \subseteq R_s(a), R_p(a) \subseteq R_{p\vee s}(a)$. Therfore $\mathscr{S}_{s\wedge p} \preceq \mathscr{S}_s, \mathscr{S}_p \preceq \mathscr{S}_{s\vee p}$ and $T_{s\wedge p} \preceq T_s, T_p \preceq T_{s\vee p}$.

Corollary 3.16 Let U be the universe and R is a preorder (reflexive and transitive) relation. Then the following conditions are provided:

(1) $\mathscr{S}_{s\wedge p} \preceq \mathscr{S}_s, \mathscr{S}_p \preceq \mathscr{S}_{s\vee p}$,

(2) $T_{s\wedge p} \preceq T_s, T_p \preceq T_{s\vee p}$.

Remark 3.17 If R is a preorder relation on U , $\mathscr{S}_{s\wedge p}$ $(\mathscr{S}_s, \mathscr{S}_p, \mathscr{S}_{s\vee p})$ form a base for $T_{s\wedge p}$ $(T_s, T_p, T_{s\vee p})$ topology on U, respectively. Moreover, this topologies are Alexandrov topologies on U.

Corollary 3.18 Let U be the universe and R is an equivalent relation. Then, the following conditions are provided:

(1) $\mathscr{S}_{s\wedge p} = \mathscr{S}_s = \mathscr{S}_p = \mathscr{S}_{s\vee p}$,

(2) $T_{s\wedge p} = T_s = T_p = T_{s\vee p}$

Remark 3.19 In the case when R is an equivalent relation on U, i.e, (U,R) is a Pawlak approximation space, then the set $\mathscr{S}_{s\wedge p}$ $(\mathscr{S}_s, \mathscr{S}_p, \mathscr{S}_{s\vee p})$ is a base for $T_{s\wedge p}$ $(T_s, T_p, T_{s\vee p})$ topology on U, respectively. In these topologies, each neighborhood operator is one equivalence class for all $x \in U$.

4.Rough approximation operators induced by relations

In this section, we investigate connection between lower and upper approximation operators using successor, predecessor, successor-and-predecessor, and successor-or-predecessor neighborhood operators by various binary relations on a universe, respectively.

Proposition 4.1 Let U be the universe and R is a binary relation. Then, for lower and upper approximation operators of $X \subseteq U$, the following conditions are provided:

1) $\underline{apr}_{p\vee s}(X) \subseteq \underline{apr}_s(X)$, $\underline{apr}_p(X) \subseteq \underline{apr}_{p\wedge s}(X)$,

2)$\overline{apr}_{p\wedge s}(X) \subseteq \overline{apr}_s(X)$, $\overline{apr}_p(X) \subseteq \overline{apr}_{p\vee s}(X)$.

Proof. (1) Let $x \in \underline{apr}_{p\vee s}(X)$ for any $x \in U$. Since $R_{p\vee s}(x) \subseteq X$ and $R_s(x) \subseteq R_{p\vee s}(x)$ then $R_s(x) \subseteq X$ and so $x \in \underline{apr}_s(X)$. Now, since $x \in \underline{apr}_s(X)$ and $R_{p\wedge s}(x) \subseteq R_s(x) \subseteq X$ then $x \in \underline{apr}_{p\wedge s}(X)$. Therefore $\underline{apr}_{p\vee s}(X) \subseteq \underline{apr}_s(X) \subseteq \underline{apr}_{p\wedge s}(X)$. Similarly, $\underline{apr}_{p\vee s}(X) \subseteq \underline{apr}_p(X) \subseteq \underline{apr}_{p\wedge s}(X)$.

(2) Let $x \in \overline{apr}_{p\wedge s}(X)$ for any $x \in U$. Since $R_{p\wedge s}(x) \cap X \neq \emptyset$ and $R_{p\wedge s}(x) \subseteq R_s(x)$, $R_p(x) \subseteq R_{p\vee s}(x)$ then $R_{p\vee s}(x) \cap X \neq \emptyset$ an so $x \in \overline{apr}_{p\vee s}(X)$. Therefore, $\overline{apr}_{p\wedge s}(X) \subseteq \overline{apr}_s(X)$, $\overline{apr}_p(X) \subseteq \overline{apr}_{p\vee s}(X)$. □

Example 4.2 Let $U = \{a,b,c,d\}$ and

$$R = \{(a,a),(a,c),(b,c),(c,a),(c,d)\}$$

be a binary relation on U. Then,

$$R_s(a) = \{a,c\},\ R_s(b) = \{c\},\ R_s(c) = \{a,d\},\ R_s(d) = \emptyset,$$

$$R_p(a) = \{a,c\},\ R_p(b) = \emptyset,\ R_p(c) = \{a,b\},\ R_p(d) = \{c\},$$

$$R_{p\wedge s}(a) = \{a,c\},\ R_{p\wedge s}(b) = \emptyset,\ R_{p\wedge s}(c) = \{a\},\ R_{p\wedge s}(d) = \emptyset,$$

$$R_{p\vee s}(a) = \{a,c\},\ R_{p\vee s}(b) = \{c\},\ R_{p\vee s}(c) = \{a,b,d\},\ R_{p\vee s}(d) = \{c\}.$$

Let $X = \{a,c,d\}$. Then,

$$\underline{apr}_s(X) = \{a,b,c,d\},$$

$$\underline{apr}_p(X) = \{a,b,d\},$$

$$\underline{apr}_{p\wedge s}(X) = \{a,b,c,d\},$$

$$\underline{apr}_{p\vee s}(X) = \{a,b,d\},$$

$$\overline{apr}_s(X) = \{a,b,c\},$$

$$\overline{apr}_p(X) = \{a,c,d\},$$

$$\overline{apr}_{p\wedge s}(X) = \{a,c\},$$

$$\overline{apr}_{p\vee s}(X) = \{a,b,c,d\}.$$

Hence, note that $\underline{apr}_s(X) \supset \overline{apr}_s(X)$.

In the original rough set theory, lower approximation of X is a subset of its upper approximation. In order to provide this condition, we need some properties to add binary relations.

The next propositions presents this conditions.

Proposition 4.3 [4] Let U be the universe and R is a binary relation. Then, for all $X \subseteq U$

$$R \text{ is serial } \Rightarrow \underline{apr}_s(X) \subseteq \overline{apr}_s(X)$$

Corollary 4.4 Let U be the universe and R is a binary relation. Then, for all $X \subseteq U$

$$R \text{ is serial } \Rightarrow \underline{apr}_{p\vee s}(X) \subseteq \underline{apr}_s(X) \subseteq \overline{apr}_s(X) \subseteq \overline{apr}_{p\vee s}(X)$$

Proof. Proof is clear from Proposition 4.1 and Proposition 4.3. □

Proposition 4.5 Let U be the universe and R is a binary relation. Then, for all $X \subseteq U$

$$R \text{ is inverse serial } \Rightarrow \underline{apr}_p(X) \subseteq \overline{apr}_p(X).$$

Proof. Let $x \in \underline{apr}_p(X)$. Then $R_p(x) \subseteq X$, which gives $R_p(x) \cap X = R_p(x) \neq \emptyset$, that is, $x \in \overline{apr}_p(X)$. □

Corollary 4.6 Let U be the universe and R is a binary relation. Then, for all $X \subseteq U$

$$R \text{ is inverse serial } \Rightarrow \underline{apr}_{p\vee s}(X) \subseteq \underline{apr}_p(X) \subseteq \overline{apr}_p(X) \subseteq \overline{apr}_{p\vee s}(X).$$

Proof. Proof is clear from Proposition 4.1 and Proposition 4.5. □

Proposition 4.7 Let U be the universe and R is a binary relation. Then, for all $X \subseteq U$

$$R \text{ is symmetric and serial (or inverse serial)} \Rightarrow \underline{apr}_{p\wedge s}(X) \subseteq \overline{apr}_{p\wedge s}(X).$$

Proof. Let $x \in \underline{apr}_{p\wedge s}(X)$. Then, from Lemma 3.5 $R_{p\wedge s}(x) = R_p(x)$ which gives $\underline{apr}_{p\wedge s}(X) = \underline{apr}_p(X)$. So, from Proposition 4.3 $x \in \overline{apr}_{p\wedge s}(X)$. □

The next propositions present a set between its lower approximation and its upper approximation conditions.

Proposition 4.8 Let U be the universe and R is a binary relation. Then, for all $X \subseteq U$

$$R \text{ is reflexive} \Rightarrow \underline{apr}_i(X) \subseteq X \subseteq \overline{apr}_i(X), \text{ where } i\text{: } s,\ p,\ p\wedge s \text{ and } p\vee s, \text{ respectively.}$$

Proof. Proof is clear from Proposition 2.2. □

Corollary 4.9 Let U be the universe and R is a reflexive or preorder binary relation. Then, for all $X \subseteq U$

(1) $\underline{apr}_{p\vee s}(X) \subseteq \underline{apr}_p(X) \subseteq \underline{apr}_{p\wedge s}(X) \subseteq X \subseteq \overline{apr}_{p\wedge s}(X) \subseteq \overline{apr}_p(X) \subseteq \overline{apr}_{p\vee s}(X)$

(2) $\underline{apr}_{p\vee s}(X) \subseteq \underline{apr}_s(X) \subseteq \underline{apr}_{p\wedge s}(X) \subseteq X \subseteq \overline{apr}_{p\wedge s}(X) \subseteq \overline{apr}_s(X) \subseteq \overline{apr}_{p\vee s}(X)$.

Proof. Proof is clear from Proposition 4.1 and Proposition 4.8. □

Proposition 4.10 Let U be the universe and R is a tolerance or equivalent binary relation. Then, for all $X \subseteq U$

$$\underline{apr}_i(X) = \underline{apr}_j(X) \subseteq X \subseteq \overline{apr}_i(X) = \overline{apr}_j(X)$$

, where $i, j : s, p, s \wedge p$ *and* $s \vee p$.

Proof. If R is a tolerance or equivalent binary relation, then for all $x \in U$,

$$R_{s\wedge p}(x) = R_s(x) = R_p x) = R_{s\vee p}(x)$$

. Thus, for all $X \subseteq U$

$$\underline{apr}_s(X) = \underline{apr}_p(X) = \underline{apr}_{p\wedge s}(X) = \underline{apr}_{p\vee s}(X)$$

and

$$\overline{apr}_s(X) = \overline{apr}_p(X) = \overline{apr}_{p\wedge s}(X) = \overline{apr}_{p\vee s}(X) \ .$$

Therefore, from Proposition

$$\underline{apr}_i(X) = \underline{apr}_j(X) \subseteq X \subseteq \overline{apr}_i(X) = \overline{apr}_j(X),$$

where $i, j : s, p, s \wedge p$ *and* $s \vee p$. □

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. A. Allam, M. Y. Bakeir, E. A. Abo-Tabl, Some Methods for Generating Topologies by Relations, Bull. Malays. Math. Sci. Soc. 31(1) (2008), 35–45.

[2] B. Qin, G. Xia, K. Yan, Similarity of binary relations based on rough set theory and topology: an application for topological structures of matroids, Soft Comput. 20 (3) (2016), 853–861.

[3] K. Janich, Topology, Springer-Verlag, New York,1984.

[4] J. Jarvinen, Properties of Rough Approximations, J. Adv. Comput. Intel. and Intelligent Inform. 9 (5) (2005), 502-505.

[5] J. Jarvinen, Approximations and Rough Sets Based on Tolerances, Lecture Notes in Artificial Intelligence (2001) 182-189.

[6] J. Jarvinen, On the Structure of Rough Approximations, Fund. Inform.53 (2002) 135-153.

[7] M. Kondo, On the structure of generalized rough sets, Inform. Sci. 176 (2006) 589–600.

[8] E.F. Lashin et al, Rough set theory for topological spaces, Int. J. Aprrox. Reason. 40 (2005) 35–43.

[9] Z. Li, T. Xie, Q. Li, Topological structure of generalized rough sets, Comput. Math. with Appl. 63 (2012) 1066–1071.

[10] G. Liu ,Y. Sai, A comparison of two types of rough setsinduced by covering, Int. J. Aprrox. Reason, 50 (2009) 521-528.

[11] Z. Pawlak, Rough sets, Internat. J. Comput. Inform. Sci. 11 (1982) 341–356.

[12] Z. Pawlak, Rough sets and intelligent data analysis, Inform. Sci. 147 (2002) 1-12.

[13] Z. Pawlak, A. Skowron, Rudiments of rough sets, Inform. Sci. 177 (2007) 3-27.

[14] Z. Pei, D. Pei , L. Zheng, Topology vs generalized rough sets, Int. J. Aprrox. Reason. 52 (2011) 231–239.

[15] L. Polkowski and A. Skowron, Eds., Rough Sets and Current Trends in Computing, vol. 1424, Springer,Berlin, Germany, 1998.

[16] K. Qin, J. Yang, Z. Pei, Generalized rough sets based on reflexive and transitive relations, Inform. Sci. 178 (2008) 4138–4141.

[17] A.S. Salama, M.M.E. Abd El-Monsef, New topological approach of rough set generalizations, Internat. J. Computer Math. 8 (7) (2011) 1347-1357.

[18] A. Skowron, On the topology in information systems, Bull. Polish Acad. Sci. Math. 36 (1988) 477–480.

[19] A. Skowron, J. Stepaniuk, Tolerance approximation spaces, Fund. Inform. 27 (1996) 245–253.

[20] M. L. Thivagar, C. Richard, N. R. Paul, Mathematical Innovations of a Modern Topology in Medical Events, Internat. J. Inform. Sci. 2(4) (2012) 33-36.

[21] M. Vlach, Algebraic and Topological Aspects of Rough Set Theory, Fourth International Workshop on Computational Intelligence Applications, IEEE SCM Hiroshima Chapter, Hiroshima University, Japan, December, 2008.

[22] A. Wiweger, On topological rough sets, Bull. Polish Acad. Sci. Math. 37 (1988) 51–62.

[23] W. Xu, X. Zhang, Q. Wang, S. Sun, On general binary relation based rough set, J. Inform. Comput. Sci. 7 (1) (2012) 054-066.

[24] Y.Y. Yao, Constructive and algebraic methods of the theory of rough sets, Inform. Sci. 109 (1998) 21- 47.

[25] Y.Y. Yao, Relational Interpretations of Neighborhood Operators and Rough Set Approximation Operators, Inform. Sci. 111 (1-4) (1998) 239- 259.

[26] W. Zhu, Relationship between generalized rough sets based on binary relation and covering, Inform. Sci. 179 (2009) 210–225.

[27] W. Zhu, Generalized rough sets based on relations, Inform. Sci. 177 (2007) 4997–5011.