Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 5, 1209-1217
ISSN: 1927-5307

COMPACT PRODUCTS OF TOEPLITZ AND HANKEL OPERATORS ON WEIGHTED BERGMAN SPACE

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Abstract. In this paper, we study the products of Toeplitz operators and Hankel operators on weighted Bergman spaces of the unit ball. We obtain the necessary and sufficient conditions for the bounded products of Toeplitz operators on the weighted Bergman spaces of the unit ball.

Keywords: Toeplitz operator; Hankel operator; unit ball; weighted Bergman space; compact operator.

2000 AMS Subject Classification: 47B35; 47B47

1. Introduction

Throughout let $n \geq 2$ be a fixed integer. Denote the unit ball in $\mathbb{C}^n$ by $\mathbb{B}_n$. Let V denote Lebesgue volume measure on $\mathbb{B}_n$, normalized so that $V(\mathbb{B}_n) = 1$. For $-1 < \alpha < \infty$, we denote by V_α the measure on $\mathbb{B}_n$ defined by $dV_\alpha(z) = (1 - |z|^2)^\alpha dV(z)$. The weighted Bergman space $A^2_\alpha(\mathbb{B}_n)$ is the space of analytic functions on $\mathbb{B}_n$ which are square-integrable with respect to measure V_α on $\mathbb{B}_n$.

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Supported by the National Natural Science Foundation of China (No.11126061) and Science & Technology Program of Shanghai Maritime University (No.20120098)

Received April 12, 2012

The reproducing kernel on $A^2_\alpha(\mathbb{B}_n)$ is given by

$$K^{(\alpha)}_\omega(z) = \frac{1}{(1-\langle z,\omega\rangle)^{(n+\alpha+1)}},$$

for $z,\omega \in \mathbb{B}_n$. If $\langle\cdot,\cdot\rangle_\alpha$ denotes the inner product in $L^2(\mathbb{B}_n, dV_\alpha)$, then $\langle h, K^{(\alpha)}_\omega\rangle_\alpha = h(\omega)$, for every $h \in A^2_\alpha(\mathbb{B}_n)$ and $\omega \in \mathbb{B}_n$.

Let P_α be the weighted Bergman orthogonal projection from $L^2(\mathbb{B}_n, dV_\alpha)$ onto $A^2_\alpha(\mathbb{B}_n)$, which is given by

$$(P_\alpha g)(\omega) = \langle g, K^{(\alpha)}_\omega\rangle_\alpha = \int_{\mathbb{B}_n} g(z)\frac{1}{(1-\langle \omega, z\rangle)^{n+\alpha+1}}dV_\alpha(z),$$

for $g \in L^2(\mathbb{B}_n, dV_\alpha)$ and $\omega \in \mathbb{B}_n$. In this paper, we use $\|\cdot\|_{\alpha,p}$ to denote the norm in $L^p(\mathbb{B}_n, dV_\alpha)$. Given f in $L^\infty(\mathbb{B}_n, dV_\alpha)$, the Toeplitz operator T_f is defined on $A^2_\alpha(\mathbb{B}_n)$ by $T_f h = P_\alpha(fh)$. We have

$$(T_f h)(\omega) = \int_{\mathbb{B}_n} \frac{f(z)h(z)}{(1-\langle \omega, z\rangle)^{(n+\alpha+1)}}dV_\alpha(z),$$

for $h \in A^2_\alpha(\mathbb{B}_n)$ and $\omega \in \mathbb{B}_n$. For a bounded measurable function f on $\mathbb{B}_n$, the Hankel operator H_f is the operator $A^2_\alpha(\mathbb{B}_n) \to (A^2_\alpha(\mathbb{B}_n))^\perp$ defined by

$$H_f h = (I-P_\alpha)(fh) = Q_\alpha(fh), \qquad h \in A^2_\alpha(\mathbb{B}_n).$$

The general problem that we are interested in is the following: When the products of Toeplitz operators and Hankel operators is compact, what is the relationship between their symbols?

2. Preliminaries

We will need the following basic facts about the Bergman space $A^2_\alpha(\mathbb{B}_n)$, see [1] for details.

The normalized reproducing kernel is given by

$$k^{(\alpha)}_\omega(z) = \frac{(1-|\omega|^2)^{\frac{(n+\alpha+1)}{2}}}{(1-\langle z,\omega\rangle)^{n+\alpha+1}},$$

for $z,\omega \in \mathbb{B}_n$.

Suppose f and g are in $A^2_\alpha(\mathbb{B}_n)$. Consider the operator $f \otimes g$ on $A^2_\alpha(\mathbb{B}_n)$ defined by

$$(f \otimes g)h = \langle h, g\rangle_\alpha f,$$

for $h \in A^2_\alpha(\mathbb{B}_n)$. It is easily proved that $f \otimes g$ is bounded on $A^2_\alpha(\mathbb{B}_n)$ and $\|f \otimes g\|_{\alpha,2} = \|f\|_{\alpha,2}\|g\|_{\alpha,2}$.

We have the following Lemma for the inner product on Bergman spaces of the unit ball proved in [2].

We observe that the Taylor expansion of the function $(1-z)^{n+\alpha+1}$ around 0,

$$(1-z)^{n+\alpha+1} = \sum_{k=0}^{\infty} C_{\alpha,k} z^k,$$

where $C_{\alpha,k} = (-1)^k \frac{(n+\alpha+1)(n+\alpha)\cdots(n+\alpha+2-k)}{k!}$, $k = 1, 2, \cdots$, $C_{\alpha,0} = 1$, is absolutely convergent on the closed unit disk in $\mathbb{C}$ for $\alpha > -1$.

The term multi-index refers to an ordered n-tuple

$$m = (m_1, \cdots, m_n)$$

of nonnegative integer m_i. The following abbreviated notations will be used:

$$z^m = z_1^{m_1} \cdots z_n^{m_n}, |m| = m_1 + \cdots + m_n, m! = m_1! \cdots m_n!.$$

We have the multinomial formula

$$(z_1 + \cdots + z_n)^N = \sum_{|m|=N} \frac{N!}{m!} z^m.$$

Now we give the representation of $k^{(\alpha)}_\omega \otimes k^{(\alpha)}_\omega$ in [3].

Lemma 2.1. *On $A^2_\alpha(\mathbb{B}_n)$, we have*

$$k^{(\alpha)}_\omega \otimes k^{(\alpha)}_\omega = \sum_{k=0}^{\infty} C_{\alpha,k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} T_{\varphi^\gamma_\omega} T_{\bar{\varphi}^\gamma_\omega},$$

for all $\omega \in \mathbb{B}_n$ and $-1 < \alpha < \infty$.

For a bounded linear operator T on $(A^2_\alpha(\mathbb{B}_n))^\perp$ and $\omega \in \mathbb{B}_n$, we define the operator $\mathcal{Y}_\omega(T)$ by

$$\mathcal{Y}_\omega(T) = \sum_{k=0}^{\infty} C_{\alpha,k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} T_{\phi^\gamma_\omega} T S_{\overline{\phi^\gamma_\omega}}.$$

Fix two real parameters a and b and define integral operators $S_{a,b}$ by

$$S_{a,b}f(z) = (1-|z|^2)^a \int_{\mathbb{B}_n} \frac{(1-|w|^2)^b}{|1-\langle z,w\rangle|^{n+1+a+b}} f(w)dV(w).$$

Using exactly the same argument as in the proof of Lemma 3.3 in [2], we have the following Lemma.

Lemma 2.2. *Let $-1<\alpha<\infty$, if S is a bounded linear operator on $(A^2_\alpha(\mathbb{B}_n))^\perp$, then*

$$\|\sum_{|\gamma|=m} \frac{m!}{\gamma!} T_{\varphi^\gamma_\omega} S S_{\bar{\varphi}^\gamma_\omega}\| \le \|S\|,$$

for every positive integer m and $\omega \in \mathbb{B}_n$.

For a bounded linear operator T on $(A^2_\alpha(\mathbb{B}_n))^\perp$ and $\omega \in \mathbb{B}_n$, we define the operator $\mathcal{Y}_\omega(T)$ by

$$\mathcal{Y}_\omega(T) = \sum_{k=0}^{\infty} C_{\alpha,k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} T_{\phi^\gamma_\omega} T S_{\overline{\phi^\gamma_\omega}}.$$

Both Toeplitz and Hankel operators are closely related to Dual Toeplitz operators. The dual Toeplitz operator with symbol f is defined by

$$S_f u = (I-P_\alpha)(fu),$$

for $u \in (A^2_\alpha(\mathbb{B}_n))^\perp$. It is clear that $S_f : (A^2_\alpha(\mathbb{B}_n))^\perp \to (A^2_\alpha(\mathbb{B}_n))^\perp$ is a bounded linear operator. In what follows Q_α denote $I-P_\alpha$.

Since the Hankel operator with a holomorphic symbol is the zero operator, by above equations we immediately obtain the following elementary properties of dual Toeplitz operators.

Lemma 2.3. *If φ is a bounded holomorphic function on $\mathbb{B}_n$ and ψ is a bounded measurable function on $\mathbb{B}_n$, then following identities hold:*

$$S_\varphi S_\psi = S_{\varphi\psi}, \quad S_{\psi\bar{\varphi}} = S_\psi S_{\bar{\varphi}}, \tag{2.4}$$

$$S_\varphi H_\psi = H_\psi T_\varphi, \quad H^*_\psi S_{\bar{\varphi}} = T_{\bar{\varphi}} H^*_\psi. \tag{2.5}$$

3. Bounded Products of Toeplitz and Hankel operators

The Lemmas below is Lemma 4.4 and Lemma 4.5 in [3].

Lemma 3.1. *Let* $-1 < \alpha < \infty$ *and* $g \in L^2(\mathbb{B}_n, dV_\alpha)$, *then*

$$\left|(H_g^* u)(\omega)\right| \le \frac{1}{(1-|\omega|^2)^{(n+\alpha+1)/2}} \|g \circ \varphi_\omega - P_\alpha(g \circ \varphi_\omega)\|_{\alpha,2} \|u\|_{\alpha,2},$$

for all $u \in (A_\alpha^2)^\perp$ *and* $\omega \in \mathbb{B}_n$.

Lemma 3.2. *Let* $-1 < \alpha < \infty$ *and* $\varepsilon > 0$. *For* $g \in L^2(\mathbb{B}_n, dV_\alpha)$, $u \in (A_\alpha^2)^\perp$ *and multi-index* γ *with* $|\gamma| = m \ge (n+\alpha+1)/2$ *we have*

$$|(H_g^* u)^{(\gamma)}(\omega)| \le C \frac{1}{(1-|\omega|^2)^m} \|g \circ \varphi_\omega - P_\alpha(g \circ \varphi_\omega)\|_{\alpha,2+\varepsilon} \bigg(S_{0,\alpha}(|u|^\delta)(\omega) \bigg)^{1/\delta}$$

for all $\omega \in \mathbb{B}_n$, *where* $\delta = (2+\varepsilon)/(1+\varepsilon)$.

Lemma 3.3. *Let* $-1 < \alpha < \infty$ *and* $f \in L^2(\mathbb{B}_n, dV_\alpha)$, *then*

$$\left|(T_f^* v)(\omega)\right| \le \frac{1}{(1-|\omega|^2)^{(n+\alpha+1)/2}} \|f \circ \varphi_\omega\|_{\alpha,2} \|v\|_{\alpha,2},$$

for all $v \in H^\infty(\mathbb{B}_n)$ *and* $\omega \in \mathbb{B}_n$.

Lemma 3.4. *Let* $-1 < \alpha < \infty$ *and* $\varepsilon > 0$. *For* $f \in L^2(\mathbb{B}_n, dV_\alpha)$, $v \in H^\infty(\mathbb{B}_n)$ *and multi-index* γ *with* $|\gamma| = m \ge (n+\alpha+1)/2$ *we have*

$$|(T_f^* v)^{(\gamma)}(\omega)| \le C \frac{1}{(1-|\omega|^2)^m} \|f \circ \varphi_\omega\|_{\alpha,2+\varepsilon} \bigg(S_{0,\alpha}(|v|^\delta)(\omega) \bigg)^{1/\delta}$$

for all $\omega \in \mathbb{B}_n$, *where* $\delta = (2+\varepsilon)/(1+\varepsilon)$.

4. Compact Haplitz Products

Using the same technique as in the proof in [2], we have the following Lemma.

Lemma 4.1. *Let* $-1 < \alpha < \infty$ *and* T *be a compact operator on* $(A_\alpha^2(B_n))^\perp$, *then* $\|\mathcal{Y}_\omega(T)\| \to 0$ *as* $|\omega| \to 1^-$.

Proof. If H_1 and H_2 are Hilbert spaces and $T : H_1 \to H_2$ is a compact operator, since operators of finite rank are dense in the set of compact operators, given $\varepsilon > 0$, there exist $f_1, \cdots, f_n \in H_1$ and $g_1, \cdots, g_n \in H_1$ such that

$$\left\| T - \sum_{i=1}^n f_i \otimes g_i \right\| < \varepsilon.$$

Thus the lemma follows once we show the Lemma for operators of rank one.

If $f \in L^2(B_n, dv_\alpha)$, as $|\omega| \to 1^-$, then for every $z \in B_n$ and multi-index γ we have $\omega^\gamma - \varphi_\omega^\gamma(z) \to 0$, so by the Lebesgue Dominated Convergence Theorem,

$$\|\omega^\gamma f - \varphi_\omega^\gamma f\|_{\alpha,2}^2 = \int_{B_n} |\omega^\gamma f(z) - \varphi_\omega^\gamma(z) f(z)|^2 dv_\alpha(z) \to 0$$

as $|\omega| \to 1^-$. It follows that $\|\xi^\gamma f - \varphi_\omega^\gamma f\|_{\alpha,2} \to 0$ as $\omega \in B_n$ tends to $\xi \in \partial B_n$.

Suppose $f \in (A_\alpha^2)$, then $P(\xi^\gamma f) = \xi^\gamma f$, so that

$$\|\xi^\gamma f - T_{\varphi_\omega^\gamma} f\|_{\alpha,2} = \|\xi^\gamma f - P\varphi_\omega^\gamma f\|_{\alpha,2} \to 0,$$

$$\|\xi^\gamma f - S_{\varphi_\omega^\gamma} f\|_{\alpha,2} = \|(I-P)(\xi^\gamma f - \varphi_\omega^\gamma f)\|_{\alpha,2} \to 0$$

as $\omega \in B_n$ tends to $\xi \in \partial B_n$. If $f \in (A_\alpha^2)$, $g \in (A_\alpha^2)^\perp$, then

$$\begin{aligned} & \|\xi^\gamma (f \otimes g)\overline{\xi}^\gamma - T_{\varphi_\omega^\gamma}(f \otimes g) S_{\overline{\varphi}_\omega^\gamma}\| \\ = \; & \|(\xi^\gamma f) \otimes (\xi^\gamma g) - (T_{\varphi_\omega^\gamma} f) \otimes (S_{\varphi_\omega^\gamma} g)\| \\ \leq \; & \|(\xi^\gamma f - T_{\varphi_\omega^\gamma} f) \otimes (\xi^\gamma g)\| + \|(T_{\varphi_\omega^\gamma} f) \otimes (\xi^\gamma g - S_{\varphi_\omega^\gamma} g)\| \\ \leq \; & \|\xi^\gamma f - T_{\varphi_\omega^\gamma} f\|_{\alpha,2} \|g\|_{\alpha,2} + \|f\|_{\alpha,2} \|\xi^\gamma g - S_{\varphi_\omega^\gamma} g\|_{\alpha,2}. \end{aligned}$$

We get

$$\|\xi^\gamma (f \otimes g)\overline{\xi}^\gamma - T_{\varphi_\omega^\gamma}(f \otimes g) S_{\overline{\varphi}_\omega^\gamma}\| \to 0$$

as $\omega \in B_n$ tends to $\xi \in \partial B_n$.

Hence we can get for any nonnegative integer k

$$\Big\| \sum_{|\gamma|=k} \frac{k!}{\gamma!} \big(\xi^\gamma (f \otimes g)\overline{\xi}^\gamma - T_{\varphi_\omega^\gamma}(f \otimes g) S_{\overline{\varphi}_\omega^\gamma}\big) \Big\| \to 0$$

as $\omega \in B_n$ tends to $\xi \in \partial B_n$. Since

$$\begin{aligned} & \|\sum_{k=0}^{\infty} C_{\alpha,k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} T_{\varphi_\omega^\gamma}(f \otimes g) S_{\overline{\varphi}_\omega^\gamma}\| \\ = \; & \|\sum_{k=0}^{\infty} C_{\alpha,k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} \big(T_{\varphi_\omega^\gamma}(f \otimes g) S_{\overline{\varphi}_\omega^\gamma} - \xi^\gamma (f \otimes g)\overline{\xi}^\gamma\big)\| \\ \leq \; & \sum_{k=0}^{\infty} |C_{\alpha,k}| \; \|\sum_{|\gamma|=k} \frac{k!}{\gamma!} \big(\xi^\gamma (f \otimes g)\overline{\xi}^\gamma - T_{\varphi_\omega^\gamma}(f \otimes g) S_{\overline{\varphi}_\omega^\gamma}\big)\| \end{aligned}$$

and by Lemma 2.4 and the series $\sum_{k=0}^{\infty}|C_{\alpha,k}|$ is convergent, we have

$$\|\sum_{k=0}^{\infty} C_{\alpha,k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}T_{\varphi_\omega^\gamma}(f\otimes g)S_{\overline{\varphi}_\omega^\gamma}\|\to 0$$

as $\omega\in B_n$ tends to $\xi\in\partial B_n$.

This completes the proof.

Theorem 4.2. *Let $f\in H^\infty(B_n,dv_\alpha)$ and $g\in L^\infty(B_n,dv_\alpha)$. Then $T_fH_g^*$ is compact if and only if*

$$\lim_{\omega\to\partial B_n}\|f\circ\varphi_\omega\|_{\alpha,2}\|g\circ\varphi_\omega-P(g\circ\varphi_\omega)\|_{\alpha,2}=0.$$

Proof. First we prove the 'if part'. Suppose $T_fH_g^*$ is compact. By Lemma 4.1, $\|\mathcal{Y}_\omega(T_fH_g^*)\|\to 0$ as $\omega\to\partial B_n$. By Lemma 2.3, 2.8, we have

$$\begin{aligned}\mathcal{Y}_\omega(T_fH_g^*) &= \|(T_fk_\omega^{(\alpha)})\otimes(H_{\bar g}k_\omega^{(\alpha)})\|\\ &= \|T_fk_\omega^{(\alpha)}\|_{\alpha,2}\|H_{\bar g}k_\omega^{(\alpha)}\|_{\alpha,2}\\ &= \|f\circ\varphi_\omega\|_{\alpha,2}\|\bar g\circ\varphi_\omega-P_\alpha(\bar g\circ\varphi_\omega)\|_{\alpha,2},\end{aligned}$$

so the 'if part' is proved.

Now we turn to the 'only part'. By formula 4.11 in [2] , we have,for $u\in(A_\alpha^2(B_n))^\perp$,$v\in H^\infty(B_n)$ and $m\geq\frac{n+\alpha+1}{2}$,

$$<(T_fH_g^*)u,v>=<H_g^*u,T_f{}^*v>=I+II+III,$$

where I,II and III are, respectively,

$$I=\sum_{j=1}^{m}b_j\int_{B_n}(1-|z|^2)^{2m+j-1}\{(H_g^*u)(z)\overline{(T_f^*v)(z)}\}dV_\alpha(z),$$

$$II=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2m+1)}\sum_{|\gamma|=m}\int_{B_n}(1-|z|^2)^{2m}\{(H_g^*u)^{(\gamma)}(z)\overline{(T_f^*v)^{(\gamma)}(z)}\}dV_\alpha(z),$$

$$III=\sum_{j=1}^{2m-1}a_j\sum_{|\gamma|=m}\int_{B_n}(1-|z|^2)^{2m+j}\{(H_g^*u)^{(\gamma)}(z)\overline{(T_f^*v)^{(\gamma)}(z)}\}dV_\alpha(z).$$

For $0 < s < 1$, $sB_n = \{sz : z \in B_n\}$ is compact subset of B_n and $B_{n,s} = B_n \backslash sB_n$, it is easy to see that there exist compact operators T_s^I, T_s^{II} and T_s^{III} on $(A_\alpha^2(B_n))^\perp$ such that

$$< T_s^I u, v >= I - I_s, < T_s^{II} u, v >= II - II_s, < T_s^{III} u, v >= III - III_s,$$

where

$$I_s = \sum_{j=1}^{m} b_j \int_{B_{n,s}} (1-|z|^2)^{2m+j-1} \{(H_g^* u)(z)\overline{(T_f^* v)(z)}\} dV_\alpha(z),$$

$$II_s = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2m+1)} \sum_{|\gamma|=m} \int_{B_{n,s}} (1-|z|^2)^{2m} \{(H_g^* u)^{(\gamma)}(z)\overline{(T_f^* v)^{(\gamma)}(z)}\} dV_\alpha(z),$$

$$III_s = \sum_{j=1}^{2m-1} a_j \sum_{|\gamma|=m} \int_{B_{n,s}} (1-|z|^2)^{2m+j} \{(H_g^* u)^{(\gamma)}(z)\overline{(T_f^* v)^{(\gamma)}(z)}\} dV_\alpha(z).$$

The operator $T_s = T_s^I + T_s^{II} + T_s^{III}$ is compact, and $< (T_f H_g^* - T_s)u, v >= I_s + II_s + III_s$, we will estimate each of the terms I_s, II_s and III_s.

Let

$$M_s = \sup_{w \in B_{n,s}} \|f \circ \varphi_w\|_{\alpha,2} \|\bar{g} \circ \varphi_w - P_\alpha(\bar{g} \circ \varphi_w)\|_{\alpha,2}.$$

It follows from Lemma 3.2 and Lemma 3.4 that there exists a constant C such that

$$|I_s| \le M_s C_{m,\alpha} \|u\|_{\alpha,2} \|v\|_{\alpha,2}.$$

Since P_α is bounded on $L^{2+2\varepsilon}(B_n, dV_\alpha)$, there exists a constant C such that

$$\|g \circ \varphi_w - P_\alpha(g \circ \varphi_w)\|_{\alpha,2+\varepsilon} \le C \|g \circ \varphi_w - P_\alpha(g \circ \varphi_w)\|_{\alpha,2}^{\frac{1}{2+\varepsilon}}.$$

An analogous estimate for $|(T_f^* v)^{(\gamma)}|$ also holds. Thus there exists a constant C such that

$$|II_s| \le C M_s^{\frac{1}{2+\varepsilon}} \int_{B_n} |S_{0,\alpha}(|u|^p)(w)|^{\frac{1}{p}} |S_{0,\alpha}(|v|^p)(w)|^{\frac{1}{p}} dV_\alpha(w).$$

Since the operator $S_{0,\alpha}$ is bounded on $L^q(B_n, dV_\alpha)$ for $q = \frac{2}{p} > 1$ by Theorem 2.10 in [5], there exists a constant C such that

$$\int_{B_n} |S_{0,\alpha}(|u|^p)(w)|^q dV_\alpha(w) \le C \|u\|_{\alpha,2}^2.$$

By the Cauchy-Schwarz inequality, there exists a constant C such that

$$|II_s| \le C M_s^{\frac{1}{2+\varepsilon}} \|u\|_{\alpha,2} \|v\|_{\alpha,2}.$$

An analogous estimate for III_s follows easily.

Then it follows from the above inequality that $T_s \to T_f H_{\bar{g}}^*$ in operator norm as $s \to 1^-$, and since each of the T_s is compact, we show that the operator is compact.

This completes the proof.

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