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HIGHER ORDER OPERATOR SPLITTING METHODS FOR AN IMAGE DE-NOISING MODEL

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Abstract. This paper is concerned with fast iterative methods with development of Euler-Lagrange equation which results from the minimization of Rudin-Osher-Fatemi (ROF) model. There are many applications of image de-noising in field of medical and astronomy. We can classify the image de-noising models into additive and multiplicative noise removal models. In case of additive noise, we have an image u corrupted with additive gaussian noise η, the main task is to recover u from the image formation model $u_0 = u + \eta$. This paper mainly focus on additive noise removal. Here semi-implicit (SIM), additive operator splitting (AOS) and additive multiplicative operator splitting (AMOS) type schemes are developed. The quality in AOS is, it treats with all coordinate axes in an equal manner. We develop a new AMOS scheme for the solution of Euler-Lagrange equation arisen from minimization of image additive noise removal model. Comparison of AMOS with SIM and AOS is also presented. Experimental results shows that by using AMOS, additive noisy image can be de-noised with best results. Numerical examples are given to show gain in CPU timing and fast convergence of AMOS-based algorithm.

Keywords: total variation; filtering; segmentation; de-noising; smoothing.

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1. Introduction

In the field of image processing, image de-noising is a significant and an extraordinary field for last decades. Through image de-noising, image is reconstructed by removing noise from a corrupted image. The noise removal method is designed in such a way that it suppresses the noise and preserves many image structures. The actual meaning of noise is an unwanted signal. Signals are the unwanted electrical fluctuations which are received by AM radios. Noise in images is a random variation of colour or brightness, it is a cause of sensor and circuitry of a digital camera or scanner. We can not avoid the noise in images. In image de-noising our main focus is on the development of such filters which maintains the compromise between the noise and the image. We consider the following image formation model

$$u_0(x,y) = u(x,y) + \eta(x,y), \tag{1}$$

in which $u_0(x,y)$ represents the observed image, $u(x,y)$ indicates the clean image, $\eta(x,y)$ denotes the additive gaussian noise. We suppose that η is distributed normally, its standard deviation is supposed to be σ and mean is 0. There are different sources in camera systems from which images are corrupted such as photon, thermal and quantization noise. In this research, we have worked upon the operator splitting methods [1-3] in terms of de-noising. There are different methods used for removing noise in images like filtering, smoothing and total variation (TV) [4-11]. Filtering has poor efficiency and edges are not preserved. TV is a technique having applications in the noise removal of digital image processing. This method is applied for reducing noise in order to preserve sharp edges in the specified signal. Compared to filtering, the results of the TV are obtained by minimizing a cost function. The main approach is based on the discretization of finite difference method. Experiments show that TV is better than other de-noising methods since not only image is de-noised but also the edges are preserved.

Weickert et al. [12] compared the performance of explicit, SIM and AOS schemes for non-linear diffusion filtering, they proposed that SIM is efficient in one dimensional case while AOS produces more stable and efficient results for all dimensions and step sizes but the main problem in AOS is, it is first order accurate in time.

Barash and Kimmel [1] extended the idea of Weickert et al. [12] and proposed a scheme which is called AMOS scheme in terms of nonlinear diffusion filtering having the second order accuracy. Rudin et al. [8] have applied an alternative method in order to descritize the minimization problem as to directly descritze PDE through gradient descent method. Goldstein and Osher [13] worked upon TV de-noising using split bregman. Strong [9] have worked upon two important properties of TV regularization, they proposed that the edges of the image have a tendency to be preserved and in particular conditions they are completely preserved. Chan et al. [14] proposed a new model for segmentation based on mumfard shah functional. Jeon et al. [15] presented an unsupervised hierarchical segmentation based on AOS scheme. D. krishnan et al. [16] minimized the TV model based on AOS methods and also compared the performance of AOS with explicit schemes, also they found that AOS scheme fails to produce good result when regularization parameter $\lambda > 4$.

In today's life, images have a broad application in our surroundings, they are used to catch criminals. Many problems in image de-noising are based on additive noise, where an image u is supposed to be corrupted with an additive noise. Rudin et al. [8] presents the first total variation based noise removal model. This model uses total variation as a regularization term for de-noising an image by minimizing

$$\min_u E(u) = \int_\Omega |\nabla u| dxdy + \frac{\lambda}{2}\int_\Omega |u-u_0|^2 dxdy, \quad where \quad |\nabla u| = \sqrt{u_x^2+u_y^2}. \tag{2}$$

The first term is the regularization and the second is the fidelity where λ is tradeoff, which balances fidelity and regularization terms. ROF model is a PDE based approach used for additive noise removal. Minimization of above equation leads to

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(\frac{u_x}{\sqrt{u_x^2+u_y^2}}\Big) + \frac{\partial}{\partial y}\Big(\frac{u_y}{\sqrt{u_x^2+u_y^2}}\Big) - \lambda\Big(u-u_0\Big); \quad t>0, x, y\in\Omega, \tag{3}$$

$u(x,y,0) = u_0(x,y)$ and $\frac{\partial u}{\partial \eta} = 0$ on $\partial\Omega$.

The steady state of eq. (4) is given by

$$0 = \frac{\partial}{\partial x}\Big(\frac{u_x}{\sqrt{u_x^2+u_y^2}}\Big) + \frac{\partial}{\partial y}\Big(\frac{u_y}{\sqrt{u_x^2+u_y^2}}\Big) - \lambda\Big(u-u_0\Big) \quad in \quad \Omega, \tag{4}$$

with $\frac{\partial u}{\partial \eta} = 0$ on $\partial\Omega$.

Equivalently, eq. (4) can be written as

$$\nabla\Big(\frac{\nabla u}{\sqrt{|\nabla u|^2+\gamma}}\Big)+\lambda(u-u_0)=\frac{\partial u}{\partial t} \quad in \quad \Omega, \tag{5}$$

with the Neumann boundary condition. We descritize eq. (5) using the finite difference method because of the discrete nature of the image.

The main goal of this work is to find a scheme which would be second order accurate in time, more efficient, stable and would produce better PSNR (peak signals to noise ratio) results than SIM and AOS schemes based on minimization of ROF model. The objective of this research is to develop fast iterative method. The paper is organized as follow:

Section 2 describes a brief survey on SIM, AOS and AMOS methods. Section 3 shows some test results and section 4 is the conclusion of the work.

2. Numerical Schemes

2.1. Semi-Implicit Scheme

We consider equation (5) with the same initial and boundary conditions. In order to descritize equation (5), consider $x_i = ih$, $y_j = jh$ and $t_n = n\Delta t$. The numerical approximation of (5) is given as

$$\begin{aligned} u_{i,j}^{n+1} &= u_{i,j}^n+\frac{\Delta t}{h}\Big[\Delta_-^x\Big(\frac{\Delta_+^x u_{i,j}^{n+1}}{\sqrt{(\Delta_+^x u_{i,j}^n)^2+(\Delta_+^y u_{i,j}^n)^2+\gamma}}\Big) \\ &+ \Delta_-^y\Big(\frac{\Delta_+^y u_{i,j}^{n+1}}{\sqrt{(\Delta_+^x u_{i,j}^n)^2+(\Delta_+^y u_{i,j}^n)^2+\gamma}}\Big)\Big]-\Delta t\lambda\Big(u_{i,j}^{n+1}-u_0\Big), \end{aligned} \tag{6}$$

where $i,j = 1,2,3,...,m-1$, $n = 1,2,3,...$, and with BCs,

$u_{0,j}^n = u_{1,j}^n, \quad u_{N,j}^n = u_{N-1,j}^n, \quad u_{i,0}^n = u_{i,N}^n = u_{i,N-1}^n.$

In our numerical calculations, we assume $h = 1$ and consider the following notations

$$u_{i,j}^n = u(x_i, y_j, t_n),$$

$$u_{i,j} = u(x_i, y_j),$$

$$\Delta_+^x = (u_{i+1,j} - u_{i,j}),$$

$$\Delta_-^x = (u_{i-1,j} - u_{i,j}),$$

$$\Delta_+^y = (u_{i,j+1} - u_{i,j}),$$

$$\Delta_-^y = (u_{i,j-1} - u_{i,j}).$$

Further descritization of equation (6) leads to

$$-\frac{\Delta t}{h}\left(C_1 u_{i+1,j}^{n+1} - C_2 u_{i-1,j}^{n+1}\right) - \frac{\Delta t}{h}\left(C_3 u_{i,j+1}^{n+1} - C_4 u_{i,j-1}^{n+1}\right) + \left(1 + \frac{\Delta t}{h}(C_1 + C_2 + C_3 + C_4) + \Delta t\lambda\right) u_{i,j}^{n+1} = u_{i,j}^n + F_{i,j}, \tag{7}$$

where C_1, C_2, C_3 and C_4 are given by

$$C_1 = \frac{1}{\sqrt{\gamma + (u_{i+1,j}^n - u_{i,j}^n)^2 + (u_{i,j+1}^n - u_{i,j}^n)^2}},$$

$$C_2 = \frac{1}{\sqrt{\gamma + (u_{i,j}^n - u_{i-1,j}^n)^2 + (u_{i-1,j+1}^n - u_{i-1,j}^n)^2}},$$

$$C_3 = \frac{1}{\sqrt{\gamma + (u_{i+1,j}^n - u_{i,j}^n)^2 + (u_{i,j+1}^n - u_{i,j}^n)^2}},$$

$$C_4 = \frac{1}{\sqrt{\gamma + (u_{i+1,j-1}^n - u_{i,j-1}^n)^2 + (u_{i,j}^n - u_{i,j-1}^n)^2}}.$$

Computing for $u_{i,j}^{n+1}$, we obtain the following vector matrix notation

$$u_{i,j}^{n+1} = \left(I - \sum_{l=1}^{m} \Delta t \left(A_l(u_{i,j}^n)\right)\right)^{-1} \left(u_{i,j}^n + F_{i,j}\right). \tag{8}$$

In eq. (8), $u_{i,j}^{n+1}$ can be obtained by inverting $\left(I - \sum_{l=1}^{m} \Delta t \left(A_l(u_{i,j}^n)\right)\right)$ using the Thomas algorithm, where $A_l(u_{i,j}^n)$ is a five-band matrix. As compare to explicit schemes, the semi-implicit schemes are more stable and efficient but when dimensions ≥ 2, the matrix in eq. (8) is no more tri-diagonal and the main draw-back is their computational cost of associated linear system of high dimensional images, that is they are less efficient for solving m-dimensional linear system. This problem was overcome by Peaceman and Rachford [17] through splitting methods.

2.2. Additive Operator Scheme (AOS)

For AOS scheme we consider eq. (8)

$$u_{i,j}^{n+1} = \Big(I - \sum_{l=1}^{m} \Delta t\Big(A_l(u_{i,j}^n)\Big)\Big)^{-1}\Big(u_{i,j}^n + F_{i,j}\Big), \tag{9}$$

the above equation can be written as

$$u_{i,j}^{n+1} = \Big(\frac{1}{m}(I+I+I+\ldots+I(m \quad times)) - \sum_{l=1}^{m} \Delta t\Big(A_l(u_{i,j}^n)\Big)\Big)^{-1}\Big(u_{i,j}^n + F_{i,j}\Big), \tag{10}$$

$$u_{i,j}^{n+1} = \Big(\frac{1}{m}(I - \Delta tm(A_1(u_{i,j}^n)) + \ldots + \frac{1}{m}(I - \Delta tm(A_l(u_{i,j}^n))\Big)^{-1}\Big(u_{i,j}^n + F_{i,j}\Big), \tag{11}$$

further simplification of above equation leads to

$$u_{i,j}^{n+1} = \Big(\frac{1}{m}\sum_{l=1}^{m}(I - \Delta tm(A_l(u_{i,j}^n))\Big)^{-1}\Big(u_{i,j}^n + F_{i,j}\Big), \tag{12}$$

we consider our desired case

$$u_{i,j}^{n+1} = \sum_{l=1}^{m}\Big(\frac{1}{m}(I - \Delta tm(A_l(u_{i,j}^n))\Big)^{-1}\Big(u_{i,j}^n + F_{i,j}\Big), \tag{13}$$

we see that the right hand side of eqs. (12) and (13) are not equal, let both to be equal when the R.H.S of eq. (13) is multiplied by a simple variable x, i.e., comparing eqs. (12) and (13) for a variable x

$$\begin{aligned}&\Big(\frac{1}{m}\sum_{l=1}^{m}(I - \Delta tm(A_l(u_{i,j}^n))\Big)^{-1}\Big(u_{i,j}^n + F_{i,j}\Big)\\ &= x\sum_{l=1}^{m}\Big(\frac{1}{m}(I - \Delta tm(A_l(u_{i,j}^n))\Big)^{-1}\Big(u_{i,j}^n + F_{i,j}\Big),\end{aligned} \tag{14}$$

$$\begin{aligned}&\Big(\frac{1}{m}(I - \Delta tm(A_1(u_{i,j}^n)) + \frac{1}{m}(I - \Delta tm(A_2(u_{i,j}^n)) + \ldots + \frac{1}{m}(I - \Delta tm(A_m(u_{i,j}^n))\Big)^{-1}\\ &\qquad \Big(u_{i,j}^n + F_{i,j}\Big)\\ &= x\Big(\big(\frac{1}{m}(I - \Delta tm(A_1(u_{i,j}^n))\big)^{-1} + \big(\frac{1}{m}(I - \Delta tm(A_2(u_{i,j}^n))\big)^{-1} + \ldots +\\ &\qquad \big(\frac{1}{m}(I - \Delta tm(A_m(u_{i,j}^n))\big)^{-1}\Big)\Big(u_{i,j}^n + F_{i,j}\Big),\end{aligned} \tag{15}$$

let us consider $U = (I - \Delta tm(A_l(u^n_{i,j}))$

$$\Big(\frac{1}{m}U + \frac{1}{m}U + ... + \frac{1}{m}U\big(m \quad times\big)\Big)^{-1}\Big(u^n_{i,j} + F_{i,j}\Big)$$

$$= x\Big(\big(\frac{1}{m}U\big)^{-1} + \big(\frac{1}{m}U\big)^{-1} + ... + \big(\frac{1}{m}U\big)^{-1}\big(m \quad times\big)\Big)\Big(u^n_{i,j} + F_{i,j}\Big), \tag{16}$$

$$\Big(\frac{m}{m}U\Big)^{-1} = xm\Big(U^{-1} + U^{-1} + ... + U^{-1}\Big), \tag{17}$$

$$U^{-1} = xm^2 U^{-1}, \tag{18}$$

inserting the value of U in equation (18), i.e.,

$$\Big(I - \Delta tm(A_l(u^n_{i,j}))\Big)^{-1} = xm^2(I - \Delta tm(A_l(u^n_{i,j}))\Big)^{-1}, \tag{19}$$

$$x = \frac{1}{m^2}, \tag{20}$$

so equation (13) becomes

$$u^{n+1}_{i,j} = \frac{1}{m^2}\sum_{l=1}^{m}\Big(\frac{1}{m}(I - \Delta tm(A_l(u^n_{i,j}))\Big)^{-1}\Big(u^n_{i,j} + F_{i,j}\Big), \tag{21}$$

which finally reduces to

$$u^{n+1}_{i,j} = \frac{1}{m}\sum_{l=1}^{m}\Big((I - \Delta tm(A_l(u^n_{i,j}))\Big)^{-1}\Big(u^n_{i,j} + F_{i,j}\Big). \tag{22}$$

The above calculation shows that AOS scheme is the modified form of the semi-implicit scheme and it uses one dimensional semi-implicit scheme in arbitrary dimensions. The numerical schemes in eq. (22) are split up in different dimensions and results are combined in an additive manner therefore eq. (22) is called Additive Operator Splitting (AOS). The final scheme in eq. (22) is tri-diagonal along each dimensions, therefore it can be solved individually by splitting schemes in an efficient manner and easy in implementation. This scheme calculates the operators in an independent manner and then sums them at each time step. It is stated without proof that the AOS scheme is an $O(\Delta t) + O(h^2)$ accurate finite difference approximation to the original equation. Eq. (22) is the Additive Operator Splitting scheme for m-dimensional case. In our case, we consider 2-dimensional case, i.e.,

$$\begin{aligned} u_{i,j}^{k+1} &= \frac{1}{2}\Big[\Big(I-2\Delta t A_1(u_i^k)\Big)^{-1}+\Big(I-2\Delta t A_2(u_j^k)\Big)^{-1}\Big]\Big(u_{i,j}^k+F_i\Big) \\ &= \frac{1}{2}\sum_{l=1}^{2}\Big(I-2\Delta t A_l(u^k)\Big)^{-1}\Big(u_{i,j}^k+F_i\Big). \end{aligned} \tag{23}$$

2.3. New AMOS Scheme for ROF Model

For AMOS scheme, we consider

$$\frac{u_i^{k+1}-u_i^k}{\Delta t}=A_1(u_i^n)u_i^{n+1}+F_i,$$

$$\frac{u_j^{k+1}-u_j^k}{\Delta t}=A_2(u_j^n)u_j^{n+1}+F_i.$$

The above both equations reduce to

$$\big(I-\Delta t A_1(u_i^n)\big)u_i^{n+1}=u_i^n+F_i, \tag{24}$$

$$\big(I-\Delta t A_2(u_j^n)\big)u_j^{n+1}=u_j^n+F_i. \tag{25}$$

From eqs. (24) and (25), we get

$$u_{i,j}^{n+1}=\big(I-\Delta t A_1(u_i^n)\big)^{-1}\big(I-\Delta t A_2(u_j^n)\big)^{-1}\big(u_{i,j}^n+F_i\big), \tag{26}$$

from eqs. (24) and (25), we can also obtain

$$u_{i,j}^{n+1}=\big(I-\Delta t A_2(u_j^n)\big)^{-1}\big(I-\Delta t A_1(u_j^n)\big)^{-1}\big(u_{i,j}^n+F_i\big). \tag{27}$$

Taking the mean of eqs. (26) and (27), i.e.,

$$\begin{aligned} u_{i,j}^{n+1}=\frac{1}{2}\Big[&\Big(I-\Delta t A_1(u_i^n)\Big)^{-1}\Big(I-\Delta t A_2(u_j^n)\Big)^{-1}+\Big(I-\Delta t A_2(u_j^n)\Big)^{-1} \\ &\times\Big(I-\Delta t A_1(u_i^n)\Big)^{-1}\Big]\Big(u_{i,j}^n+F_i\Big). \end{aligned} \tag{28}$$

Eq. (28) is called AMOS (additive multiplicative operator splitting) scheme because it is both additive as well as multiplicative scheme, additivity is important to make the splitting symmetric. AMOS scheme is more accurate than the AOS scheme and is unconditionally stable, also it consists the merits of AOS and MOS (Multiplicative Operator Splitting Scheme). AMOS scheme is considered to be more accurate than the AOS and preserves the symmetry and accuracy as well.

3. Experimental Results

Here we perform experiments on grey color images of different pixel sizes. Results of experiments are given to compare the performance of AMOS with AOS and SIM. All the de-noising algorithms are implemented on additive noisy images.

Now in order to show gain in CPU timing and PSNR of SIM, AOS and AMOS, the numerical examples are given below.

FIGURE 1. Experimental results of SIM for problem 1, no. of iterations=3000

FIGURE 2. Experimental results of SIM for problem 2, no. of iterations=3000

FIGURE 3. Experimental results of SIM for problem 3, no. of iterations=3000

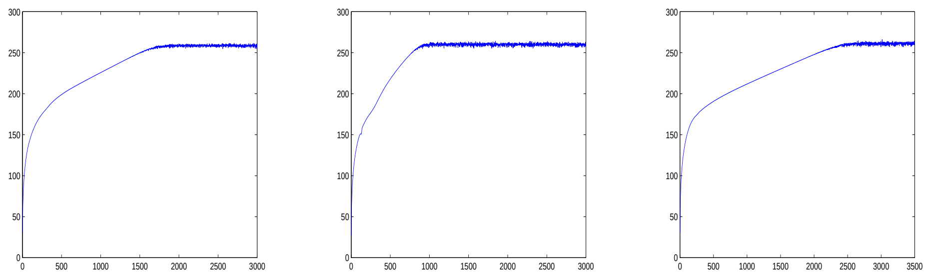

FIGURE 4. Experimental results of SIM for problem1, no. of iterations=3000

FIGURE 5. Experimental results of AOS for problem 1, no. of iterations=400

FIGURE 6. Experimental results of AOS for problem 2, no. of iterations=400

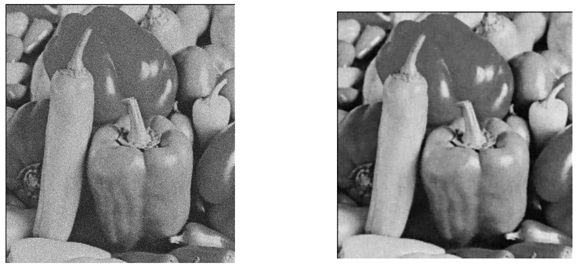

FIGURE 7. Experimental results of AOS for problem 3, no. of iterations=400

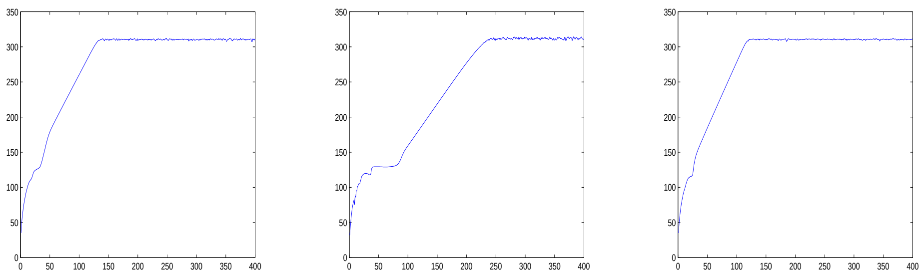

FIGURE 8. PSNR results of AOS for problem 1, problem 2 and problem 3

FIGURE 9. Experimental results of AMOS for problem 1, no. of iterations=250

FIGURE 10. Experimental results of AMOS for problem 2, no. of iterations=250

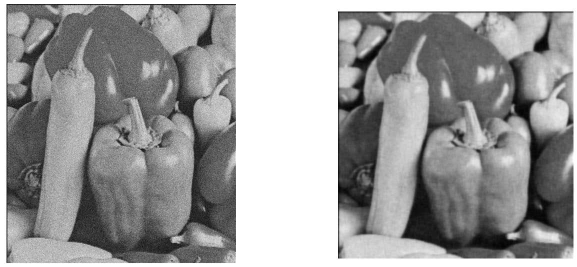

FIGURE 11. Experimental results of AMOS for problem 3, no. of iterations=250

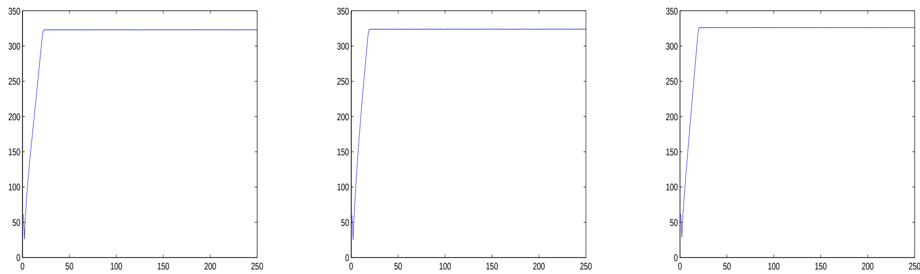

FIGURE 12. PSNR results of AMOS for problem 1, problem 2 and problem 3. The above figures show that AMOS gives us much better results in terms of highest PSNR value than SIM and AOS within 25 iterations.

Problem1:	β	λ	dt	No. of iterations	CPU(S)	PSNR(dB)
SIM	0.001	70	10	3000	2623	258
AOS	0.003	70	6	400	118	311
AMOS	0.001	75	18	250	87	323
Problem2:						
SIM	0.001	70	13	3000	3061	260
AOS	0.003	80	6	400	126	312
AMOS	0.001	80	18	250	86	323.8
Problem3:						
SIM	0.001	70	13	3000	3199	261
AOS	0.003	65	6	400	117	311
AMOS	0.001	90	18	250	83	327

TABLE 1. Comparison of SIM, AOS and AMOS for Peak Signal-to-Noise Ratio (PSNR) on grey color noisy images.

SIM:	β	λ	dt	No. of iterations	CPU(S)	PSNR(dB)
256^2	0.001	70	13	3000	2623	258
512^2	0.001	70	13	3000	13788	247
768^2	0.001	70	13	3000	30208	234
1024^2	0.001	70	13	3000	66679	189
AOS:						
256^2	0.003	65	10	400	123	310
512^2	0.003	65	10	400	484	104
768^2	0.003	65	10	400	1186	111
1024^2	0.003	65	10	400	2074	143
AMOS:						
256^2	0.001	90	18	250	95	326
512^2	0.001	90	18	250	361	325
768^2	0.001	90	18	250	816	324
1024^2	0.001	90	18	250	1848	318

TABLE 2. Comparison of SIM, AOS and AMOS with TV de-noising algorithms for additive noisy images ($256^2 - 1024^2$).

Three iterative schemes namely SIM, AOS and AMOS for additive noise suppression are compared. It is found that by using the AMOS algorithm, the technique has the advantage of highest PSNR results, speed of computation and effectiveness in de-noising the images over other iterative techniques of SIM and AOS. In a nutshell, AMOS is more effective and efficient than SIM and AOS.

4. Conclusion

In this paper, the de-noising algorithms for operator splitting methods such as SIM, AOS and AMOS are presented based on minimization of ROF model related with total variation approach. This amounts to discretized a time dependent PDE by the constraints. Comparison of AMOS with other iterative schemes such as SIM and AOS is also presented. These methods successively de-noise the image until the steady state solutions are obtained. Experiments show that by using AMOS, noisy images with different dimensions ($256^2 - 1024^2$) can be recovered with better PSNR results than SIM and AOS. It is concluded that AMOS is efficient, effective, fast and stable than AOS and SIM. The extension of this work is to develop a multigrid algorithm for our proposed model which is expected to be more enhanced than the existing splitting methods in case of stability and CPU time.

Conflict of Interests

The authors declare that there is no conflict of interests.

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