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MODIFIED OF HOUSEHOLDER ITERATIVE METHOD FOR SOLVING NONLINEAR SYSTEMS

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Abstract: In this paper, an efficient method is constructed and used for solving system of nonlinear equations. The method based on a Householder iterative method (HIM). This technique is revised and modified to solve a system of nonlinear equations of *n*-dimension with *n*-variables. In addition, the proposed method has been tested on a series of examples published in the literature and show good results.

Keywords: Householder iterative method; System of nonlinear algebraic equations; Iterative method; Taylor series.

2000 AMS Subject Classification: 47H17; 47H05; 47H09

1. Introduction

Recently, several iterative methods have been made on the development for solving nonlinear equations and system of nonlinear equations. These methods have been improved using several different techniques including Taylor series, quadrature formulas, homotopy and decomposition techniques, see [1–9] and references therein. He [10] suggested an iterative method for solving the nonlinear equations by rewriting the given nonlinear equation as a system of coupled equations. This technique has been used by Chun [11] and Noor [12,13] to suggest some higher order convergent

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iterative methods for solving nonlinear equations. Newton method is the well-known iterative method for finding the solution of the nonlinear equations. There exist several classical multipoint methods with fourth-order and sixth-order convergence for solving nonlinear equations. It is well known [9] that the two-step Newton method has fourth-order convergence, which has been suggested by using the technique of updating the solution. Noor [6] have suggested and analyzed a two-step Halley method using the Newton method as a predictor and Halley method as corrector. It has been shown that this two-step Halley method is of sixth-order convergence and is an efficient one. Noor et al [14] modified Householder iterative method for nonlinear equations. He also show that this new method includes famous two step Newton method as a special case. He proved that this new method is of sixth-order convergence. In this paper, a new iterative method was constructed by using the Noor's decomposition technique [14]. This technique, however, needs to be revised to solve the system of nonlinear equations. Some illustrative examples have been presented, to demonstrate our method and the results are compared with those derived from the previous methods. All test problems reveals the accuracy and fast convergence of the new method.

2. Householder iterative method

Suppose we have system of nonlinear equations of the following form

$$\begin{aligned} f_1(x_1,x_2,\cdots,x_n) &= 0, \\ f_2(x_1,x_2,\cdots,x_n) &= 0, \\ &\vdots \\ f_n(x_1,x_2,\cdots,x_n) &= 0, \end{aligned}$$

where $f_i : R^n \to R$ and the functions f_i is differentiable up to any desired order [15], can be thought of as mapping a vector $\mathrm{X} = (x_1, x_2, \ldots, x_n)^T$ of the n-dimensional space R^n, into the real line R. The system can alternatively be represented by defining a functional $F : R^n \to R^n$, by

$$F(x_1,x_2,\ldots x_n) = (f_1(x_1,x_2,\ldots x_n),\ldots,f_n(x_1,x_2,\ldots x_n))^T$$

Using vector notation to represent the variables $x_1, x_2, \ldots x_n$, the previous system then assumes the form:

$$F(\mathrm{x}) = 0 \tag{1}$$

For simplicity, we assume that X^* is a simple root of Eq. (1) and X_0 is an initial guess sufficiently close to X^*. Using the Taylor's series expansion of the function $f_k(x)$, we have

$$f_k(X_0)+\frac{1}{1!}[\sum_{i=1}^{n}(x_i-x_i^{(0)})f_{k,i}(X_0)]+\frac{1}{2!}[\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-x_i^{(0)})(x_j-x_j^{(0)})f_{k,ij}(X_0)]+\ldots=0 \qquad (2)$$

where $k=1,2,\ldots,n,\ f_{k,i}=\dfrac{\partial f_k}{\partial x_i},\ f_{k,ij}=\dfrac{\partial^2 f_k}{\partial x_i \partial x_j}$ and $X_0=[x_1^{(0)},x_2^{(0)},\ldots,x_n^{(0)}]^T$ is the initial approximation of Eq. (1). Matrices of first and second partial derivatives appearing in equation (2) are Jacobian J and Hessian matrix H respectively. In matrix notation

$$F(X_0)+J(X_0)[X^{(1)}-X_0]+\frac{1}{2!}\sum_{i=1}^{n}e_i\otimes[X^{(1)}-X_0]^T H_i(X_0)[X^{(1)}-X_0]=0$$

from which we have

$$X^{(1)}=X_0-J^{-1}(X_0)F(X_0)-\frac{1}{2!}J^{-1}(X_0)\sum_{i=1}^{n}e_i\otimes[X^{(1)}-X_0]^T H_i(X_0)[X^{(1)}-X_0] \qquad (3)$$

where H_i is the Hessian matrix of the function f_i, $\otimes$ is the Kronecker product and e_i is a $n\times 1$ vector of zero except for a 1 in the position *i*. First two terms of the equation (3) gives the first approximation, as

$$Y=X_0-J^{-1}(X_0)F(X_0) \qquad (4)$$

Substitution again of (4) into the right hand side of (3) gives the second approximation

$$X=X_0-J^{-1}(X_0)F(X_0)-\frac{1}{2!}J^{-1}(X_0)\sum_{i=1}^{n}e_i\otimes\left[Y-X_0\right]^T H_i(X_0)\left[Y-X_0\right] \qquad (5)$$

This formulation allows us to suggest the following iterative methods for solving system of nonlinear equations (1).

3. Algorithms

In this section, we suggest four algorithms for solving system of nonlinear equations:

Algorithm.1. For a given $X^{(k)}=[x_1^{(k)},x_2^{(k)},...,x_n^{(k)}]^T$ calculate the approximation solution $X^{(k+1)}=[x_1^{(k+1)},x_2^{(k+1)},...,x_n^{(k+1)}]^T$ for k =0,1, 2, … by the iterative scheme

$$X^{(k+1)}=\begin{bmatrix} x_1^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{bmatrix}=\begin{bmatrix} x_1^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix}-\begin{bmatrix} f_{1,1}(X^k) & \cdots & f_{1,n}(X^k) \\ \vdots & \ddots & \vdots \\ f_{n,1}(X^k) & \cdots & f_{n,n}(X^k) \end{bmatrix}^{-1}\begin{bmatrix} f_1(X^k) \\ \vdots \\ f_n(X^k) \end{bmatrix}, \tag{6}$$

which is the Newton–Raphson method for n dimension.

Algorithm.2. For a given $X^{(k)}=[x_1^{(k)},x_2^{(k)},...,x_n^{(k)}]^T$ calculate the approximation solution $X^{(k+1)}=[x_1^{(k+1)},x_2^{(k+1)},...,x_n^{(k+1)}]^T$ for k =0, 1, 2, … by the iterative schemes

$$\begin{bmatrix} z_1^{(k)} \\ \vdots \\ z_n^{(k)} \end{bmatrix}=-\begin{bmatrix} f_{1,1}(X^{(k)}) & \cdots & f_{1,n}(X^{(k)}) \\ \vdots & \ddots & \vdots \\ f_{n,1}(X^{(k)}) & \cdots & f_{n,n}(X^{(k)}) \end{bmatrix}^{-1}\begin{bmatrix} f_1(X^{(k)}) \\ \vdots \\ f_n(X^{(k)}) \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} x_1^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{bmatrix}=\begin{bmatrix} x_1^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix}-\begin{bmatrix} f_{1,1}(X^{(k)}) & \cdots & f_{1,n}(X^{(k)}) \\ \vdots & \ddots & \vdots \\ f_{n,1}(X^{(k)}) & \cdots & f_{n,n}(X^{(k)}) \end{bmatrix}^{-1}\begin{bmatrix} f\,(z_1^{(k)})+\frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} z_i^{(k)} z_j^{(k)} f_{1,ij}(X^{(k)}) \\ \vdots \\ f\,(z_n^{(k)})+\frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} z_i^{(k)} z_j^{(k)} f_{n,ij}(X^{(k)}) \end{bmatrix}. \tag{8}$$

We also remark that if $f_{k,ij}=0,\ \forall k,i,j=1,2,...,n$, then Algorithm 2 reduces to the Newton method, that is, Algorithm 2 which is the generalized of Householder iterative method for solving system of nonlinear equations. Now using Algorithm 1 as a predictor and Algorithm 2 as a corrector, we suggest and analyze a new two-step iterative method for solving system of nonlinear equations.

Algorithm.3. For a given $X^{(k)}=[x_1^{(k)},x_2^{(k)},...,x_n^{(k)}]^T$ calculate the approximation solution $X^{(k+1)}=[x_1^{(k+1)},x_2^{(k+1)},...,x_n^{(k+1)}]^T$ for k =0,1, 2, … by the iterative schemes

$$Z^{(k)}=-J^{-1}(X^{(k)})F(X^{(k)}), \qquad Y^{(k)}=X^{(k)}+Z^{(k)}, \tag{9}$$

$$\begin{bmatrix} x_1^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{bmatrix} = \begin{bmatrix} y_1^{(k)} \\ \vdots \\ y_n^{(k)} \end{bmatrix} - \begin{bmatrix} f_{1,1}(\mathrm{Y}^{(k)}) & \cdots & f_{1,n}(\mathrm{Y}^{(k)}) \\ \vdots & \ddots & \vdots \\ f_{n,1}(\mathrm{Y}^{(k)}) & \cdots & f_{n,n}(\mathrm{Y}^{(k)}) \end{bmatrix}^{-1} \begin{bmatrix} f_1(\mathrm{Y}^{(k)}) + \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} z_i^{(k)} z_j^{(k)} f_{1,ij}(\mathrm{Y}^{(k)}) \\ \vdots \\ f_n(\mathrm{Y}^{(k)}) + \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} z_i^{(k)} z_j^{(k)} f_{n,ij}(\mathrm{Y}^{(k)}) \end{bmatrix}. \quad (10)$$

Algorithm 3 is a modified to Householder method. It is clear that for $\mathrm{Y}^{(k)} = \mathrm{X}^{(k)}$ Algorithm 3 is exactly Algorithm 2. If, $f_{k,ij} = 0,\ \forall k, i, j = 1, 2, \ldots, n$ then Algorithm 3 reduce to the following two-step method.

Algorithm.4. For a given $\mathrm{X}^{(k)} = [x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}]^T$ calculate the approximation solution $\mathrm{X}^{(k+1)} = [x_1^{(k+1)}, x_2^{(k+1)}, \ldots, x_n^{(k+1)}]^T$ for k =0,1, 2, … by the iterative schemes

$$\mathrm{Y}^{(k)} = \mathrm{X}^{(k)} - J^{-1}(\mathrm{X}^{(k)}) F(\mathrm{X}^{(k)}), \quad (11)$$

$$\mathrm{X}^{(k+1)} = \mathrm{Y}^{(k)} - \begin{bmatrix} f_{1,1}(\mathrm{Y}^{(k)}) & \cdots & f_{1,n}(\mathrm{Y}^{(k)}) \\ \vdots & \ddots & \vdots \\ f_{n,1}(\mathrm{Y}^{(k)}) & \cdots & f_{n,n}(\mathrm{Y}^{(k)}) \end{bmatrix}^{-1} \begin{bmatrix} f_1(\mathrm{Y}^{(k)}) \\ \vdots \\ f_n(\mathrm{Y}^{(k)}) \end{bmatrix}. \quad (12)$$

4. Applications

We present some examples to illustrate the efficiency of our proposed method. We apply the Algorithm 2.2 and compare the results with the standard Newton–Raphson method (NM). Here, the algorithm is performed by Maple 15 with 20 digits. In Tables 1, 2 we list the results obtained by generalized Householder method (HIM), modified generalized Householder method (MHIM) and comparison them with Newton–Raphson method (NM). In the present study, we use $\| X^{(n+1)} - X^{(n)} \| < \varepsilon$ or $\| F(X^{(n)}) \| < \varepsilon$ and $\varepsilon = 10^{-15}$ as stop criteria.

4.1. Small systems of nonlinear equations.

Example 1. In a case of one dimension, consider the following nonlinear functions [16], $f_1(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$, with $x_0 = -2$ and $f_2(x) = e^{x^2+7x-30} - 1$ with $x_0 = 3.5$.

Example 2. In a case two dimension, consider the following systems of nonlinear functions [17],

$$F_3(\mathrm{x}) = \begin{cases} f_1(x,y) = x^2 - 10x + y^2 + 8 = 0 \\ f_2(x,y) = xy^2 + x - 10y + 8 = 0 \end{cases}, \qquad (x_0, y_0)=(0.8,\ 0.8).$$

$$F_4(\mathrm{x}) = \begin{cases} f_1(x,y) = x^4 y - xy + 2x - y - 1 = 0 \\ f_2(x,y) = ye^{-x} + x - y - e^{-1} = 0 \end{cases}, \quad (x_0, y_0)=(0.8,\ 0.8).$$

Example 3. In a case three dimension, consider the following systems of nonlinear functions [18-20].

$$F_5(\mathrm{x}) = \begin{cases} f_1(x,y,z) = 15x + y^2 - 4z - 13 = 0 \\ f_2(x,y,z) = x^2 + 10y - e^{-z} - 11 = 0, \\ f_3(x,y,z) = y^3 - 25z + 22 = 0 \end{cases} \qquad (x_0, y_0, \mathrm{z}_0)=(5,\ 4,\ 2).$$

$$F_6(\mathrm{x}) = \begin{cases} f_1(x,y,z) = 3x - \cos(yz) - 0.5 = 0 \\ f_2(x,y,z) = x^2 - 81(y+0.1)^2 + \sin z + 1.06 = 0 \\ f_3(x,y,z) = e^{-xy} + 20z + \dfrac{10\pi - 3}{3} = 0 \end{cases}, \qquad (x_0, y_0, \mathrm{z}_0)=(2,\ 2,\ 2).$$

4.2 Large systems of nonlinear equations.

In this subsection, we test HPM with some sparse systems with *m* unknowns variables. In examples 4 through 6, we compare the NR method with the proposed method HPM focusing on iteration numbers.

Example 4. Consider the following system of nonlinear equations [21]:

$$F_7 : f_i = e^{x_i} - 1, \quad i = 1, 2, \ldots, n.$$

The exact solution of this system is $\mathrm{X}^* = [0,0,...,0]^{\mathrm{T}}$. To solve this system, we set $\mathrm{X}_0 = [0.5, 0.5, \ldots, 0.5]^T$ as an initial value. Table 6 is shown the result.

Example 5. Consider the following system of nonlinear equations:

$$F_8 : f_i = x_i^2 - \cos(x_i - 1), \quad i = 1, 2, \ldots, n.$$

One of the exact solutions of this system is $\mathrm{X}^* = [1,1,...,1]^{\mathrm{T}}$. To solve this system, we set $\mathrm{X}_0 = [2,2.,...,2]^{\mathrm{T}}$ as an initial value. The results are presented in Table 6.

Example 6. Consider the following system of nonlinear equations [21]:

$$F_9 : f_i = \cos x_i - 1, \quad i = 1, 2, ..., n.$$

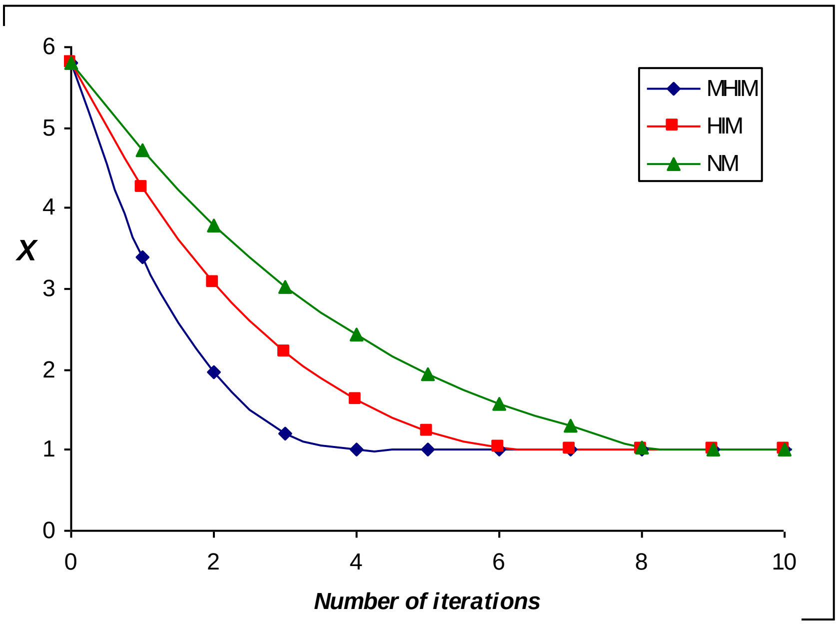
One of the exact solutions of this system is $X^* = [0,0,...,0]^T$. To solve this system, we set $X_0 = [2,2.,...,2]^T$ as an initial guess. The results are presented in Table 6.

Table 1. Number of iterations for Examples 4-6.

Number of iterations	F_7	F_8	F_9	F_7	F_8	F_9	F_7	F_8	F_9
$\varepsilon = 10^{-13}$	$n = 50$			$n = 75$			$n = 100$		
NM	6	7	26	6	7	26	6	7	26
HIM	4	5	18	4	5	18	4	5	18
MHIM	2	3	11	2	3	11	2	3	11

In Tables 1-2, we list the results obtained by modified Householder iteration method. As we see from this Tables, it is clear that the result obtained by MHIM is very superior to that obtained by HIM and NM. Fig. 1 confirm this result.

Fig. 1. Approximate solution against the number of iterations



5. Conclusions

Householder iteration method is generalized and modified and applied to numerical solution for solving nonlinear system equations. The numerical examples show that our method is very effective and efficient. Moreover, our proposed method provides highly accurate results in a less number of iterations as compared with Newton–Raphson method.

Table 2.Numerical results for Examples 1-6

	IT	x_n	y_n	z_n
F_1, x_0 =-2.				
NM	8	-1.2076478271309189270	-	-
HIM	5	-1.2076478271309189336	-	-
MHIM	4	-1.2076478271309189270	-	-
$F2$, x_0 =3.5				
NM	12	3	-	-
HIM	8	3	-	-
MHIM	5	3	-	-
F_3, X_0=(0.8, 0.8)				
NM	5	1	1	-
HIM	4	1	1	-
MHIM	3	1	1	-
F_4, X_0=(0.8, 0.8)				
NM	8	1	1	-
HIM	6	1	1	-
MHIM	4	1	1	-
F_5, X_0=(5, 4, 2)				
NM	6	1.0421495605769383383	1.0310912718394023591	.92384815487936785148
HIM	5	1.0421495605769383383	1.0310912718394023591	.92384815487936785149
MHIM	4	1.0421495605769383383	1.0310912718394023591	.92384815487936785147
F_6, X_0=(2, 2, 2)				
NM	9	0.5	.47716958178424068e-21	-.52359877559829887308
HIM	6	0.5	.35372140894085690e-21	-.52359877559829887309
MHIM	5	0.5	.24110457563735448e-20	-.52359877559829887309

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