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EIGENVALUES AND EIGENVECTORS OF BRUALDI-LI TOURNAMENT MATRICES

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Abstract. In this paper, we obtain the result that the characteristic polynomial and the right(left) eigenvectors of Brualdi-Li tournament matrices by new methods, and that the Brualdi-Li tournament matrix has exactly one positive eigenvalue and exactly one negative eigenvalue and the others are complex numbers. In addition, we give that some properties for the eigenvalues and the right(left) eigenvector of Brualdi-Li tournament matrices.

Keywords: tournament matrix; Brualdi-Li matrix; eigenvalue; eigenvector.

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1. Introduction

The symbols $\mathbb{C}$ and $\mathbb{R}$ will respectively denote the complex field and the real field. $\mathbb{C}^n = \overbrace{\mathbb{C}\times\mathbb{C}\times\cdots\times\mathbb{C}}^{n}$, $\mathbb{R}^n = \overbrace{\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}}^{n}$. We will denote the set of integers as $\mathbb{Z}$. The real and imaginary parts of a complex number $\lambda\in\mathbb{C}$ will be respectively denoted as $Re(\lambda)$ and $Im(\lambda)$. The complex conjugate of λ will be denoted as $\overline{\lambda}$.

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Throughout this paper we will use the following notations to denote particular matrices. The $n\times n$ identity matrix will be denoted as I_n. The $n\times n$ all-ones matrix will be denoted as J_n, which can be expressed as $J_n=\mathbf{1}_n\mathbf{1}_n^t$, where $\mathbf{1}_n$ is the all-ones $n\times 1$ vector.

The characteristic polynomial of a matrix A is defined as $P(A,\lambda)=det(\lambda I-A)$. The equation $P(A,\lambda)=det(\lambda I-A)=0$ has n roots $\lambda_1,\lambda_2,\cdots,\lambda_n\in\mathbb{C}$, and these roots are defined as the eigenvalues of A. The spectral radius of a matrix A is defined as $\rho=max\{|\lambda_1|,|\lambda_2|,\cdots,|\lambda_n|\}$.

The trace for an $n\times n$ matrix is defined as the sum of its diagonal entries. It is well known that the trace and determinant of A can be respectively expressed as $tr(A)=\lambda_1+\lambda_2+\cdots+\lambda_n$ and $det(A)=\lambda_1\lambda_2\cdots\lambda_n$. The transpose of A is denoted by A^t.

A tournament matrix of order n is a $(0,1)$ matrix T_n satisfying the equation $T_n+T_n^t=J_n-I_n$. Let

$$\mathscr{B}_{2m}=\begin{pmatrix} U_m & U_m^t \\ I_m+U_m^t & U_m \end{pmatrix},$$

where U_m is strictly upper triangular tournament matrix (all of whose entries above the main diagonal are equal to one), i.e.

$$U_m=\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{m\times m}$$

is the tournament matrix of order $2m$.

The matrix $\mathscr{B}_{2m}$ has been dubbed by the Brualdi-Li tournament matrix. In 1983 Brualdi and Li conjectured that the maximal spectral radius for tournaments of order $2m$ is attained by the Brualdi-Li matrix [1]. This conjecture has been confirmed in [2]. The properties of Brualdi-Li tournament matrix have been investigated in[3, 4, 5, 6, 7, 8, 9].

In Section 2, we obtain that the characteristic polynomial of $\mathscr{B}_{2m}$ and the roots of the characteristic polynomial are simple by new methods, and that $\mathscr{B}_{2m}$ has exactly one positive eigenvalue and exactly one negative eigenvalue and the others are complex numbers.

In Section 3, we give that the right(left) eigenvectors of $\mathscr{B}_{2m}$ and some properties for the right(left) eigenvectors of $\mathscr{B}_{2m}$.

2. Eigenvalues of Brualdi-Li Tournament Matrices

By simple calculation, we have

Lemma 2.1. *Let $2 \le m \in \mathbb{Z}$, and $X = (1, x, x^2, \cdots, x^{m-1})^t$, where $x \neq 1$ is real variable. Then*

$$\begin{aligned} (1)\ \ X^t U_m &= -\frac{1}{1-x} X^t + \frac{1}{1-x} \mathbf{1}_m^t, \\ (2)\ \ X^t U_m^t &= \frac{x}{1-x} X^t - \frac{x^m}{1-x} \mathbf{1}_m^t. \end{aligned}$$

Lemma 2.2. *Let $2 \le m \in \mathbb{Z}$, $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathscr{B}_{2m}$, and let $\xi = \binom{v}{w}$ and $\widehat{\xi} = \binom{\widehat{v}}{\widehat{w}}$ denote the right and left eigenvector of $\mathscr{B}_{2m}$ corresponding to λ, respectively, where $v, w, \widehat{v}, \widehat{w} \in \mathbb{C}^m$. Let x be real variable, $X = (1, x, x^2, \cdots, x^{m-1})^t$, $f(x) = X^t v, g(x) = X^t w$, $\widehat{f}(x) = X^t \widehat{v}$, and $\widehat{g}(x) = X^t \widehat{w}$. Then*

$$\begin{aligned} (1)\ \ f(x) &= \frac{a(1+\lambda) + (a+b)(x + x^2 + \cdots + x^{m-1}) + (a - b\lambda)x^m}{(1+\lambda)^2 - \lambda^2 x}, \\ (2)\ \ g(x) &= \frac{a + b(1+\lambda) + (a+b)(x + x^2 + \cdots + x^{m-1}) - a\lambda x^m}{(1+\lambda)^2 - \lambda^2 x}, \\ (3)\ \ \widehat{f}(x) &= \frac{\widehat{b}\lambda - (\widehat{a} + \widehat{b})(x + x^2 + \cdots + x^{m-1}) - (\widehat{a}(1+\lambda) + \widehat{b})x^m}{\lambda^2 - (1+\lambda)^2 x}, \\ (4)\ \ \widehat{g}(x) &= \frac{\widehat{a}\lambda - \widehat{b} - (\widehat{a} + \widehat{b})(x + x^2 + \cdots + x^{m-1}) - \widehat{b}(1+\lambda)x^m}{\lambda^2 - (1+\lambda)^2 x}, \end{aligned}$$

where $a = \mathbf{1}_m^t v, b = \mathbf{1}_m^t w, \widehat{a} = \mathbf{1}_m^t \widehat{v}, \widehat{b} = \mathbf{1}_m^t \widehat{w}$.

Proof. Let λ be an eigenvalue, with the right eigenvector $\xi = \binom{v}{w}$, of $\mathscr{B}_{2m}$, then $\mathscr{B}_{2m}\xi = \lambda\xi$ can be expanded to

$$\begin{pmatrix} U_m & U_m^t \\ I_m + U_m^t & U_m \end{pmatrix} \binom{v}{w} = \lambda \binom{v}{w}.$$

Therefore

$$\begin{cases} U_m v & + & U_m^t w & = & \lambda v \\ (I_m + U_m^t) v & + & U_m w & = & \lambda w. \end{cases}$$

We have

$$\begin{cases} X^t U_m v & + & X^t U_m^t w & = & \lambda X^t v \\ X^t (I_m + U_m^t) v & + & X^t U_m w & = & \lambda X^t w. \end{cases}$$

By Lemma 2.1, we have

$$\begin{cases} (1 + (1-x)\lambda) f(x) - x g(x) & = & a - b x^m \\ -f(x) + (1 + (1-x)\lambda) g(x) & = & b - a x^m. \end{cases}$$

Notice that this equation holds for $x = 1$ too. Hence,

$$\begin{aligned} f(x) &= \frac{(a - bx^m)(1 + (1-x)\lambda) + x(b - ax^m)}{(1 + (1-x)\lambda)^2 - x} \\ &= \frac{(1-x)\Big(a(1+\lambda) + (a+b)(x + x^2 + \cdots + x^{m-1}) + (a - b\lambda)x^m\Big)}{(1-x)\Big((1+\lambda)^2 - \lambda^2 x\Big)} \\ &= \frac{a(1+\lambda) + (a+b)(x + x^2 + \cdots + x^{m-1}) + (a - b\lambda)x^m}{(1+\lambda)^2 - \lambda^2 x}, \end{aligned}$$

$$\begin{aligned} g(x) &= \frac{(b - ax^m)(1 + (1-x)\lambda) + a - bx^m}{(1 + (1-x)\lambda)^2 - x} \\ &= \frac{(1-x)\Big(a + b(1+\lambda) + (a+b)(x + x^2 + \cdots + x^{m-1}) - a\lambda x^m\Big)}{(1-x)\Big((1+\lambda)^2 - \lambda^2 x\Big)} \\ &= \frac{a + b(1+\lambda) + (a+b)(x + x^2 + \cdots + x^{m-1}) - a\lambda x^m}{(1+\lambda)^2 - \lambda^2 x}. \end{aligned}$$

Let $\widehat{\xi} = \binom{\widehat{v}}{\widehat{w}}$ be the left eigenvector of $\mathscr{B}_{2m}$ corresponding to λ, then
$\widehat{\xi}^t \mathscr{B}_{2m} = \lambda \widehat{\xi}^t$ can be expanded to

$$\begin{pmatrix} U_m^t & I_m + U_m \\ U_m & U_m^t \end{pmatrix} \begin{pmatrix} \widehat{v} \\ \widehat{w} \end{pmatrix} = \lambda \begin{pmatrix} \widehat{v} \\ \widehat{w} \end{pmatrix}.$$

Using a similar approach, we arrive at

$$\begin{aligned} \widehat{f}(x) &= \frac{\widehat{b}\lambda - (\widehat{a}+\widehat{b})(x+x^2+\cdots+x^{m-1}) - \left(\widehat{a}(1+\lambda)+\widehat{b}\right)x^m}{\lambda^2-(1+\lambda)^2 x}, \\ \widehat{g}(x) &= \frac{\widehat{a}\lambda - \widehat{b} - (\widehat{a}+\widehat{b})(x+x^2+\cdots+x^{m-1}) - \widehat{b}(1+\lambda)x^m}{\lambda^2-(1+\lambda)^2 x}. \end{aligned}$$

We are done.

Lemma 2.3. *Under the assumptions and in the notation of Lemma 2.2,*

$$\begin{aligned} &(1) \qquad a+b = \mathbf{1}_m^t v + \mathbf{1}_m^t w \neq 0, a = \mathbf{1}_m^t v \neq 0, b = \mathbf{1}_m^t w \neq 0, \\ &(2) \qquad \widehat{a}+\widehat{b} = \mathbf{1}_m^t \widehat{v} + \mathbf{1}_m^t \widehat{w} \neq 0, \widehat{a} = \mathbf{1}_m^t \widehat{v} \neq 0, \widehat{b} = \mathbf{1}_m^t \widehat{w} \neq 0. \end{aligned}$$

Proof. In Lemma 2.2(1), by setting $x = 1$, we have

$$a = f(1) = \frac{a(1+\lambda)+(a+b)(m-1)+a-b\lambda}{(1+\lambda)^2-\lambda^2}.$$

Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathscr{B}_{2m}$. It is easy to see that $\lambda \neq -\frac{1}{2}, m-1, m$. It follows that

$$(a+b)(\lambda - m + 1) = a.$$

If $a+b = 0$, then $a = 0$, and $b = (a+b) - a = 0$. By Lemma 2.2 (1), (2),

$$f(x) \quad = \quad \sum_{k=1}^{m} v_k x^{k-1} \equiv 0,$$

$$g(x) = \sum_{k=1}^{m} w_k x^{k-1} \equiv 0,$$

for an arbitrary real variable x. It is not possible. Hence $a+b \neq 0$. It is easy to see that $a \neq 0$, and $b \neq 0$.

Using a similar approach, we have $\widehat{a}+\widehat{b} = \mathbf{1}_m^t\widehat{v}+\mathbf{1}_m^t\widehat{w} \neq 0, \widehat{a} = \mathbf{1}_m^t\widehat{v} \neq 0, \widehat{b} = \mathbf{1}_m^t\widehat{w} \neq 0$.

We give a new proof of the following theorem.

Theorem 2.4.(Theorem1 in [4]) *Let $2 \leq m \in \mathbb{Z}$, $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathscr{B}_{2m}$, and*

$$P(\mathscr{B}_{2m},\lambda) = \frac{\left(2\lambda^2 - 2(m-1)\lambda - (m-1)\right)\left(\lambda^{2m} + (1+\lambda)^{2m}\right) - \lambda^{2m}}{(1+2\lambda)^2}. \tag{1}$$

Then $P(\mathscr{B}_{2m},\lambda)$ is the characteristic polynomial of $\mathscr{B}_{2m}$.

Proof. Let $2 \leq m \in \mathbb{Z}$ and $\xi = \binom{v}{w}$ be the right eigenvector of $\mathscr{B}_{2m}$ corresponding to $\lambda \in \mathbb{C}$. Let x be a real variable, $X = (1,x,x^2,\cdots,x^{m-1})^t$, $v = (v_1,v_2,\cdots,v_m)^t, f(x) = X^t v, a = \mathbf{1}_m^t v$, and $b = \mathbf{1}_m^t w$.

By Lemma 2.3, we set $a+b = 1$, hence

$$a = \lambda - m + 1, \tag{2}$$

$$b = m - \lambda. \tag{3}$$

Denoting $f_0 = a(1+\lambda), f_1 = f_2 = \cdots = f_{m-1} = 1, f_m = a - b\lambda$, and $d = \left(\frac{\lambda}{1+\lambda}\right)^2$.

By Lemma 2.2(1), we have

$$\begin{aligned} f(x) &= X^t v = \sum_{k=1}^{m} v_k x^{k-1} \\ &= \frac{a(1+\lambda) + (a+b)(x+x^2+\cdots+x^{m-1}) + (a-b\lambda)x^m}{(1+\lambda)^2 - \lambda^2 x} \\ &= \frac{1}{(1+\lambda)^2}\frac{1}{(1-dx)}\sum_{k=0}^{m} f_k x^k \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(1+\lambda)^2}\sum_{k=0}^{\infty} d^k x^k \sum_{k=0}^{m} f_k x^k \\
&= \frac{1}{(1+\lambda)^2}\sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} f_j d^{k-j}\Big) x^k.
\end{aligned}
$$

It must be that the coefficient of x^m, $\frac{1}{(1+\lambda)^2}\sum_{j=0}^{m} f_j d^{m-j} = 0$. Note that $\lambda \neq -1, -\frac{1}{2}$, hence

$$
\begin{aligned}
0 &= \sum_{j=0}^{m} f_j d^{m-j} \\
&= a(1+\lambda)d^m + d^{m-1} + \cdots + d^2 + d + (a-b\lambda)d^0 \\
&= a(1+\lambda)d^m + \frac{d-d^m}{1-d} + a - b\lambda.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
d^m &= \frac{(1-d)(b\lambda-a)-d}{a(1-d)(1+\lambda)-1}, \\
1+\frac{1}{d^m} &= 1+\frac{a(1-d)(1+\lambda)-1}{(1-d)(b\lambda-a)-d} \\
&= \frac{(1-d)\Big(a(1+\lambda)+b\lambda-a\Big)-(1+d)}{(1-d)(b\lambda-a)-d} \\
&= \frac{(1-d)\lambda-(1+d)}{(1-d)(b\lambda-a)-d} \\
&= \frac{\Big(1-(\frac{\lambda}{1+\lambda})^2\Big)\lambda-\Big(1+(\frac{\lambda}{1+\lambda})^2\Big)}{\Big(1-(\frac{\lambda}{1+\lambda})^2\Big)(b\lambda-a)-(\frac{\lambda}{1+\lambda})^2} \\
&= \frac{(1+2\lambda)\lambda-(1+2\lambda+2\lambda^2)}{(1+2\lambda)(b\lambda-a)-\lambda^2} \\
&= \frac{-(1+\lambda)}{(1+2\lambda)\Big((m-\lambda)\lambda-\lambda+m-1\Big)-\lambda^2}
\end{aligned}
$$

$$= \frac{-(1+\lambda)}{-(1+\lambda)\Big(2\lambda^2-2(m-1)\lambda-(m-1)\Big)}$$

$$= \frac{1}{2\lambda^2-2(m-1)\lambda-(m-1)},$$

$$\Big(2\lambda^2-2(m-1)\lambda-(m-1)\Big)\Big(1+\frac{1}{d^m}\Big)-1=0.$$

Note that $\lambda \neq 0$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathscr{B}_{2m}$, then λ satisfies the equation

$$\Big(2\lambda^2-2(m-1)\lambda-(m-1)\Big)\Big(\lambda^{2m}+(1+\lambda)^{2m}\Big)-\lambda^{2m}=0.$$

Now $-\frac{1}{2}$ is a root of multiplicity 2 of the equation. On the other hand, $-\frac{1}{2}$ is not an eigenvalue of $\mathscr{B}_{2m}$. We have that the characteristic polynomial of $\mathscr{B}_{2m}$ is

$$P(\mathscr{B}_{2m},\lambda)=\frac{\Big(2\lambda^2-2(m-1)\lambda-(m-1)\Big)\Big(\lambda^{2m}+(1+\lambda)^{2m}\Big)-\lambda^{2m}}{(1+2\lambda)^2}.$$

The vectors $v_1,v_2,\cdots,v_n$ are said to be linearly dependent, if there exists a finite numbers $a_1,a_2,\cdots,a_n$, not all zero, such that $\sum_{k=1}^{m} a_k v_k = 0$. Otherwise, the vectors $v_1,v_2,\cdots,v_n$ are said to be linearly independent.

The rank of a matrix A is the size of the largest collection of linearly independent columns of A (the column rank) or the size of the largest collection of linearly independent rows of A(the row rank). The rank of a matrix A will be denoted as $rank(A)$.

Let $p(x)$, $q(x)$ be polynomials. A greatest common divisor of $p(x)$ and $q(x)$ is a monic polynomial $d(x)$ that divides $p(x)$ and $q(x)$ such that every common divisor of $p(x)$ and $q(x)$ also divides $d(x)$. The greatest common divisor of $p(x)$ and $q(x)$ is denoted by $gcd(p(x),q(x))$. In particular, $gcd(p(x),q(x))=1$ means that the invertible constants are the only common divisors, and thus $p(x)$ and $q(x)$ are coprime. It is well known that if $gcd(f(x),f'(x))=1$ then $f(x)$ has no multiple divisor, where $f'(x)$ is the derivative of $f(x)$.

We give a new proof of the following theorem.

Theorem 2.5.(Theorem3 in [4]) *For $2 \leq m \in \mathbb{Z}$, the roots of $P(\mathscr{B}_{2m},\lambda)$ are simple.*

Proof. Let $2 \leq m \in \mathbb{Z}$, $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathscr{B}_{2m}$, by Theorem 2.4, λ satisfies the equation

$$\left(2\lambda^2 - 2(m-1)\lambda - (m-1)\right)\left(\left(1+\frac{1}{\lambda}\right)^{2m}+1\right)-1=0. \tag{4}$$

Let $y = 1+\frac{1}{\lambda}$, then $\lambda = \frac{1}{y-1}$. Suppose $y \neq -1$. The equation (4) is

$$(m-1)y^{2m+2}-(m+1)y^{2m}+my^2-2y-m=0. \tag{5}$$

Let

$$\begin{aligned}
F(y) &= (m-1)y^{2m+2}-(m+1)y^{2m}+my^2-2y-m,\\
F_1(y) &= (m+1)y^{2m}-m^2y^2+(2m+1)y+m^2+m,\\
F_2(y) &= (m-1)y^{2m+2}+(m-m^2)y^2+(2m-1)y+m^2.
\end{aligned}$$

The derivative of $F(y)$ is defined as $F'(y)$, then

$$F'(y) = 2(m^2-1)y^{2m+1}-2m(m+1)y^{2m-1}+2my-2.$$

Notice that if $gcd\left(p(x),h(x)\right)=1$, then $gcd\left(p(x),q(x)\right)=gcd\left(p(x),h(x)q(x)\right)$, where $q(x)$ is an arbitrary polynomial, $p(x)\neq 0$.

Obviously, $gcd\left(F(y),\frac{y}{2(m+1)}\right)=1$, therefore

$$\begin{aligned}
& gcd\left(F(y),F'(y)\right)\\
= \;& gcd\left(F(y),\frac{y}{2(m+1)}F'(y)\right)\\
= \;& gcd\left(F(y),F(y)-\frac{y}{2(m+1)}F'(y)\right)\\
= \;& gcd\left(F(y),-y^{2m}+\frac{m^2}{m+1}y^2-\frac{2m+1}{m+1}y-m\right)
\end{aligned}$$

$$
\begin{aligned}
&= gcd\Big(F(y), F_1(y)\Big)\\
&= gcd\Big(F(y)+F_1(y), F_1(y)\Big)\\
&= gcd\Big(F_2(y), F_1(y)\Big).
\end{aligned}
$$

Let

$$
\begin{aligned}
S(y) &= \sum_{k=0}^{2m-2} s_k y^k, \;\; s=(s_0,s_1,s_2,\cdots,s_{2m-2})^t \in \mathbb{R}^{2m-1},\\
Z(y) &= \sum_{k=0}^{2m} z_k y^k, \;\; z=(z_0,z_1,z_2,\cdots,z_{2m})^t \in \mathbb{R}^{2m+1},
\end{aligned}
$$

$\delta = (1,1,0,0,\cdots,0)^t \in \mathbb{R}^{4m+1}$,
$\alpha_i = (\overbrace{0,0,\cdots,0}^{i}, m^2, 2m-1, m-m^2, \overbrace{0,0,\cdots,0}^{2m-2}, m-1, \overbrace{0,0,\cdots,0}^{2m-1-i})^t \in \mathbb{R}^{4m+1}$, $\beta_j = (\overbrace{0,0,\cdots,0}^{j}, m^2 + m, 2m+1, -m^2, \overbrace{0,0,\cdots,0}^{2m-4}, m+1, \overbrace{0,0,\cdots,0}^{2m+1-j})^t \in \mathbb{R}^{4m+1}$, $i=0,1,2,\cdots,2m-2$, $j=0,1,2,\cdots,2m$.

Suppose

$$F_2(y)S(y)+F_1(y)Z(y) = 1+y,$$

i.e.

$$
\begin{aligned}
&\Big((m-1)y^{2m+2}+(m-m^2)y^2+(2m-1)y+m^2\Big)\sum_{k=0}^{2m-2} s_k y^k\\
+&\Big((m+1)y^{2m}-m^2y^2+(2m+1)y+m^2+m\Big)\sum_{k=0}^{2m} z_k y^k = 1+y.
\end{aligned}
$$

We denote the coefficients of y^k in polynomial $F_1(y)S(y)+F_2(y)Z(y)$ as c_k, $k=0,1,2,\cdots,4m+1$. It must be that $c_0=c_1=1, c_k=0, k=2,3,\cdots,4m$. We have the equations:

$$\Big(\alpha_0,\alpha_1,\cdots,\alpha_{2m-2},\beta_0,\beta_1,\cdots,\beta_{2m}\Big)\begin{pmatrix} s\\ z\end{pmatrix} = \delta. \tag{6}$$

i.e.

$$A\begin{pmatrix} s\\ z\end{pmatrix} = \delta, \tag{7}$$

where $A=(\alpha_0,\alpha_1,\cdots,\alpha_{2m-2},\beta_0,\beta_1,\cdots,\beta_{2m})_{(4m+1)\times(4m)}$ and $(A\mid\delta)$ are the coefficient matrix and the augmented matrix of equations (7), respectively.

For the augmented matrix $(A\mid\delta)$, add $(-1)^{k-1}$ times the $k-th$ row to the first row, $k=2,3,\cdots,4m+1$. It is obvious to see that all of whose entries in the first row are equal to zero. Hence

$$rank(A\mid\delta)<4m+1.$$

Let

$$\sum_{k=0}^{2m-2}p_k\alpha_k+\sum_{k=0}^{2m}q_k\beta_k=0. \tag{8}$$

Suppose $\gamma_k=(\overbrace{0,0,\cdots,0}^{k+1},(-1)^{k+2},(-1)^{k+3},\cdots,(-1)^{4m})^t\in\mathbb{R}^{4m+1}, 0\le k\le 2m$. By simple calculation, we have

$\gamma_k^t\alpha_j=0$, if $k<j$, $\gamma_k^t\alpha_j\neq 0$, if $k=j, k=0,1,\cdots,2m-2$,
$\gamma_k^t\beta_j=0$, if $k<j$, $\gamma_k^t\beta_j\neq 0$, if $k=j, k=0,1,\cdots,2m$.

By formula (8), we have

$$\gamma_0^t\left(\sum_{k=0}^{2m-2}p_k\alpha_k+\sum_{k=0}^{2m}q_k\beta_k\right)=p_0\alpha_0+q_0\beta_0=\gamma_0^t0=0.$$

Obviously, α_0 and β_0 are linearly independent, hence

$$p_0=q_0=0.$$

Formula (8) yields

$$\sum_{k=1}^{2m-2}p_k\alpha_k+\sum_{k=1}^{2m}q_k\beta_k=0.$$

$$\gamma_1^t\left(\sum_{k=1}^{2m-2}p_k\alpha_k+\sum_{k=1}^{2m}q_k\beta_k\right)=p_1\alpha_1+q_1\beta_1=\gamma_1^t0=0.$$

We have

$$p_1=q_1=0.$$

Using the similar method to obtain

$p_k=0$, for $k=2,3,\cdots,2m-2, q_j=0$, for $j=2,3,\cdots,2m$.

Therefore, $\left\{\alpha_0, \alpha_1, \cdots, \alpha_{2m-2}, \beta_0, \beta_1, \cdots, \beta_{2m}\right\}$ are linearly independent. Further, $rank(A) = rank\Big(\{\alpha_0, \alpha_1, \cdots, \alpha_{2m-2}, \beta_0, \beta_1, \cdots, \beta_{2m})\}\Big) = 4m$. Hence,

$$rank(A) = rank\Big((A \mid \delta)\Big) = 4m.$$

Notice that the equation $AX = b$ has its solution if and only if $rank(A) = rank(A \mid b)$. By the above result, there exists a solution of the equations (7) with $S(y)$ and $Z(y)$ satisfying the equation

$$F_2(y)S(y) + F_1(y)Z(y) = 1 + y.$$

Because $1 + y$ is a common divisor of $F_2(y)$ and $F_1(y)$, then

$$gcd\Big(F_2(y), F_1(y)\Big) = 1 + y,$$

$$gcd\left(\frac{F(y)}{1+y}, \frac{F'(y)}{1+y}\right) = gcd\left(\frac{F_2(y)}{1+y}, \frac{F_1(y)}{1+y}\right) = 1.$$

It is easy to show that the roots of $\frac{F(y)}{1+y}$ are simple. Hence, the roots of $P(\mathscr{B}_{2m}, \lambda)$ are simple.

Lemma 2.6.(Lemma 2 and Corollary 2 in[9]) *Let $2 \le m \in \mathbb{Z}$, ρ_{2m} be the spectral radius of $\mathscr{B}_{2m}$. Then*

$$m - \frac{1}{2} - \frac{1}{5m} < \rho_{2m} < m - \frac{1}{2} - \frac{1}{4\tau m}$$

where $e = 2.71828\cdots, \tau = \frac{e^2+1}{e^2-1}$.

Theorem 2.7. *Let $2 \le m \in \mathbb{Z}$, the Brualdi-Li Matrix $\mathscr{B}_{2m}$ has exactly one positive eigenvalue ρ_{2m} and exactly one negative eigenvalue λ_0 and others are complex λ_k for $k = 1, 2, \cdots, 2m-2$, satisfying*

(1) ρ_{2m} is the spectral radius of $\mathscr{B}_{2m}$, $m - \frac{1}{2} - \frac{1}{5m} < \rho_{2m} < m - \frac{1}{2} - \frac{1}{4\tau m}$,

(2) $-\frac{1}{2} < Re(\lambda_k) < -\frac{1}{2} + \frac{1}{5m}$ for $k = 0, 1, 2, \cdots, 2m-2$,

where $e = 2.71828\cdots, \tau = \frac{e^2+1}{e^2-1}$.

Proof. Let $2 \le m \in \mathbb{Z}$, $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathscr{B}_{2m}$, ξ be the right eigenvector of $\mathscr{B}_{2m}$ corresponding to λ, and ρ_{2m} be the spectral radius of $\mathscr{B}_{2m}$. by the Perron-Frobenius theorem,

ρ_{2m} is simple, by Lemma 2.6,

$$m-\frac{1}{2}-\frac{1}{5m}<\rho_{2m}<m-\frac{1}{2}-\frac{1}{4\tau m}.$$

By Lemma 2.3, We set $\mathbf{1}_{2m}^t\xi=1$. Hence

$$\begin{aligned}
1 &= \overline{\xi}^t J\xi \\
&= \overline{\xi}^t(I+\mathscr{B}_{2m}+\mathscr{B}_{2m}^t)\xi \\
&= \overline{\xi}^t\xi+\overline{\xi}^t\mathscr{B}_{2m}\xi+\overline{\xi}^t\mathscr{B}_{2m}^t\xi \\
&= \overline{\xi}^t\xi+\lambda\overline{\xi}^t\xi+\overline{\lambda\xi}^t\xi \\
&= (1+\lambda+\overline{\lambda})\overline{\xi}^t\xi \\
&= \Big(1+2Re(\lambda)\Big)\overline{\xi}^t\xi,
\end{aligned}$$

$$Re(\lambda)=-\frac{1}{2}+\frac{1}{\overline{\xi}^t\xi}>-\frac{1}{2}.$$

Let $\rho_{2m},\lambda_{k_0},\lambda_{k_1},\cdots,\lambda_{k_{s-1}},\lambda_{k_s},\cdots,\lambda_{k_{2m-2}}\in\mathbb{C}$ are all eigenvalues of $\mathscr{B}_{2m}$, where $0\le s\in\mathbb{Z},Re(\lambda_{k_0}),Re(\lambda_{k_1}),\cdots,Re(\lambda_{k_{s-1}})\ge-\frac{1}{2}+\frac{1}{5m}$,
$Re(\lambda_{k_s}),Re(\lambda_{k_{s+1}}),\cdots,Re(\lambda_{k_{2m-2}})<-\frac{1}{2}+\frac{1}{5m}$.
It is well known that

$$\rho_{2m}+\sum_{i=0}^{2m-2}\lambda_{k_i}=\rho_{2m}+\sum_{i=0}^{2m-1}Re(\lambda_{k_i})=tr(\mathscr{B}_{2m})=0.$$

Then,

$$\rho_{2m}+\sum_{i=0}^{s-1}\lambda_{k_i} \;=\; -\sum_{i=s}^{2m-2}Re(\lambda_{k_i})<\sum_{i=s}^{2m-2}\frac{1}{2}=\frac{2m-1-s}{2},$$

On the other hand,

$$\rho_{2m}+\sum_{i=0}^{s-1}\lambda_{k_i} \;\ge\; \rho_{2m}>m-\frac{1}{2}-\frac{1}{5m}+s(-\frac{1}{2}+\frac{1}{5m}).$$

Hence,

$$\begin{aligned} m-\frac{1}{2}-\frac{1}{5m}+s(-\frac{1}{2}+\frac{1}{5m}) &< \frac{2m-1-s}{2}, \\ \frac{s}{5m} &< \frac{1}{5m}. \end{aligned}$$

We have $s=0$. Therefore,

$-\frac{1}{2} < Re(\lambda_{k_i}) < -\frac{1}{2}+\frac{1}{5m}$ for $i=0,1,2,\cdots,2m-2$ and $\mathscr{B}_{2m}$ has exactly one nonegative real eigenvalue ρ_{2m}.

Notice that

$$\begin{aligned} F(y) &= (m-1)y^{2m+2}-(m+1)y^{2m}+my^2-2y-m \\ &= a_0y^{2m+2}+a_1y^{2m}+a_2y^2+a_3y+a_4, \\ F(-y) &= a_0y^{2m+2}+a_1y^{2m}+a_2y^2-a_3y+a_4, \end{aligned}$$

where $a_0=(m-1)>0, a_1=-(m+1)<0, a_2=m>0, a_3=-2<0, a_4=-m<0$.
By Descartes?? Rule of Signs, $F(y)$ has exactly three negative real roots

$$y_0, -1, -1.$$

But -1 is not a eigenvalue of $\mathscr{B}_{2m}$, hence $\mathscr{B}_{2m}$ has exactly one negative real eigenvalue $\lambda_0 = \frac{1}{y_0-1}$. We are done.

Theorem 2.8. *Let $2\le m\in\mathbb{Z}$, $\mathscr{B}_{2m}$ have $2m$ eigenvalues $\lambda_k, k=1,2,3,\cdots,2m$. Then,*

$$(1)\ \sum_{k=1}^{2m}\frac{1}{\lambda_k}=-2m-2,$$

$$(2)\ \prod_{k=1}^{2m}\left(1+\lambda_k\right)=m.$$

Proof. $\mathscr{B}_{2m}$ have $2m$ eigenvalues $\lambda_k, k=1,2,3,\cdots,2m$, by formula (4)and (5), the polynomial $F(y)=(m-1)y^{2m+2}-(m+1)y^{2m}+my^2-2y-m$ have roots $y_k=1,1,1+\frac{1}{\lambda_k}$, $k=1,2,3,\cdots,2m$. Hence,

$$2+\sum_{k=1}^{2m}\left(1+\frac{1}{\lambda_k}\right)=0.$$

and

$$\prod_{k=1}^{2m}\left(1+\frac{1}{\lambda_k}\right)=\frac{-m}{m-1}.$$

Therefore,

$$\sum_{k=1}^{2m}\frac{1}{\lambda_k}=-2m-2.$$

It is well known that $det(\mathscr{B}_{2m})=1-m$, hence

$$\begin{aligned}\prod_{k=1}^{2m}\left(1+\lambda_k\right) &= \prod_{k=1}^{2m}\left(1+\frac{1}{\lambda_k}\right)\prod_{k=1}^{2m}\lambda_k\\ &= \frac{-m}{m-1}det(\mathscr{B}_{2m})\\ &= m.\end{aligned}$$

3. Eigenvectors of Brualdi-Li Tournament Matrices

Theorem 3.1. *Let $2\leq m\in\mathbb{Z}$, $\lambda\in\mathbb{C}$ be an eigenvalue of $\mathscr{B}_{2m}$, and let $\xi=\binom{v}{w}$ and $\widehat{\xi}=\binom{\widehat{v}}{\widehat{w}}$ denote the right and left eigenvector of $\mathscr{B}_{2m}$ corresponding to λ, respectively, $\mathbf{1}_m^t v+\mathbf{1}_m^t w=\mathbf{1}_m^t\widehat{v}+\mathbf{1}_m^t\widehat{w}=1$. Then,*

$$\begin{aligned}(1)\ v &= Q_m(\lambda)\eta_v,\\ (2)\ w &= Q_m(\lambda)\eta_w,\\ (3)\ \widehat{v} &= \widehat{Q}_m(\lambda)\widehat{\eta}_v,\\ (4)\ \widehat{w} &= \widehat{Q}_m(\lambda)\widehat{\eta}_w,\end{aligned}$$

where

$$Q_m(\lambda) = \begin{pmatrix} 1 & & & & \\ d & 1 & & & \\ d^2 & d & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ d^{m-1} & d^{m-2} & d^{m-3} & \cdots & 1 \end{pmatrix}_{m\times m,}$$

$$\widehat{Q}_m(\lambda) = \begin{pmatrix} 1 & & & & \\ d^{-1} & 1 & & & \\ d^{-2} & d^{-1} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ d^{-(m-1)} & d^{-(m-2)} & d^{-(m-3)} & \cdots & 1 \end{pmatrix}_{m\times m,}$$

$$\begin{aligned} \eta_v &= \frac{1}{(1+\lambda)^2}\Big((1+\lambda)(\lambda-m+1),1,1,\cdots,1\Big)^t \in \mathbb{C}^m, \\ \eta_w &= \frac{1}{(1+\lambda)^2}\Big(\lambda(m-\lambda)+1,1,1,\cdots,1\Big)^t \in \mathbb{C}^m, \\ \widehat{\eta}_v &= -\frac{1}{\lambda^2}\Big(-\lambda(\lambda-m+1),1,1,\cdots,1\Big)^t \in \mathbb{C}^m, \\ \widehat{\eta}_w &= -\frac{1}{\lambda^2}\Big(-(1+\lambda)(m-\lambda)+1,1,1,\cdots,1\Big)^t \in \mathbb{C}^m, \\ d &= \left(\frac{\lambda}{1+\lambda}\right)^2. \end{aligned}$$

Proof. Let $\lambda \in \mathbb{C}$ be an arbitrary eigenvalue of $\mathscr{L}\mathscr{B}_{2m}$, and let $\xi = \binom{v}{w}$ and $\widehat{\xi} = \binom{\widehat{v}}{\widehat{w}}$ denote the right and left eigenvector of $\mathscr{B}_{2m}$ corresponding to λ, respectively,

$$v = (v_1, v_2, \cdots, v_m)^t, w = (w_1, w_2, \cdots, w_m)^t \in \mathbb{C}^m,$$

$$\widehat{v} = (\widehat{v}_1, \widehat{v}_2, \cdots, \widehat{v}_m)^t, \widehat{w} = (\widehat{w}_1, \widehat{w}_2, \cdots, \widehat{w}_m)^t \in \mathbb{C}^m.$$

Let x be a real variable,

$$\begin{aligned}
X &= (1, x, x^2, \cdots, x^{m-1})^t, \\
f(x) &= X^t v, \quad g(x) = X^t w, \\
\widehat{f}(x) &= X^t \widehat{v}, \quad g(x) = X^t \widehat{w}, \\
a &= \mathbf{1}_m^t v, \quad b = \mathbf{1}_m^t w, \\
\widehat{a} &= \mathbf{1}_m^t \widehat{v}, \quad \widehat{b} = \mathbf{1}_m^t \widehat{w}.
\end{aligned}$$

In theorem 2.4, it is shown that

$$f(x) = \sum_{k=1}^{m} v_k x^{k-1} = \frac{1}{(1+\lambda)^2} \sum_{k=0}^{\infty} (\sum_{j=0}^{k} f_j d^{k-j}) x^k,$$

where $f_0 = a(1+\lambda), f_1 = f_2 = \cdots = f_{m-1} = 1, f_m = a - b\lambda,$ and $d = \left(\frac{\lambda}{1+\lambda}\right)^2$. It must be that

$$v_k = \frac{1}{(1+\lambda)^2} \sum_{j=0}^{k-1} f_j d^{k-1-j}, \ \ k = 1, 2, \cdots, m.$$

We set $a+b=1$, by formula (2),(3), $f_0 = a(1+\lambda) = (\lambda - m + 1)(\lambda + 1)$. Hence,

$$v_k = \frac{1}{(1+\lambda)^2} \sum_{j=0}^{k-1} f_j d^{k-1-j} = \frac{1}{(1+\lambda)^2} \left((\lambda - m + 1)(\lambda + 1) d^{k-1} + \sum_{j=1}^{k-1} d^{k-1-j} \right) \tag{9}$$

$k = 1, 2, \cdots, m$. Therefore, $v = Q_m(\lambda)\eta_v$ holds. Using a similar approach, we obtain $w = Q_m(\lambda)\eta_w$.

By lemma 2.2(3), setting $x = 1$, we have

$$\widehat{a} = \widehat{f}(1) = \frac{\widehat{b}\lambda - (m-1) - (\widehat{a}(1+\lambda) + \widehat{b})}{\lambda^2 - (1+\lambda)^2}.$$

Setting $\widehat{a} + \widehat{b} = 1$, we have

$$\widehat{a} = m - \lambda,$$

$$\widehat{b} = \lambda - m + 1.$$

Denoting $\widehat{f}_0 = -\widehat{b}\lambda = -\lambda(\lambda - m + 1), \widehat{f}_1 = \widehat{f}_2 = \cdots = f_{m-1} = 1, \widehat{f}_m = \widehat{a}(1+\lambda) + \widehat{b}$. By lemma 2.2(3), we have

$$\begin{aligned}
\widehat{f}(x) &= X^t \widehat{v} = \sum_{k=1}^{m} \widehat{v}_k x^{k-1} \\
&= \frac{\widehat{b}\lambda - (x + x^2 + \cdots + x^{m-1}) - (\widehat{a}(1+\lambda) + \widehat{b})x^m}{\lambda^2 - (1+\lambda)^2 x}, \\
&= -\frac{1}{\lambda^2} \frac{1}{(1 - d^{-1}x)} \sum_{j=0}^{m} \widehat{f}_j x^j \\
&= -\frac{1}{\lambda^2} \sum_{i=0}^{\infty} d^{-i} x^i \sum_{j=0}^{m} \widehat{f}_j x^j \\
&= -\frac{1}{\lambda^2} \sum_{k=0}^{\infty} (\sum_{j=0}^{k} \widehat{f}_j d^{-(k-j)}) x^k.
\end{aligned}$$

It must be that the coefficient of x^k,

$$\widehat{v}_k = -\frac{1}{\lambda^2} \sum_{j=0}^{k-1} \widehat{f}_j d^{-(k-1-j)} \tag{10}$$

$k = 1, 2, \cdots, m$. We have $\widehat{v} = \widehat{Q}_m(\lambda)\widehat{\eta}_v$.

Using a similar approach, we obtain $\widehat{w} = \widehat{Q}_m(\lambda)\widehat{\eta}_w$, thereby giving the desired results.

As a consequence of Theorem 3.1, we can express $v_k, w_k, \widehat{v}_k$ and $\widehat{w}_k$ in terms of λ.

Corollary 3.2. *Under the assumptions and in the notation of Theorem 3.1, for $k = 1, 2, \cdots, m$,*

$$\begin{aligned}
(1)\ v_k &= \frac{1}{1+2\lambda}\left(1 + \frac{G-1}{1+\lambda} d^{k-1}\right), \\
(2)\ w_k &= \frac{1}{1+2\lambda}\left(1 - \frac{\lambda(G-1)}{(1+\lambda)^2} d^{k-1}\right), \\
(3)\ \widehat{v}_k &= w_{m+1-k} = \frac{1}{1+2\lambda}\left(1 + \frac{G}{\lambda} d^{-(k-1)}\right), \\
(4)\ \widehat{w}_k &= v_{m+1-k} = \frac{1}{1+2\lambda}\left(1 - \frac{(1+\lambda)G}{\lambda^2} d^{-(k-1)}\right),
\end{aligned}$$

where $d = \left(\frac{\lambda}{1+\lambda}\right)^2, G = 2\lambda^2 - 2(m-1)\lambda - m + 1$.

Proof. By formula (9), it follows that for $k = 1, 2, \cdots, m$,

$$\begin{aligned} v_k &= \frac{1}{(1+\lambda)^2}\left((\lambda - m + 1)(\lambda + 1)d^{k-1} + \sum_{j=1}^{k-1} d^{k-1-j}\right) \\ &= \frac{1}{(1+\lambda)^2}\left((\lambda - m + 1)(\lambda + 1)d^{k-1} + \frac{1-d^{k-1}}{1-d}\right) \\ &= \frac{1}{(1+\lambda)^2(1-d)}\left(1 + \Big((1-d)(\lambda - m + 1)(\lambda + 1) - 1\Big)d^{k-1}\right) \\ &= \frac{1}{1+2\lambda}\left(1 + \frac{2\lambda^2 - 2(m-1)\lambda - m}{1+\lambda}d^{k-1}\right) \\ &= \frac{1}{1+2\lambda}\left(1 + \frac{G-1}{1+\lambda}d^{k-1}\right). \end{aligned}$$

Using a similar approach, we obtain

$$w_k = \frac{1}{1+2\lambda}\left(1 - \frac{\lambda(G-1)}{(1+\lambda)^2}d^{k-1}\right).$$

By formula (10), it follows that for $k = 1, 2, \cdots, m$,

$$\begin{aligned} \widehat{v}_k &= -\frac{1}{\lambda^2}\left(-(\lambda - m + 1)\lambda d^{-(k-1)} + \sum_{j=1}^{k-1} \widehat{f}_j d^{-(k-1-j)}\right) \\ &= -\frac{1}{\lambda^2}\left(-(\lambda - m + 1)\lambda d^{-(k-1)} + \frac{1-d^{-(k-1)}}{1-d^{-1}}\right) \\ &= -\frac{1}{\lambda^2(1-d^{-1})}\left(1 - \Big((1-d^{-1})(\lambda - m + 1)\lambda + 1\Big)d^{-(k-1)}\right) \\ &= -\frac{1}{\lambda^2(1-d^{-1})}\left(1 - \Big((1-d^{-1})(\lambda - m + 1)\lambda + 1\Big)d^{-(k-1)}\right) \\ &= \frac{1}{1+2\lambda}\left(1 - \Big((1-d^{-1})(\lambda - m + 1)\lambda + 1\Big)d^{-(k-1)}\right) \\ &= \frac{1}{1+2\lambda}\left(1 + \frac{2\lambda^2 - 2(m-1)\lambda - m + 1}{\lambda}d^{-(k-1)}\right) \\ &= \frac{1}{1+2\lambda}\left(1 + \frac{G}{\lambda}d^{-(k-1)}\right). \end{aligned}$$

By formula (4),

$$G=\frac{d^m}{1+d^m},\quad 1+d^m=\frac{1}{1-G}.$$

We have

$$\begin{aligned}
\widehat{v}_k &= \frac{1}{1+2\lambda}\left(1+\frac{G}{\lambda}d^{-(k-1)}\right)\\
&= \frac{1}{1+2\lambda}\left(1+\frac{d^m}{\lambda(1+d^m)}d^{-(k-1)}\right)\\
&= \frac{1}{1+2\lambda}\left(1+\frac{d(1-G)}{\lambda}d^{(m+1-k)-1}\right)\\
&= \frac{1}{1+2\lambda}\left(1-\frac{\lambda(G-1)}{(1+\lambda)^2}d^{(m+1-k)-1}\right)\\
&= w_{m+1-k}.
\end{aligned}$$

Using a similar approach, we obtain

$$\widehat{w}_k=v_{m+1-k}=\frac{1}{1+2\lambda}\left(1-\frac{(1+\lambda)G}{\lambda^2}d^{-(k-1)}\right).$$

Theorem 3.3. *Let $2\le m\in\mathbb{Z}$, $\lambda\in\mathbb{C}$ be an eigenvalue of $\mathscr{B}_{2m}$, and let $\xi=\binom{v}{w}$ and $\widehat{\xi}=\binom{\widehat{v}}{\widehat{w}}$ denote the right and left eigenvector of $\mathscr{B}_{2m}$ corresponding to λ, respectively,*

$$v=(v_1,v_2,\cdots,v_m)^t, w=(w_1,w_2,\cdots,w_m)^t\in\mathbb{C}^m,$$

$$\widehat{v}=(\widehat{v}_1,\widehat{v}_2,\cdots,\widehat{v}_m)^t, \widehat{w}=(\widehat{w}_1,\widehat{w}_2,\cdots,\widehat{w}_m)^t\in\mathbb{C}^m,$$

$$\mathbf{1}_m^t v+\mathbf{1}_m^t w=\mathbf{1}_m^t\widehat{v}+\mathbf{1}_m^t\widehat{w}=1.$$

Then

$$(1)\ \ v_{k+1}-dv_k=w_{k+1}-dw_k=\frac{1}{(1+\lambda)^2},\ \ k=1,2,\cdots,m-1,$$

$$(2)\ \ v_{k+1}-v_k=\frac{1-G}{\lambda^2(1+\lambda)}d^k,\ \ w_{k+1}-w_k=-\frac{1-G}{\lambda(1+\lambda)^2}d^k,\ \ k=1,2,\cdots,m-1,$$

$$(3)\ \ v_k-w_k=\frac{G-1}{\lambda^2}d^k,\ \ \widehat{v}_k-\widehat{w}_k=\frac{G}{(1+\lambda)^2}d^{-k},\ \ k=1,2,\cdots,m,$$

$$(4)\ \ \frac{v_{k+1}-v_k}{w_{k+1}-w_k}=\frac{\widehat{w}_{k+1}-\widehat{w}_k}{\widehat{v}_{k+1}-\widehat{v}_k}=-\frac{1+\lambda}{\lambda},\ \ \ k=1,2,\cdots,m-1,$$

where $d=\left(\frac{\lambda}{1+\lambda}\right)^2, G=2\lambda^2-2(m-1)\lambda-m+1.$

Proof. By Corollary 3.2(1), it follows that for $k=1,2,\cdots,m-1$,
$(1+2\lambda)v_k-1=\frac{G-1}{1+\lambda}d^{k-1}$ and $(1+2\lambda)v_{k+1}-1=\frac{G-1}{1+\lambda}d^k$, hence

$$\frac{(1+2\lambda)v_{k+1}-1}{(1+2\lambda)v_k-1}=d.$$

We have

$$v_{k+1}-dv_k=\frac{1-d}{1+2\lambda}=\frac{1}{(1+\lambda)^2}.$$

Using a similar approach, we obtain

$$w_{k+1}-dw_k=\frac{1}{(1+\lambda)^2}.$$

By Corollary 3.2(1), it follows that for $k=1,2,\cdots,m-1$,

$$\begin{aligned}
v_{k+1}-v_k &= \frac{1}{1+2\lambda}\left(\frac{G-1}{1+\lambda}d^k-\frac{G-1}{1+\lambda}d^{k-1}\right)\\
&= \frac{1}{1+2\lambda}\left(\frac{G-1}{1+\lambda}d^k-\frac{G-1}{1+\lambda}d^{k-1}\right)\\
&= \frac{1}{1+2\lambda}\left(1-\frac{1}{d}\right)\frac{G-1}{1+\lambda}d^k\\
&= \frac{1-G}{\lambda^2(1+\lambda)}d^k.
\end{aligned}$$

Using a similar approach, we obtain

$$w_{k+1}-w_k=-\frac{1-G}{\lambda(1+\lambda)^2}d^k.$$

By Corollary 3.2(1),(2), it follows that for $k=1,2,\cdots,m$,

$$\begin{aligned}
v_k-w_k &= \frac{1}{1+2\lambda}\left(1+\frac{G-1}{1+\lambda}d^{k-1}\right)-\frac{1}{1+2\lambda}\left(1-\frac{\lambda(G-1)}{(1+\lambda)^2}d^{k-1}\right)\\
&= \frac{1}{1+2\lambda}\left(\frac{1}{1+\lambda}-\frac{\lambda}{(1+\lambda)^2}\right)(G-1)d^{k-1}\\
&= \frac{G-1}{\lambda^2}d^k.
\end{aligned}$$

Using a similar approach, we obtain

$$\widehat{v}_k-\widehat{w}_k=\frac{G}{(1+\lambda)^2}d^{-k}.$$

It is clear

$$\frac{v_{k+1}-v_k}{w_{k+1}-w_k}=-\frac{1+\lambda}{\lambda}.$$

By Corollary 3.2(3),(4), we have

$$\frac{\widehat{w}_{k+1}-\widehat{w}_k}{\widehat{v}_{k+1}-\widehat{v}_k}=\frac{v_{m-k}-v_{m+1-k}}{w_{m-k}-w_{m+1-k}}=-\frac{1+\lambda}{\lambda}.$$

Theorem 3.4. *Let $2\le m\in\mathbb{Z}$, $\lambda\in\mathbb{C}$ be an eigenvalue of $\mathscr{B}_{2m}$, and let $\xi=\binom{v}{w}$ and $\widehat{\xi}=\binom{\widehat{v}}{\widehat{w}}$ denote the right and left eigenvector of $\mathscr{B}_{2m}$ corresponding to λ, respectively, $\mathbf{1}_m^t v+\mathbf{1}_m^t w=\mathbf{1}_m^t\widehat{v}+\mathbf{1}_m^t\widehat{w}=1$. Then*

$$\xi^t\widehat{\xi}=2\left(m+\frac{1}{1+2\lambda}-\frac{m^2(m+1)}{1+\lambda}+\frac{m^2(m-1)}{\lambda}\right)$$

Proof. Let $d=\left(\frac{\lambda}{1+\lambda}\right)^2, G=2\lambda^2-2(m-1)\lambda-m+1$,

$$v=(v_1,v_2,\cdots,v_m)^t, w=(w_1,w_2,\cdots,w_m)^t\in\mathbb{C}^m,$$

$$\widehat{v}=(\widehat{v}_1,\widehat{v}_2,\cdots,\widehat{v}_m)^t, \widehat{w}=(\widehat{w}_1,\widehat{w}_2,\cdots,\widehat{w}_m)^t\in\mathbb{C}^m,$$

By formula (4), we have

$$d^m = \frac{G}{1-G}, \quad 1-d^m = \frac{1-2G}{1-G}.$$

By Corollary 3.2, we have

$$\begin{aligned}
\xi^t \widehat{\xi} &= \sum_{k=1}^m v_k \widehat{v}_k + \sum_{k=1}^m w_k \widehat{w}_k \\
&= \sum_{k=1}^m v_k w_{m+1-k} + \sum_{k=1}^m w_k v_{m+1-k} \\
&= 2\sum_{k=1}^m v_k w_{m+1-k} \\
&= 2\sum_{k=1}^m \frac{1}{(1+2\lambda)^2}\left(1+\frac{G-1}{1+\lambda}d^{k-1}\right)\left(1-\frac{\lambda(G-1)}{(1+\lambda)^2}d^{m-k}\right) \\
&= \frac{2}{(1+2\lambda)^2}\sum_{k=1}^m \left(1+\frac{G-1}{1+\lambda}d^{k-1}-\frac{\lambda(G-1)}{(1+\lambda)^2}d^{m-k}-\frac{(G-1)^2}{\lambda(1+\lambda)}d^m\right) \\
&= \frac{2}{(1+2\lambda)^2}\left(m+\frac{(G-1)(1-d^m)}{(1+\lambda)(1-d)}-\frac{\lambda(G-1)(1-d^m)}{(1+\lambda)^2(1-d)}-\frac{m(G-1)^2}{\lambda(1+\lambda)}d^m\right) \\
&= \frac{2}{(1+2\lambda)^2}\left(m+\frac{(G-1)(1-d^m)}{1+2\lambda}-\frac{m(G-1)^2}{\lambda(1+\lambda)}d^m\right) \\
&= \frac{2}{(1+2\lambda)^2}\left(m+\frac{(G-1)(1-d^m)}{1+2\lambda}-\frac{m(G-1)^2}{\lambda(1+\lambda)}d^m\right) \\
&= \frac{2}{(1+2\lambda)^2}\left(m+\frac{2G-1}{1+2\lambda}+\frac{m(G-1)G}{(1+\lambda)\lambda}\right).
\end{aligned}$$

Notice that $G = 2\lambda^2 - 2(m-1)\lambda - m + 1 = (1+2\lambda)(\lambda - m + 1) - \lambda$. Therefore,

$$\begin{aligned}
\xi^t \widehat{\xi} &= \frac{2}{(1+2\lambda)^2}\left(m+\frac{2G-1}{1+2\lambda}+\frac{m(G-1)G}{(1+\lambda)\lambda}\right) \\
&= \frac{2}{(1+2\lambda)^2}\left(m+(1+2\lambda-2m)+\frac{m(1+2\lambda)^2(\lambda-m+1)(\lambda-m)}{(1+\lambda)\lambda}+m\right) \\
&= 2\left(\frac{1}{1+2\lambda}+\frac{m(\lambda-m+1)(\lambda-m)}{(1+\lambda)\lambda}\right) \\
&= 2\left(m+\frac{1}{1+2\lambda}-\frac{m^2(m+1)}{1+\lambda}+\frac{m^2(m-1)}{\lambda}\right).
\end{aligned}$$

Theorem 3.5. *Let $2 \le m \in \mathbb{Z}$, $\lambda \in \mathbb{R}$ be an eigenvalue of $\mathscr{B}_{2m}$, and let $\xi = \binom{v}{w}$ denote the right eigenvector of $\mathscr{B}_{2m}$ corresponding to λ,*

$$\begin{aligned} v = (v_1, v_2, \cdots, v_m)^t, w = (w_1, w_2, \cdots, w_m)^t \in \mathbb{C}^m, \\ \mathbf{1}_m^t v + \mathbf{1}_m^t w = 1. \end{aligned}$$

(1) If $\lambda > 0$, then

$$v_1 < v_2 < \cdots < v_m < w_m < w_{m-1} < \cdots < w_1,$$

(2) If $\lambda < 0$, then

$$v_1 < w_1 < v_2 < w_2 < \cdots < v_{m-1} < w_{m-1} < v_m < w_m.$$

Proof. Let $d = \left(\frac{\lambda}{1+\lambda}\right)^2, G = 2\lambda^2 - 2(m-1)\lambda - m + 1$.
If $\lambda > 0$, by Theorem 2.7(1), λ is the spectral radius of $\mathscr{B}_{2m}$ with $m - \frac{1}{2} - \frac{1}{5m} < \lambda < m - \frac{1}{2} - \frac{1}{4\gamma m}$. It is easy to verify that $0 < G < 1$. According to Theorem 3.3(2) , it follows that for $k = 1, 2, \cdots, m-1$,

$$\begin{aligned} v_{k+1} - v_k &= \frac{1-G}{\lambda^2(1+\lambda)} d^k > 0, \\ w_{k+1} - w_k &= -\frac{1-G}{\lambda(1+\lambda)^2} d^k < 0. \end{aligned}$$

According to Theorem 3.3(3) , it follows that

$$v_m - w_m = \frac{G-1}{\lambda^2} d^m < 0,$$

Therefore,

$$v_1 < v_2 < \cdots < v_m < w_m < w_{m-1} < \cdots < w_1.$$

If $\lambda < 0$, by Theorem 2.7(2), $-\frac{1}{2} < \lambda < -\frac{1}{2} + \frac{1}{5m}$. It is easy to verify that $0 < G < \frac{1}{2}$. According to Theorem 3.3(2),(3) , it follows that

$$\begin{aligned} v_{k+1} - v_k &= \frac{1-G}{\lambda^2(1+\lambda)} d^k > 0 \quad (k = 1, 2, \cdots, m-1), \\ v_k - w_k &= \frac{G-1}{\lambda^2} d^k < 0 \quad (k = 1, 2, \cdots, m). \end{aligned}$$

Hence, for $k = 1, 2, \cdots, m-1$

$$\begin{aligned} v_{k+1} - w_k &= (v_{k+1} - v_k) + (v_k - w_k) \\ &= \frac{1-G}{\lambda^2(1+\lambda)} d^k + \frac{G-1}{\lambda^2} d^k \\ &= \frac{\lambda(G-1)}{\lambda^2(1+\lambda)} d^k > 0. \end{aligned}$$

Therefore,

$$v_1 < w_1 < v_2 < w_2 < \cdots < v_{m-1} < w_{m-1} < v_m < w_m.$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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