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SOME NEW FIXED POINT THEOREMS IN RECTANGULAR METRIC SPACES

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Abstract. In this paper, we introduce the concept of $\xi-(\psi,F,\varphi)$ weakly contractive mappings endowed with C-class functions via a $\alpha-$orbital attractive mapping and present new fixed point theorems for such mappings in rectangular metric spaces. Furthermore, we provide some example and applications to illustrate the usability of our obtained results.

Keywords: C-class functions; fixed point; $\alpha-$admissible; $\alpha-$orbital attractive; rectangular metric spaces.

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1. Introduction and Preliminaries

Fixed point theory is a vital and genuine theme of nonlinear analysis. Furthermore, it's well established that the contraction mapping principle substantiated doctoral thesis of Banach [12] is one of the most prominent theorems in functional analysis. Since 2010, this theorem has exposed to multifarious generalization either by easing circumstance on contractivity or

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by revoking the condition of completeness or ocassionaly even both as well. Recently, many challenging generalization was attained in [2] by substituting triangle inequality by a three-term expression. Moreover, Bracciari showed an analog of Banach theorem in such spaces. For more, the reader can refer to [5], [13-17].

In 2014, Isık et al. [10] stated and proved some common fixed point theorems for $(\psi, F, \alpha, \beta)-$ weakly contractive mappings in rectangular metric spaces via new functions. They also provided interesting example to support the usability of their results. In 2016, Latif et al. [9] established the concept of cyclic admissible generalized contractions involving C-class functions and presented some common fixed point theorems. In a recent paper, Yolacan [7] introduced fixed point theorems for mappings satisfying a modified $\gamma - \psi -$contractive mappings in rectangular metric space.

Henceforward, let A be a nonempty set. Let A^2 be the product space $A \times A$. Unless indicated otherwise, "for all n" will imply "for all $n \geq 0$".

Definition 1.1. [2] Let $\gamma : A^2 \to [0, \infty)$ satisfy the following terms for all a, $b \in A$ and all distinct c, $d \in A$ each of which is dissimilar to a and b. (1) $\gamma(a,b) = 0 \Leftrightarrow a = b$, (2) $\gamma(a,b) = \gamma(b,a)$, (3) $\gamma(a,b) \leq \gamma(a,c) + \gamma(c,d) + \gamma(d,b)$. Then the map γ is called a rectangular metric (briefly, RM). Here, the pair (A, γ) is called rectangular metric space (briefly, RMS).

Definition 1.2. [2] Let (A, γ) a RMS and $\{a_n\}$ be sequence in A. (1) $\{a_n\}$ is called RMS convergent to a limit a iff $\gamma(a_n, a) \to 0$ as $n \to \infty$. (2) $\{a_n\}$ is called RMS Cauchy sequence iff for every $\varepsilon > 0$ there exists positive integer $N(\varepsilon)$ such that $\gamma(a_n, a_m) < \varepsilon$ for all $n > m > N(\varepsilon)$. (3) A rectangular metric spaces (A, γ) is called complete if every RMS Cauchy sequence is RMS convergent. (4) A mapping $T : (A, \gamma) \to (A, \gamma)$ is continuous if for any sequence $\{a_n\}$ in A such that $\gamma(a_n, a) \to 0$ as $n \to \infty$, we have $\gamma(Ta_n, Ta) \to 0$ as $n \to \infty$.

Lemma 1.1. [18] *Let (A, γ) be a RMS, and let $\{a_n\}$ be a Cauchy sequence in A such that $\gamma(a_n, a) \to 0$ when $n \to \infty$ for some $a \in A$. Then $\gamma(a_n, a) \to \gamma(a, b)$ when $n \to \infty$ for all $b \in A$. In particular, $\{a_n\}$ does not convergence to b if $b \neq a$.*

Definition 1.3. [1] Let $T : A \to A$ and $\alpha : A^2 \to [0, \infty)$ be given mappings. We say that T is $\alpha-$admissible if for all a, $b \in A$, we have

$$\alpha(a,b) \geq 1 \Rightarrow \alpha(Ta, Tb) \geq 1.$$

The notion of $\alpha-$*orbital admissible* and $\alpha-$*orbital attractive* mappings were investigated by Popescu [3] as follows.

Definition 1.4. [3] Let $T : A \to A$ be a mapping and $\alpha : A^2 \to [0,\infty)$ be a function. We say that T is $\alpha-$orbital admissible (briefly, $\alpha-OA$) if

$$a \in A,\ \alpha(a,Ta) \geq 1 \Rightarrow \alpha\left(Ta,T^2a\right) \geq 1.$$

Definition 1.5. [3] Let $T : A \to A$ be a mapping and $\alpha : A^2 \to [0,\infty)$ be a function. We say that T is $\alpha-$orbital attractive if

$$a \in A,\ \alpha(a,Ta) \geq 1 \text{ implies } \alpha(a,b) \geq 1 \text{ or } \alpha(b,Ta) \geq 1$$

for every $b \in A$.

Definition 1.6. [4] Let T be a self-mapping on a metric space (A,γ) and let $\alpha,\ \eta : A^2 \to [0,\infty)$ be two functions. We say that T is an $\alpha-$admissible with respect to η mapping if

$$a,\ b \in A,\ \alpha(a,b) \geq \eta(a,b) \Rightarrow \alpha(Ta,Tb) \geq \eta(Ta,Tb).$$

Note that if we take $\eta(a,b) = 1$, then this definition reduces to Definition 1.3. If we also take $\alpha(a,b) = 1$, then we say that T is an $\eta-$subadmissible mapping.

Ansari [6] initiated the notion of C-class functions and furnish new fixed point theorem in 2014. Since then several papers have dealt with fixed point theory for C-class function in metric space (see [8-10] and references therein).

Definition 1.7. [6] A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called C-class function if it is continuous and satisfies following axioms:

(1) $F(x,y) \leq x$;

(2) $F(x,y) = x$ implies that either $x = 0$ or $y = 0$; for all $x,\ y \in [0,\infty)$.

Note that $F(0,0) = 0$.

We indicate C-class functions as $\mathscr{C}$.

Example 1.1. [6] The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathscr{C}$, for all $x,\ y \in [0,\infty)$:

(1) $F(x,y) = x - y$, $F(x,y) = x \Rightarrow y = 0$;

(2) $F(x,y) = mx$, $0<m<1$, $F(x,y) = x \Rightarrow x = 0$;

(3) $F(x,y) = \frac{x}{(1+y)^r}$; $r \in (0,\infty)$, $F(x,y) = x \Rightarrow x = 0$ or $y = 0$;

(4) $F(x,y) = \log(y+\delta^x)/(1+y)$, $\delta > 1$, $F(x,y) = x \Rightarrow x = 0$ or $y = 0$;

(5) $F(x,y) = \ln(1+\delta^x)/2$, $\delta > e$, $F(x,1) = x \Rightarrow x = 0$;

(6) $F(x,y) = (x+l)^{(1/(1+y)^r)} - l$, $l > 1$, $r \in (0,\infty)$, $F(x,y) = x \Rightarrow y = 0$;

(7) $F(x,y) = x\log_{y+\delta}\delta$, $\delta > 1$, $F(x,y) = x \Rightarrow x = 0$ or $y = 0$;

(8) $F(x,y) = x - (\frac{1+x}{2+x})(\frac{y}{1+y})$, $F(x,y) = x \Rightarrow y = 0$;

(9) $F(x,y) = x\beta(x)$, $\beta : [0,\infty) \to [0,1)$, $F(x,y) = x \Rightarrow x = 0$;

(10) $F(x,y) = x - \frac{y}{k+y}$, $F(x,y) = x \Rightarrow y = 0$;

(11) $F(x,y) = x - \varphi(x)$, $F(x,y) = x \Rightarrow x = 0$, here $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(y) = 0 \Leftrightarrow y = 0$;

(12) $F(x,y) = xh(x,y)$, $F(x,y) = x \Rightarrow x = 0$, here $h : [0,\infty)^2 \to [0,\infty)$ is a continuous function such that $h(y,x) < 1$ for all $y,x > 0$;

(13) $F(x,y) = x - (\frac{2+y}{1+y})y$, $F(x,y) = x \Rightarrow y = 0$;

(14) $F(x,y) = \sqrt[n]{\ln(1+x^n)}$, $F(x,y) = x \Rightarrow x = 0$;

(15) $F(x,y) = \phi(x)$, $F(x,y) = x \Rightarrow x = 0$, here $\phi : [0,\infty) \to [0,\infty)$ is a upper semicontinuous function such that $\phi(0) = 0$, and $\phi(y) < y$ for $y > 0$;

(16) $F(x,y) = \frac{x}{(1+x)^r}$; $r \in (0,\infty)$, $F(x,y) = x \Rightarrow x = 0$.

Definition 1.8. [11] Let Φ denote the class of functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfying

(φi) φ is continuous;

(φii) $\varphi(t) < t$ for all $t > 0$.

Note that by (φi) and (φii), we have $\varphi(t) = 0$ if and only if $t = 0$.

Definition 1.9. [6] Let Φ_u denote ultra distance functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfying

(φ_ui) φ is continuous and nondecreasing mapping

($\varphi_u$ii) $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \geq 0$.

In this paper, we introduce the concept of $\xi - (\psi, F, \varphi)$ weakly contractive mappings endowed with C-class functions via a $\alpha-$orbital attractive mapping and present new fixed point theorems for such mappings in rectangular metric spaces. Furthermore, we provide some example and applications to illustrate the usability of our obtained results.

2. Main Results

Let Ψ be the set of all the functions $\psi : [0,+\infty) \to [0,+\infty)$ such that

(1) ψ is continuous and nondecreasing,

(2) $\psi(t) = 0$ iff $t = 0$.

Let Φ^* denote the set of functions $\varphi : [0,+\infty) \to [0,+\infty)$ such that

(1) $\liminf_{t\to r^+} \varphi(t) > 0$ for all $r > 0$,

(2) $\varphi(t) = 0$ iff $t = 0$.

Let Φ_u^* denote the set of functions $\varphi : [0,+\infty) \to [0,+\infty)$ such that

(1) $\liminf_{t\to r^+} \varphi(t) > 0$ for all $r > 0$,

(2) $\varphi(0) \geq 0$.

Theorem 2.1. *Let (A,γ) be a complete RMS, and let T be a mapping. Assume that for $\psi \in \Psi$, $\varphi \in \Phi_u^*$ and $F \in \mathscr{C}$,*

$$a,\ b \in A,\ \xi(a,b) \geq 1 \Rightarrow \psi(\gamma(Ta,Tb)) \leq F(\psi(\max\{\gamma(a,b),\gamma(a,Ta),\gamma(b,Tb)\}),\varphi(\gamma(a,b))). \tag{2.1}$$

Also suppose that the following assertions hold:

(i) T is $\xi-OA$;

(ii) there exists $a_0 \in A$ such that $\xi(a_0,Ta_0) \geq 1$ and $\xi(a_0,T^2a_0) \geq 1$;

(iii) T is $\xi-$orbital attractive mappings.

Then T has a unique fixed point $\omega_* \in A$ and $\{T^n a_0\}$ converges to ω_*.

Proof. Let $a_0 \in A$ be such that $\xi(a_0,Ta_0) \geq 1$ and $\xi(a_0,T^2a_0) \geq 1$. We define the iterative sequence $\{a_n\}$ in A by the rule $a_n = T^n a_0 = Ta_{n-1}$ for all n. Obviously, if $a_{n+1} = a_n$ for some n, then $a = a_n$ is a fixed point for T. Suppose also that $a_{n+1} \neq a_n$ for each n. Since T is $\xi-OA$, *we have* $\xi(a_0,a_1) = \xi(a_0,Ta_0) \geq 1$ implies $\xi(Ta_0,T^2a_0) \geq 1$ and $\xi(a_0,T^2a_0) \geq 1$ implies $\xi(Ta_0,T^3a_0) \geq 1$.

By continuing this process, we have

$$\xi(a_n,a_{n+1}) \geq 1 \text{ for all } n \tag{2.2}$$

and

$$\xi\left(a_n, a_{n+2}\right) \geq 1 \text{ for all } n. \tag{2.3}$$

From assumptions (2.1) and (2.2), then for every n, we get

$$\begin{aligned}
\psi\left(\gamma\left(a_{n+1}, a_{n+2}\right)\right) &= \psi\left(\gamma\left(Ta_n, Ta_{n+1}\right)\right) \\
&\leq F\left(\psi\left(\max\left\{\begin{array}{c}\gamma\left(a_n, a_{n+1}\right), \gamma\left(a_n, Ta_n\right), \\ \gamma\left(a_{n+1}, Ta_{n+1}\right)\end{array}\right\}\right), \varphi\left(\gamma\left(a_n, a_{n+1}\right)\right)\right) \\
&< \psi\left(\max\left\{\gamma\left(a_n, a_{n+1}\right), \gamma\left(a_n, Ta_n\right), \gamma\left(a_{n+1}, Ta_{n+1}\right)\right\}\right) \\
&= \psi\left(\max\left\{\gamma\left(a_n, a_{n+1}\right), \gamma\left(a_{n+1}, a_{n+2}\right)\right\}\right). (2.4)
\end{aligned}$$

By (2.4), using property of ψ, we have

$$\gamma\left(a_{n+1}, a_{n+2}\right) < \max\left\{\gamma\left(a_n, a_{n+1}\right), \gamma\left(a_{n+1}, a_{n+2}\right)\right\} \text{ for all } n.$$

If for some n, $\gamma\left(a_n, a_{n+1}\right) < \gamma\left(a_{n+1}, a_{n+2}\right)$, then $\max\left\{\gamma\left(a_n, a_{n+1}\right), \gamma\left(a_{n+1}, a_{n+2}\right)\right\} = \gamma\left(a_{n+1}, a_{n+2}\right) > 0$, thus inequality (2.4) turns into

$$0 < \psi\left(\gamma\left(a_{n+1}, a_{n+2}\right)\right) < \psi\left(\gamma\left(a_{n+1}, a_{n+2}\right)\right),$$

which is a contradiction. Therefore, $\max\left\{\gamma\left(a_n, a_{n+1}\right), \gamma\left(a_{n+1}, a_{n+2}\right)\right\} = \gamma\left(a_n, a_{n+1}\right)$ for all n. Hence, inequality (2.4) becomes

$$\psi\left(\gamma\left(a_{n+1}, a_{n+2}\right)\right) < \gamma\left(a_n, a_{n+1}\right). \tag{2.5}$$

From (2.5), the sequence $\left\{\gamma\left(x_n, x_{n+1}\right)\right\}$ is nonincreasing and ultimately, there exists $z \geq 0$ such that $\lim_{n\to\infty} \gamma\left(a_n, a_{n+1}\right) = z$. We claim that $\lim_{n\to\infty} \gamma\left(a_n, a_{n+1}\right) = z = 0$. Conversely, assume that $z > 0$. Taking limit when $n \to \infty$ in (2.4) and from the continuity of ψ and the property (1) of function $\varphi \in \Phi_u^*$, we have

$$\psi(z) \leq F\left(\psi(z), \liminf_{\gamma\left(a_n, a_{n+1}\right)\to z^+} \varphi(z)\right) < \psi(z).$$

For this reason, $\psi(z) = 0$ or $\liminf_{\gamma\left(a_n, a_{n+1}\right)\to z^+} \varphi(z) = 0$, then $z = 0$ is a contradiction. Under the circumstances, we have

$$\gamma\left(a_n, a_{n+1}\right) \to 0 \text{ as } n \to \infty. \tag{2.6}$$

From assumptions (2.1) and (2.3), then for every n, we get

$$\begin{aligned} \psi(\gamma(a_{n+1},a_{n+3})) &= \psi(\gamma(Ta_n,Ta_{n+2})) \\ &\leq F(\psi\left(\max\left\{\begin{array}{c}\gamma(a_n,a_{n+2}),\\ \gamma(a_n,Ta_n),\gamma(a_{n+2},Ta_{n+2})\end{array}\right\}\right),\varphi(\gamma(a_n,a_{n+2}))) \\ &< \psi(\max\{\gamma(a_n,a_{n+2}),\gamma(a_n,Ta_n),\gamma(a_{n+2},Ta_{n+2})\})\,(2.7) \end{aligned}$$

Hence, from (2.7), for each $n \in N$, either

$$\psi(\gamma(a_{n+1},a_{n+3})) < \psi(\gamma(a_n,a_{n+2})) \tag{2.8}$$

or

$$\psi(\gamma(a_{n+1},a_{n+3})) < \psi(\max\{\gamma(a_n,Ta_n),\gamma(a_{n+2},Ta_{n+2})\}). \tag{2.9}$$

Suppose at first that there is some $n \in N$ such that (2.8) holds for all $n \geq n_0$. Using property of ψ, we get that

$$\gamma(a_{n+1},a_{n+3}) < \gamma(a_n,a_{n+2}) \text{ for all } n.$$

Thus, the sequence of positive reals $\{\gamma(x_n,x_{n+2})\}$ is monotone decreasing and ultimately, there exists $y \geq 0$ such that $\lim_{n\to\infty}\gamma(a_n,a_{n+2}) = y$. We claim that $\lim_{n\to\infty}\gamma(a_n,a_{n+2}) = y = 0$. Conversely, assume that $y > 0$. Taking limit when $n \to \infty$ in (2.7) and from the continuity of ψ and the property (1) of function $\varphi \in \Phi_u^*$, we have

$$\psi(y) \leq F(\psi(y), \liminf_{\gamma(a_n,a_{n+2})\to y^+} \varphi(y)) < \psi(y).$$

For this reason, $\psi(y) = 0$ or $\liminf_{\gamma(a_n,a_{n+2})\to y^+} \varphi(y) = 0$, then $y = 0$ is a contradiction. Under the circumstances, we have

$$\gamma(a_n,a_{n+2}) \to 0 \text{ as } n \to \infty. \tag{2.10}$$

Suppose that now (2.9) holds for some infinite subset $\{n_l\}$ of positive integers. Then by (2.9) we obtain that

$$\psi\left(\gamma\left(a_{n_l+1},a_{n_l+3}\right)\right) < \psi\left(\max\left\{\gamma\left(a_{n_l},a_{n_l+1}\right),\gamma\left(a_{n_l+2},a_{n_l+3}\right)\right\}\right)$$

for all $n_l \in N$. Hence, due to property of ψ,

$$\gamma\left(a_{n_l+1},a_{n_l+3}\right) < \max\left\{\gamma\left(a_{n_l},a_{n_l+1}\right),\gamma\left(a_{n_l+2},a_{n_l+3}\right)\right\} \text{ for all } n_l \in N. \tag{2.11}$$

Taking limit when $l \to \infty$ in (2.11) and from (2.6), we have

$$\limsup_{l\to\infty} \gamma\left(a_{n_l+1}, a_{n_l+3}\right) < \lim_{l\to\infty} \max\left\{\gamma\left(a_{n_l}, a_{n_l+1}\right), \gamma\left(a_{n_l+2}, a_{n_l+3}\right)\right\} = 0.$$

Therefore, we get that $\limsup_{l\to\infty} \gamma\left(a_{n_l+1}, a_{n_l+3}\right) = 0$. This implies $\gamma(a_n, a_{n+2}) \to 0$ as $n \to \infty$. Hence we showed that (2.10) holds.

Next, we shall show that $\{a_n\}$ is a RMS Cauchy sequence. Suppose, on the contrary, that $\{a_n\}$ is not a Cauchy sequence. Then there is $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers k,

$$n_k > m_k > k,\ \gamma(a_{m_k}, a_{n_k}) \geq \varepsilon \text{ and } \gamma\left(a_{m_k}, a_{n_k-1}\right) < \varepsilon.$$

Next, by the rectangular inequality, since a_{m_k}, a_{n_k}, a_{n_k-1}, a_{n_k-2} are distinct points, we obtain

$$\begin{aligned} \varepsilon &\leq \gamma(a_{m_k}, a_{n_k}) \leq \gamma\left(a_{m_k}, a_{n_k-1}\right) + \gamma\left(a_{n_k-1}, a_{n_k-2}\right) + \gamma\left(a_{n_k-2}, a_{n_k}\right) \\ &< \varepsilon + \gamma\left(a_{n_k-1}, a_{n_k-2}\right) + \gamma\left(a_{n_k-2}, a_{n_k}\right). (2.12) \end{aligned}$$

Taking limit when $k \to \infty$ in (2.12), from (2.6) and (2.10), we have

$$\gamma(a_{m_k}, a_{n_k}) \to \varepsilon. \tag{2.13}$$

Similarly, we get

$$\begin{aligned} &\gamma(a_{m_k}, a_{n_k}) - \gamma\left(a_{m_k-1}, a_{m_k}\right) - \gamma\left(a_{n_k-1}, a_{n_k}\right) \\ \leq\ &\gamma\left(a_{m_k-1}, a_{n_k-1}\right) \\ \leq\ &\gamma\left(a_{m_k-1}, a_{m_k}\right) + \gamma(a_{m_k}, a_{n_k}) + \gamma\left(a_{n_k-1}, a_{n_k}\right). (2.14) \end{aligned}$$

Taking limit when $k \to \infty$ in (2.14), by (2.6) and (2.13), we have

$$\gamma\left(a_{m_k-1}, a_{n_k-1}\right) \to \varepsilon. \tag{2.15}$$

Again, by the rectangular inequality, we get

$$\begin{aligned} &\gamma\left(a_{m_k}, a_{m_k-2}\right) - \gamma\left(a_{m_k-2}, a_{m_k-1}\right) - \gamma\left(a_{m_k-1}, a_{n_k-1}\right) \\ \leq\ &\gamma\left(a_{m_k}, a_{n_k-1}\right) \\ \leq\ &\gamma\left(a_{m_k}, a_{m_k-2}\right) + \gamma\left(a_{m_k-2}, a_{m_k-1}\right) + \gamma\left(a_{m_k-1}, a_{n_k-1}\right) (2.16) \end{aligned}$$

Taking limit when $k\to\infty$ in (2.16), from (2.6), (2.10) and (2.15), we have

$$\gamma\left(a_{m_k},a_{n_k-1}\right)\to\varepsilon. \tag{2.17}$$

Similarly, we obtain

$$\lim_{n\to\infty}\gamma\left(a_{n_k},a_{m_k-1}\right)=\lim_{n\to\infty}\gamma\left(a_{n_k+1},a_{m_k}\right)=\lim_{n\to\infty}\gamma\left(a_{n_k+1},a_{m_k-1}\right)=\varepsilon. \tag{2.18}$$

Since $\xi\left(a_{n_k-1},Ta_{n_k-1}\right)\geq 1$ and T is $\xi-$orbital attractive mappings we have

$$\xi\left(a_{n_k-1},a_{m_k-1}\right)\geq 1 \text{ or } \xi\left(a_{m_k-1},Ta_{n_k-1}\right)\geq 1.$$

Thus, we have two cases as follows.

(1) There exists an infinite subset P of N such that $\xi\left(a_{n_k-1},a_{m_k-1}\right)\geq 1$ for every $k\in P$.

(2) There exists an infinite subset Q of N such that $\xi\left(a_{m_k-1},Ta_{n_k-1}\right)\geq 1$ for every $k\in Q$.

Case 1.

From (2.6) and (2.15), we obtain that

$$\max\left\{\gamma\left(a_{n_k-1},a_{m_k-1}\right),\gamma\left(a_{n_k-1},a_{n_k}\right),\gamma\left(a_{m_k-1},a_{m_k}\right)\right\}\to\varepsilon \text{ as } n\to\infty. \tag{2.19}$$

Taking $a=a_{n_k-1}$ and $b=a_{m_k-1}$ in (2.1) and regarding $\xi\left(a_{n_k-1},a_{m_k-1}\right)\geq 1$, we get that

$$\begin{aligned}\psi(\gamma(a_{n_k},a_{m_k})) &= \psi\left(\gamma\left(Ta_{n_k-1},Ta_{m_k-1}\right)\right)\\ &\leq F\left(\psi\left(\max\left\{\begin{array}{c}\gamma\left(a_{n_k-1},a_{m_k-1}\right),\\ \gamma\left(a_{n_k-1},a_{n_k}\right),\gamma\left(a_{m_k-1},a_{m_k}\right)\end{array}\right\}\right),\varphi\left(\gamma\left(a_{n_k-1},a_{m_k-1}\right)\right)\right).\end{aligned}$$

Taking limit when $k\to\infty$, $k\in P$, from (2.13) and (2.19) we have

$$\psi(\varepsilon)\leq F(\psi(\varepsilon),\lim_{\gamma\left(a_{n_k-1},a_{m_k-1}\right)\to\varepsilon^+}\varphi(\varepsilon))<\psi(\varepsilon),$$

which is a contradiction.

Case 2.

From (2.6) and (2.18), we obtain that

$$\max\left\{\gamma\left(a_{m_k-1},a_{n_k}\right),\gamma\left(a_{m_k-1},Ta_{m_k-1}\right),\gamma\left(a_{n_k},Ta_{n_k}\right)\right\}\to\varepsilon \text{ as } n\to\infty. \tag{2.20}$$

Taking $a = a_{m_k-1}$ and $b = a_{n_k}$ in (2.1) and regarding $\xi\left(a_{m_k-1}, Ta_{n_k-1}\right) \geq 1$, we get that

$$\begin{aligned}\psi\left(\gamma\left(a_{m_k}, a_{n_k+1}\right)\right) &= \psi\left(\gamma\left(Ta_{m_k-1}, Ta_{n_k}\right)\right) \\ &\leq F\left(\psi\left(\max\left\{\begin{array}{c}\gamma\left(a_{m_k-1}, a_{n_k}\right), \\ \gamma\left(a_{m_k-1}, Ta_{m_k-1}\right), \gamma\left(a_{n_k}, Ta_{n_k}\right)\end{array}\right\}\right), \varphi\left(\gamma\left(a_{m_k-1}, a_{n_k}\right)\right)\right).\end{aligned}$$

Taking limit when $k \to \infty$, $k \in Q$, from (2.18) and (2.20) we have

$$\psi(\varepsilon) \leq F(\psi(\varepsilon), \lim_{\gamma\left(a_{m_k-1}, a_{n_k}\right) \to \varepsilon^+} \varphi(\varepsilon)) < \psi(\varepsilon),$$

which is a contradiction.

Hence, we obtain that $\{a_n\}$ is a Cauchy sequence. From the completeness of A, there exists $\omega_* \in A$ such that $\gamma(x_n, \omega_*) \to 0$ when $n \to \infty$.

Next, we shall show that $\omega_* = T\omega_*$. Assume, on the contrary, that $\omega_* \neq T\omega_*$. As T is $\xi-$orbital attractive mappings, we have

$$\xi\left(a_n, \omega_*\right) \geq 1 \text{ or } \xi\left(\omega_*, a_{n+1}\right) \geq 1$$

for all n. Thus, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that

$$\xi\left(a_{n_k}, \omega_*\right) \geq 1 \text{ for all } k \tag{2.21}$$

or

$$\xi\left(\omega_*, a_{n_k}\right) \geq 1 \text{ for all } k. \tag{2.22}$$

Using properties of ψ, φ and F, by (2.1) and (2.21), we have

$$\begin{aligned}\psi\left(\gamma\left(a_{n_k+1}, T\omega_*\right)\right) &\leq F(\psi\left(\max\left\{\begin{array}{c}\gamma(a_{n_k}, \omega_*), \\ \gamma(a_{n_k}, Ta_{n_k}), \gamma(\omega_*, T\omega_*)\end{array}\right\}\right), \varphi(\gamma(a_{n_k}, \omega_*))) \\ &< \psi\left(\max\left\{\begin{array}{c}\gamma(a_{n_k}, \omega_*), \\ \gamma(a_{n_k}, Ta_{n_k}), \gamma(\omega_*, T\omega_*)\end{array}\right\}\right).\end{aligned}$$

Letting $k \to \infty$ in the above equality, from Lemma 1.1, we get

$$\psi(\gamma(\omega_*, T\omega_*)) < \psi(\gamma(\omega_*, T\omega_*)),$$

which is a contradiction. Similarly, using assumptions (2.1) and (2.22), we have $\omega_* = T\omega_*$.

If v_* is another fixed point of T such that $\omega_* \neq v_*$, as T is $\xi-$orbital attractive mapping, we conclude that

$$\xi\left(a_n, v_*\right) \geq 1 \text{ for all } n$$

or

$$\xi\left(v_*, a_{n+1}\right) \geq 1 \text{ for all } n.$$

Thus, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that

$$\xi\left(a_{n_k}, v_*\right) \geq 1 \text{ for all } k$$

or

$$\xi\left(v_*, a_{n_k}\right) \geq 1 \text{ for all } k.$$

In the first condition, from properties of ψ, φ and F, we get

$$\begin{aligned}\psi\left(\gamma\left(a_{n_k+1}, Tv_*\right)\right) &\leq F(\psi\left(\max\left\{\begin{array}{c}\gamma(a_{n_k}, v_*),\\ \gamma(a_{n_k}, Ta_{n_k}), \gamma(v_*, Tv_*)\end{array}\right\}\right), \varphi\left(\gamma(a_{n_k}, v_*)\right))\\ &< \psi\left(\max\left\{\begin{array}{c}\gamma(a_{n_k}, v_*),\\ \gamma(a_{n_k}, Ta_{n_k}), \gamma(v_*, Tv_*)\end{array}\right\}\right).\end{aligned}$$

Letting $k \to \infty$ in the above equality, we deduce that

$$\psi\left(\gamma(\omega_*, v_*)\right) < \psi\left(\gamma(\omega_*, v_*)\right),$$

so $\gamma(\omega_*, v_*) = 0$. This is a contradiction. The second condition is similar.

Corollary 2.2. *Let (A, γ) be a complete RMS, and let T be a mapping. Assume that for $\psi \in \Psi$, $\varphi \in \Phi_u^*$ and $F \in \mathscr{C}$,*

$$a,\ b \in A,\ \xi\left(a, b\right) \geq 1 \Rightarrow \psi\left(\gamma(Ta, Tb)\right) \leq F\left(\psi\left(\gamma(a, b)\right), \varphi\left(\gamma(a, b)\right)\right). \tag{2.23}$$

Also suppose that the following assertions hold:

(i) T is $\xi-OA$;

(ii) there exists $a_0 \in A$ such that $\xi\left(a_0, Ta_0\right) \geq 1$ and $\xi\left(a_0, T^2 a_0\right) \geq 1$;

(iii) T is $\xi-$orbital attractive mappings.

Then T has a unique fixed point $\omega_* \in A$ and $\{T^n a_0\}$ converges to ω_*.

Clearly, Theorem 2.1 implies the following results.

Corollary 2.3. *Let (A,γ) be a complete RMS, and let T be a mapping. Assume that for $\psi \in \Psi$, $\varphi \in \Phi_u^*$ and $F \in \mathscr{C}$,*

$$a,\ b \in A,\ \xi(a,b)\,\psi(\gamma(Ta,Tb)) \le F(\psi(\max\{\gamma(a,b),\gamma(a,Ta),\gamma(b,Tb)\}),\varphi(\gamma(a,b))).$$

Also suppose that the following assertions hold:

(i) T is $\xi-OA$;

(ii) there exists $a_0 \in A$ such that $\xi(a_0,Ta_0) \ge 1$ and $\xi(a_0,T^2a_0) \ge 1$;

(iii) T is $\xi-$orbital attractive mappings.

Then T has a unique fixed point $\omega_* \in A$ and $\{T^n a_0\}$ converges to ω_*.

From Corollary 2.3, if the function $\xi : A^2 \to [0,\infty)$ is such that $\xi(a,b) = 1$ for all $a,\ b \in A$, we deduce the following corollary.

Corollary 2.4. *Let (A,γ) be a complete RMS, and let T be a mapping. Assume that for $\psi \in \Psi$, $\varphi \in \Phi_u^*$ and $F \in \mathscr{C}$,*

$$a,\ b \in A,\ \psi(\gamma(Ta,Tb)) \le F(\psi(\max\{\gamma(a,b),\gamma(a,Ta),\gamma(b,Tb)\}),\varphi(\gamma(a,b))).$$

Also suppose that the following assertions hold:

(i) T is $\xi-OA$;

(ii) there exists $a_0 \in A$ such that $\xi(a_0,Ta_0) \ge 1$ and $\xi(a_0,T^2a_0) \ge 1$;

(iii) T is $\xi-$orbital attractive mappings.

Then T has a unique fixed point $\omega_* \in A$ and $\{T^n a_0\}$ converges to ω_*.

Example 2.1. Let $A = \{0,1,2,3\}$ and define $\gamma : A^2 \to [0,\infty)$ as follows:

$$\begin{aligned}
\gamma(0,1) &= \gamma(1,0) = 1.3,\ \gamma(1,2) = \gamma(2,1) = 0.7,\\
\gamma(0,2) &= \gamma(2,0) = 0.2,\ \gamma(1,3) = \gamma(3,1) = 1.1,\\
\gamma(0,3) &= \gamma(3,0) = 0.4,\ \gamma(2,3) = \gamma(3,2) = 0.8,\\
\gamma(0,0) &= \gamma(1,1) = \gamma(2,2) = \gamma(3,3) = 0.
\end{aligned}$$

Then it easy to show that (A,γ) is a complete RMS, but it is not a metric space. Indeed,

$$1.3 = \gamma(0,1) > \gamma(0,2) + \gamma(2,1) = 0.2 + 0.7.$$

Now, define $T : A \to A$, $T0 = T1 = T2 = 2$, $T3 = 1$ and $\xi(a,b) = \begin{cases} 1 & \text{if } a,\ b \in A/\{3\} \\ \frac{5}{9} & \text{otherwise} \end{cases}$.

Define also the mappings $F : [0,\infty)^2 \to \mathbb{R}$ by $F(x,y) = \frac{x}{(1+y)^2}$ and $\psi, \varphi : [0,+\infty) \to [0,+\infty)$ by $\psi(t) = 2t$ and $\varphi(t) = \frac{t}{2}$.

(1) T is $\xi-OA$;

(2) T is $\xi-$orbital attractive mappings;

(3) there exists $a_0 \in A$ such that $\xi(a_0, Ta_0) \geq 1$ and $\xi(a_0, T^2 a_0) \geq 1$;

(4) T has a fixed point $\omega_* \in A$.

Proof. 1. Let $a \in A$ such that $\xi(a, Ta) \geq 1$ implies $\xi(Ta, T^2 a) \geq 1$. Then, by the definition of ξ, we have $a \in A/\{3\}$, therefore, we obtain

$$\begin{aligned} \xi(0,T0) &= \xi(0,2) \geq 1 \text{ implies } \xi(T0,T^2 0) = \xi(2,2) \geq 1; \\ \xi(1,T1) &= \xi(1,2) \geq 1 \text{ implies } \xi(T1,T^2 1) = \xi(2,2) \geq 1; \\ \xi(2,T2) &= \xi(2,2) \geq 1 \text{ implies } \xi(T2,T^2 2) = \xi(2,2) \geq 1. \end{aligned}$$

We have also shown that T is $\xi - OA$.

2. Let $a,\ b \in A$ such that $\xi(a, Ta) \geq 1$ implies $\xi(a,b) \geq 1$ or $\xi(b, Ta) \geq 1$. Again the definition of ξ gives $a,\ b \in A/\{3\}$, hence we obtain

$$\begin{aligned} \xi(0,T0) &= \xi(0,2) \geq 1 \text{ implies } \xi(0,b) \geq 1 \text{ or } \xi(b,T0) = \xi(b,2) \geq 1; \\ \xi(1,T1) &= \xi(1,2) \geq 1 \text{ implies } \xi(1,b) \geq 1 \text{ or } \xi(b,T1) = \xi(b,2) \geq 1; \\ \xi(2,T2) &= \xi(2,2) \geq 1 \text{ implies } \xi(2,b) \geq 1 \text{ or } \xi(b,T2) = \xi(b,2) \geq 1; \end{aligned}$$

Thereby, T is $\xi-$orbital attractive mappings.

3. Taking $a_0 = 2$, we have $\xi(2, T2) = \xi(2,2) \geq 1$ and $\xi(2, T^2 2) = \xi(2,2) \geq 1$.

4. Clearly, T has a fixed point $2 \in A$.

Next, we claim that there exists $\psi \in \Psi$, $\varphi \in \Phi_u^*$ and $F \in \mathscr{C}$ and such that for all $a,\ b \in A$

$$a,\ b \in A,\ \xi(a,b) \geq 1 \Rightarrow \psi(\gamma(Ta,Tb)) \leq F(\psi(\gamma(a,b)), \varphi(\gamma(a,b))).$$

Firstly, $\xi(a,b) \geq 1$ implies $a,\ b \in A/\{3\}$.

Moreover, let $a,\ b \in A$ with $a \neq b$ and consider the following possible cases.

Case 1. If $a,\, b \in \{0,1,2\}$, then $\gamma(Ta,Tb) = \gamma(2,2) = 0$ and thus (2.1) trivially holds.

Case 2. If $a = 3$, $b \in \{0,1,2\}$, then $\gamma(Ta,Tb) = \gamma(1,2) = 0.7$.

If $b = 0$, then

$$\begin{aligned} & \psi(\gamma(Ta,Tb)) - F(\psi(\gamma(a,b)), \varphi(\gamma(a,b))) \\ = & \psi(\gamma(T3,T0)) - F(\psi(\gamma(3,0)), \varphi(\gamma(3,0))) \\ = & 2.0.7 - \frac{2.0.4}{(1+0.2)^2} \\ = & 1.4 - \frac{0.8}{1.44} = 0.85 > 0. \end{aligned}$$

If $b = 1$, then

$$\begin{aligned} & \psi(\gamma(Ta,Tb)) - F(\psi(\gamma(a,b)), \varphi(\gamma(a,b))) \\ = & \psi(\gamma(T3,T1)) - F(\psi(\gamma(3,1)), \varphi(\gamma(3,1))) \\ = & 2.0.7 - \frac{2.1.1}{(1+0.55)^2} \\ = & 1.4 - \frac{2.2}{2.40} = 0.49 > 0. \end{aligned}$$

If $b = 2$, then

$$\begin{aligned} & \psi(\gamma(Ta,Tb)) - F(\psi(\gamma(a,b)), \varphi(\gamma(a,b))) \\ = & \psi(\gamma(T3,T2)) - F(\psi(\gamma(3,2)), \varphi(\gamma(3,2))) \\ = & 2.0.7 - \frac{2.0.8}{(1+0.4)^2} \\ = & 1.4 - \frac{1.6}{1.96} = 0.59 > 0. \end{aligned}$$

Case 3. Let $a \in \{0,1,2\}$, $b = 3$. Since γ is symmetric, thus (2.23) holds trivially by Case 2.

Taking $F(x,y) = x - y$ in Theorem 2.1, we obtain the following statement.

Corollary 2.5. *Let (A,γ) be a complete RMS, and let T be a mapping. Assume that for $\psi \in \Psi$, $\varphi \in \Phi_u^*$ and $F \in \mathscr{C}$,*

$$a,\ b \in A,\ \xi(a,b) \geq 1 \Rightarrow \psi(\gamma(Ta,Tb)) \leq \psi(\max\{\gamma(a,b), \gamma(a,Ta), \gamma(b,Tb)\}) - \varphi(\gamma(a,b)).$$

Also suppose that the following assertions hold:

(i) T is $\xi-OA$;

(ii) there exists $a_0 \in A$ such that $\xi(a_0, Ta_0) \geq 1$ and $\xi\left(a_0, T^2 a_0\right) \geq 1$;

(iii) T is $\xi-$orbital attractive mappings.

Then T has a unique fixed point $\omega_* \in A$ and $\{T^n a_0\}$ converges to ω_*.

Taking $F(x,y) = \frac{x}{(1+y)^r}$, $r \in (0,\infty)$ in Theorem 2.1, we obtain the following statement.

Corollary 2.6. *Let (A,γ) be a complete RMS, and let T be a mapping. Assume that for $r \in (0,\infty)$, $\psi \in \Psi$, $\varphi \in \Phi_u^*$ and $F \in \mathscr{C}$,*

$$a,\ b \in A,\ \xi(a,b) \geq 1 \Rightarrow \psi(\gamma(Ta,Tb)) \leq \frac{\psi(\max\{\gamma(a,b),\gamma(a,Ta),\gamma(b,Tb)\})}{(1+\varphi(\gamma(a,b)))^r}.$$

Also suppose that the following assertions hold:

(i) T is $\xi-OA$;

(ii) there exists $a_0 \in A$ such that $\xi(a_0, Ta_0) \geq 1$ and $\xi\left(a_0, T^2 a_0\right) \geq 1$;

(iii) T is $\xi-$orbital attractive mappings.

Then T has a unique fixed point $\omega_* \in A$ and $\{T^n a_0\}$ converges to ω_*.

Taking $F(x,y) = \log(y+\delta^x)/(1+y)$, $\delta > 1$ in Theorem 2.1, we obtain the following statement.

Corollary 2.7. *Let (A,γ) be a complete RMS, and let T be a mapping. Assume that for $\delta > 1$, $\psi \in \Psi$, $\varphi \in \Phi_u^*$ and $F \in \mathscr{C}$,*

$$a,\ b \in A,\ \xi(a,b) \geq 1 \Rightarrow \psi(\gamma(Ta,Tb)) \leq \log(\varphi(\gamma(a,b)) + \delta^{\psi(\max\{\gamma(a,b),\gamma(a,Ta),\gamma(b,Tb)\})})/(1+\varphi(\gamma(a,b))).$$

Also suppose that the following assertions hold:

(i) T is $\xi-OA$;

(ii) there exists $a_0 \in A$ such that $\xi(a_0, Ta_0) \geq 1$ and $\xi\left(a_0, T^2 a_0\right) \geq 1$;

(iii) T is $\xi-$orbital attractive mappings.

Then T has a unique fixed point $\omega_* \in A$ and $\{T^n a_0\}$ converges to ω_*.

Taking $F(c,d) = \ln(1+\delta^x)/2$, $\delta > e$ in Theorem 2.1, we obtain the following statement.

Corollary 2.8. *Let (A,γ) be a complete RMS, and let T be a mapping. Assume that for $\delta > e$, $\psi \in \Psi$, $\varphi \in \Phi_u^*$ and $F \in \mathscr{C}$,*

$$a,\ b \in A,\ \xi(a,b) \geq 1 \Rightarrow \psi(\gamma(Ta,Tb)) \leq \ln(1+\delta^{\psi(\max\{\gamma(a,b),\gamma(a,Ta),\gamma(b,Tb)\})})/2.$$

Also suppose that the following assertions hold:

(i) T is $\xi-OA$;

(ii) there exists $a_0 \in A$ such that $\xi\left(a_0, Ta_0\right) \geq 1$ and $\xi\left(a_0, T^2 a_0\right) \geq 1$;

(iii) T is $\xi-$orbital attractive mappings.

Then T has a unique fixed point $\omega_* \in A$ and $\{T^n a_0\}$ converges to ω_*.

Taking $F(c,d) = (x+l)^{(1/(1+y)^r)} - l$, $l > 1$ in Theorem 2.1, we obtain the following statement.

Corollary 2.9. *Let (A,γ) be a complete RMS, and let T be a mapping. Assume that for $l > 1$, $r \in (0,\infty)$, $\psi \in \Psi$, $\varphi \in \Phi_u^*$ and $F \in \mathscr{C}$,*

$$a,\ b \in A,\ \xi\left(a,b\right) \geq 1 \Rightarrow \psi\left(\gamma(Ta,Tb)\right) \leq \left(\psi\left(\max\left\{\gamma(a,b), \gamma(a,Ta), \gamma(b,Tb)\right\}\right) + l\right)^{(1/(1+\varphi(\gamma(a,b)))^r)} - l.$$

Also suppose that the following assertions hold:

(i) T is $\xi-OA$;

(ii) there exists $a_0 \in A$ such that $\xi\left(a_0, Ta_0\right) \geq 1$ and $\xi\left(a_0, T^2 a_0\right) \geq 1$;

(iii) T is $\xi-$orbital attractive mappings.

Then T has a unique fixed point $\omega_* \in A$ and $\{T^n a_0\}$ converges to ω_*.

Taking $F(x,y) = x \log_{y+\delta} \delta$, $\delta > 1$ in Theorem 2.1, we obtain the following statement.

Corollary 2.10. *Let (A,γ) be a complete RMS, and let T be a mapping. Assume that for $\delta > 1$, $\psi \in \Psi$, $\varphi \in \Phi_u^*$ and $F \in \mathscr{C}$,*

$$a,\ b \in A,\ \xi\left(a,b\right) \geq 1 \Rightarrow \psi\left(\gamma(Ta,Tb)\right) \leq \left(\psi\left(\max\left\{\gamma(a,b), \gamma(a,Ta), \gamma(b,Tb)\right\}\right)\right) \log_{\varphi(\gamma(a,b))+\delta} \delta.$$

Also suppose that the following assertions hold:

(i) T is $\xi-OA$;

(ii) there exists $a_0 \in A$ such that $\xi\left(a_0, Ta_0\right) \geq 1$ and $\xi\left(a_0, T^2 a_0\right) \geq 1$;

(iii) T is $\xi-$orbital attractive mappings.

Then T has a unique fixed point $\omega_* \in A$ and $\{T^n a_0\}$ converges to ω_*.

Taking $F(x,y) = \sqrt[n]{\ln(1+x^n)}$ in Theorem 2.1, we obtain the following statement.

Corollary 2.11. *Let (A,γ) be a complete RMS, and let T be a mapping. Assume that for $\psi \in \Psi$, $\varphi \in \Phi_u^*$ and $F \in \mathscr{C}$,*

$$a,\ b \in A,\ \xi\left(a,b\right) \geq 1 \Rightarrow \psi\left(\gamma(Ta,Tb)\right) \leq \sqrt[3]{\ln(1+(\psi\left(\max\left\{\gamma(a,b), \gamma(a,Ta), \gamma(b,Tb)\right\}\right))^3)}$$

Also suppose that the following assertions hold:

(i) T is $\xi-OA$;

(ii) there exists $a_0 \in A$ such that $\xi(a_0, Ta_0) \geq 1$ and $\xi\left(a_0, T^2 a_0\right) \geq 1$;

(iii) T is $\xi-$orbital attractive mappings.

Then T has a unique fixed point $\omega_* \in A$ and $\{T^n a_0\}$ converges to ω_*.

3. Applications

Let Λ be the set of functions $\kappa : [0,+\infty) \to [0,+\infty)$ such that

(1) κ is Lebesgue integrable mapping on each compact subset of $[0,+\infty)$;

(2) $\int_0^{\varepsilon} \kappa(s)\, ds > 0$ for every $\varepsilon > 0$.

For this class of functions, we can express the following results.

Theorem 3.1. *Let (A,γ) be a complete RMS, and let T be a mapping satisfying*

$$\int_0^{\gamma(Ta,Tb)} \kappa_1(s)\, ds \leq F\left(\int_0^{\max\{\gamma(a,b),\gamma(a,Ta),\gamma(b,Tb)\}} \kappa_1(s)\, ds, \int_0^{\gamma(a,b)} \kappa_2(s)\, ds\right),$$

for all $a, b \in A$ and κ_1, $\kappa_2 \in \Lambda$ and $F \in \mathscr{C}$.

Then T has a unique fixed point $\omega_* \in A$ and $\{T^n a_0\}$ converges to ω_*.

Proof. Let $\psi(s) = \int_0^s \kappa_1(v)\, dv$ and $\varphi(s) = \int_0^s \kappa_2(v)\, dv$. Then $\psi \in \Psi$, $\varphi \in \Phi_u^*$, and furthermore, the function ψ is nondecreasing. By Corollary 2.4, T has a fixed point.

Taking $F(x,y) = x - y$ in Corollary 2.4, we obtain the following statement.

Corollary 3.2. *Let (A,γ) be a complete RMS, and let T be a mapping satisfying*

$$\int_0^{\gamma(Ta,Tb)} \kappa_1(s)\, ds \leq \int_0^{\max\{\gamma(a,b),\gamma(a,Ta),\gamma(b,Tb)\}} \kappa_1(s)\, ds - \int_0^{\gamma(a,b)} \kappa_2(s)\, ds,$$

for all $a, b \in A$ and κ_1, $\kappa_2 \in \Lambda$ and $F \in \mathscr{C}$.

Then T has a unique fixed point $\omega_* \in A$ and $\{T^n a_0\}$ converges to ω_*.

Corollary 3.3. *Let (A,γ) be a complete RMS, and let T be a mapping satisfying*

$$\int_0^{\gamma(Ta,Tb)} \kappa_1(s)\, ds \leq \int_0^{\gamma(a,b)} \kappa_1(s)\, ds - \int_0^{\gamma(a,b)} \kappa_2(s)\, ds,$$

for all $a, b \in A$ and κ_1, $\kappa_2 \in \Lambda$ and $F \in \mathscr{C}$.

Then T has a unique fixed point $\omega_* \in A$ and $\{T^n a_0\}$ converges to ω_*.

Corollary 3.4. *Let (A,γ) be a complete RMS, and let T be a mapping satisfying*

$$\int_0^{\gamma(Ta,Tb)} \kappa_1(s)\,ds \leq m \int_0^{\gamma(a,b)} \kappa_1(s)\,ds,$$

for all a, $b \in A$ and some $0 \leq m < 1$, κ_1, $\kappa_2 \in \Lambda$ and $F \in \mathscr{C}$.

Then T has a unique fixed point $\omega_* \in A$ and $\{T^n a_0\}$ converges to ω_*.

Proof. Let $\kappa_2(s) = (1-m)\kappa_1(s)$. Then by Corollary 3.3, T has a fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests.

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