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OSCILLATORY SOLUTIONS FOR DYNAMIC EQUATIONS WITH NON-MONOTONE ARGUMENTS

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Abstract. Consider the first-order delay dynamic equation

$$x^{\Delta}(t)+p(t)x(\tau(t))=0,\ t\in[t_0,\infty)_{\mathbb{T}}$$

where $p\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$, $\tau\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ is non-monotone, and $\tau(t)\leq t$, $\lim_{t\to\infty}\tau(t)=\infty$. Under the assumption that the τ is non-monotone, we present sufficient conditions for the oscillation of first-order delay dynamic equations on time scales. An example illustrating the result is also given.

Keywords: dynamic equations; time scale; non-monotone argument; retarded argument; oscillatory solutions.

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1. Introduction

The problem of establishing sufficient conditions for the oscillation of all solutions to the differential/difference and dynamic equations have been the subject of many investigations. See, for example, [1−32] and the references cited therein. Consider the first-order delay dynamic

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equation

$$x^{\Delta}(t)+p(t)x(\tau(t))=0,\ \ t\in[t_0,\infty)_{\mathbb{T}} \tag{E}$$

where $\mathbb{T}$ is a time scale unbounded above with $t_0\in\mathbb{T}$, p is rd-continuous and nonnegative, the delay function $\tau:\mathbb{T}\to\mathbb{T}$ is non-monotone and satisfies

$$\tau(t)\leq t\ \text{ for all } t\in\mathbb{T},\ \ \lim_{t\to\infty}\tau(t)=\infty, \tag{1.1}$$

and $\sup\mathbb{T}=\infty$.

First we give a short review on the time scales calculus extracted from [3]. A time scale, which inherits the standard topology on $\mathbb{R}$, is a nonempty closed subset of reals. Here and later throughout this paper, a time scale will be denoted by the symbol $\mathbb{T}$, and the intervals with a subscript $\mathbb{T}$ are used to denoted the intersection of the usual interval with $\mathbb{T}$. For $t\in\mathbb{T}$, we define the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ by $\sigma:=\inf(t,\infty)_{\mathbb{T}}$ while the backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ is defined by $\rho:=\sup(-\infty,t)_{\mathbb{T}}$, and the graininess function $\mu:\mathbb{T}\to\mathbb{R}_0^+$ is defined to be $\mu(t):=\sigma(t)-t$. A point $t\in\mathbb{T}$ is called right-dense if $\sigma(t)=(t)$ and/or equivalently $\mu(t)=0$ holds; otherwise it is called right-scattered, and similarly left-dense and left scattered points are defined with respect to the backward jump operator. We also need the set $\mathbb{T}^{\kappa}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa}=\mathbb{T}-\{m\}$. Otherwise, $\mathbb{T}^{\kappa}=\mathbb{T}$. A function $f:\mathbb{T}\to\mathbb{R}$ is said to be Δ-differentiable at the point $t\in\mathbb{T}^{\kappa}$ provided that there exists $f^{\Delta}(t)$ such that for every $\varepsilon>0$ there exists a neighborhood U of t such that

$$\left|[f(\sigma(t)-f(s)]-f^{\Delta}(t)\left[\sigma(t)-s\right]\right|\leq\varepsilon\left|\sigma(t)-s\right|\ \text{for all } s\in U.$$

We shall mean the Δ-derivative of a function when we only say derivative unless otherwise is specified. A function $f:\mathbb{T}\to\mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$, and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f:\mathbb{T}\to\mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T},\mathbb{R})$.

The set of functions $f:\mathbb{T}\to\mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $C_{rd}^1(\mathbb{T},\mathbb{R})$. For $s,t\in\mathbb{T}$ and a function $f\in C_{rd}(\mathbb{T},\mathbb{R})$, the Δ-integral of is defined by

$$\int_s^t f(\eta)\Delta(\eta) = F(t) - F(s)$$

where $F \in C^1_{rd}(\mathbb{T},\mathbb{R})$ is an anti-derivative of f, i.e., $F^\Delta = f$ on $\mathbb{T}^\kappa$. Every rd-continuous function has an antiderivative. In paticular, if $t_0 \in \mathbb{T}$ then F defined by

$$F(t) = \int_s^t f(\eta)\Delta(\eta) \text{ for } t \in \mathbb{T}$$

is an antiderivative of f. And, for $t \in \mathbb{T}^\kappa$

$$\int_t^{\sigma(t)} f(\eta)\Delta(\eta) = \mu(t).f(t).$$

It is obvious that if $f^\Delta \geq 0$, then f is nondecreasing.

A function $f \in C_{rd}(\mathbb{T},\mathbb{C})$ is called regressive if $1 + f\mu \neq 0$ on $\mathbb{T}^\kappa$, and $f \in C_{rd}(\mathbb{T},\mathbb{C})$ is called positively regressive if $1 + f\mu > 0$ on $\mathbb{T}^\kappa$. The set of regressive functions and the set of positively regressive functions are denoted by $\mathscr{R}(\mathbb{T},\mathbb{C})$ and $\mathscr{R}^+(\mathbb{T},\mathbb{R})$, respectively, $\mathscr{R}^-(\mathbb{T},\mathbb{R})$ is defined similarly. For simplicity, we denote by $\mathscr{R}_c(\mathbb{T},\mathbb{C})$ the set of regressive constants, and similarly we define the sets $\mathscr{R}_c^+(\mathbb{T},\mathbb{R})$ and $\mathscr{R}_c^-(\mathbb{T},\mathbb{R})$.

A function $x : \mathbb{T} \to \mathbb{R}$ is called a solution of the equation (E), if $x(t)$ is delta differentiable for $t \in \mathbb{T}^\kappa$ and satisfies equation (E) for $t \in \mathbb{T}$. We say that a solution x of equation (E) has a generalized zero at t if $x(t) = 0$ or if $\mu(t) > 0$ and $x(t)x(\sigma(t)) < 0$. Let $\sup \mathbb{T} = \infty$ and then a nontrivial solution x of equation (E) is called oscillatory on $[t,\infty)$ if it has arbitrarirly large generalized zeros in $[t,\infty)$.

Next, let us recall some known oscillation results on this subject. For $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, equation (E) reduces to

$$x'(t) + p(t)x(\tau(t)) = 0, \;\; t \in \mathbb{R}_0^+ \tag{1.2}$$

and

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \;\; n \in \mathbb{N}_0^+, \tag{1.3}$$

respectively.

In 1972, Ladas, Lakshmikantham and Papadakis [20] proved that if $\tau(t)$ is nondecreasing and

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > 1, \tag{1.4}$$

then all solutions of (1.2) oscillate.

In 1982, Koplatadze and Canturija [19] established the following result.

If $\tau(t)$ is non-monotone or nondecreasing, and

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}, \tag{1.5}$$

then all solutions of (1.2) oscillate.

Assume that the argument $\tau(t)$ is non-monotone. Set

$$h(t) := \sup_{s\le t}\tau(s), \quad t\ge 0. \tag{1.6}$$

Clearly, $h(t)$ is nondecreasing, and $\tau(t)\le h(t)$ for all $t\ge 0$.

In 2011, Braverman and Karpuz [5], proved that, if $\tau(t)$ is non-monotone and

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left\{\int_{\tau(s)}^{h(t)} p(\xi)d\xi\right\}ds > 1, \tag{1.7}$$

then all solutions of (1.2) oscillate.

Very recently, Chatzarakis and Öcalan [9], proved that, if $\tau(t)$ is non-monotone and

$$\liminf_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left\{\int_{\tau(s)}^{h(t)} p(\xi)d\xi\right\}ds > \frac{1}{e}, \tag{1.8}$$

then all solutions of (1.2) oscillate.

In 1998, Zhang and Tian [30], studied the equation (1.3) and proved that, if $(\tau(n))$ is non-monotone, and

$$\limsup_{n\to\infty} p(n) > 0 \quad \text{and} \quad \liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e} \tag{1.9}$$

then all sollutions of (1.3) oscillate.

In 2006, Chatzarakis, Koplatadze and Stavroulakis [6,7], when $(\tau(n))$ is non-monotone or nondecreasing, studied the equation (1.3) and proved that, if one of the following conditions

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) > 1, \quad \text{where } h(n) = \max_{0\le s\le n} \tau(s),\, n \ge 0, \tag{1.10}$$

or

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) < \infty \quad \text{and} \quad \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e} \tag{1.11}$$

is satisfied, then all sollutions of (1.3) oscillate.

Assume that the argument $(\tau(n))$ is non-monotone. Set

$$h(n) := \max_{s\le n} \tau(s), \quad n \ge 0. \tag{1.12}$$

Clearly, h is nondecreasing, and $\tau(n) \le h(n) \le n-1$ for all $n \ge 0$.

In 2016, Öcalan [26], proved that, if $(\tau(n))$ is non-monotone and

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) \left(\frac{j-\tau(j)+1}{j-\tau(j)} \right)^{j-\tau(j)+1} > 1, \tag{1.13}$$

then all solutions of (1.3) oscillate.

In 2011, Braverman and Karpuz [5], proved that, if $(\tau(n))$ is non-monotone and

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} > 1, \tag{1.14}$$

then all solutions of (1.3) oscillate.

Very recently, Chatzarakis and Öcalan [8], proved that, if $(\tau(n))$ is non-monotone and

$$\liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} > \frac{1}{e}, \tag{1.15}$$

then all solutions of (1.3) oscillate.

For Equation (E), in 2002, Zhang and Deng [31], proved the following result by the help of cylinder transforms.

Define

$$\alpha = \limsup_{t_0\to\infty} \sup_{\lambda\in E} \left\{ \lambda \exp_{-\lambda p}(\tau(t), t) \right\} \tag{1.16}$$

where

$$\exp_{-\lambda p}(\tau(t),t) = \exp\int_{\tau(t)}^{t} \xi_{\mu(s)}(-\lambda p(s))\Delta s,$$

$E = \{\lambda : \lambda > 0,\ 1-\lambda p(t)\mu(t) > 0\}$, and

$$\xi_h(z) = \begin{cases} \frac{Log(1+hz)}{h} & ,\text{ if } h \neq 0 \\ z & ,\text{ if } h = 0 \end{cases} .$$

If $\alpha < 1$, then all solutions of equation (E) are oscillatory.

In 2005, Bohner [4], proved that, using exponential functions notation for any time scale $\mathbb{T}$, if Eq. (E) has an eventually positive solution, then α defined by (1.16) satisfies $\alpha \geq 1$.

In 2005, Zhang et al. [32], and in 2006, Şahiner and Stavroulakis [28], using by different technique, obtained that if $\tau(t)$ is nondecreasing and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s > 1, \tag{1.17}$$

then all solutions of equation (E) are oscillatory.

2. Main results

In this section, we present a new sufficient condition for the oscillation of all solutions of (E), under the assumption that the argument $\tau(t)$ is non-monotone. Set

$$h(t) := \sup_{s\leq t} \tau(s), \quad t \geq 0. \tag{2.1}$$

Clearly, $h(t)$ is nondecreasing, and $\tau(t) \leq h(t)$ for all $t \geq 0$.

The following lemma was given in [28].

Lemma 2.1. *Assume that $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, $g : \mathbb{T} \to \mathbb{R}$ is nonincreasing and $\tau : \mathbb{T} \to \mathbb{T}$ is nondecreasing. If $b < u$, then*

$$\int_{b}^{\sigma(u)} f(s)g(\tau(s))\Delta s \geq g(\tau(u)) \int_{b}^{\sigma(u)} f(s))\Delta s. \tag{2.2}$$

Theorem 2.2. *Assume that* (1.1) *holds. If* $\tau(t)$ *is non-monotone and*

$$\limsup_{t\to\infty}\int_{h(t)}^{\sigma(t)} p(s)\Delta s > 1, \tag{2.3}$$

where $h(t)$ *is defined (2.1), then all solutions of* (E) *oscillate.*

Proof. *Assume, for the sake of contradiction, that there exists a nonoscillatory solution* $x(t)$ *of (E). Since* $-x(t)$ *is also a solution of (E), we can confine our discussion only to the case where the solution* $x(t)$ *is eventually positive. Then there exists a* $t_1 > t_0$ *such that* $x(t),\ x(\tau(t)),\ x(h(t)) > 0$, *for all* $t \geq t_1$. *Thus, from (E) we have*

$$x^{\Delta}(t) = -p(t)x(\tau(t)) \leq 0, \quad \textit{for all } t \geq t_1,$$

which means that $x(t)$ *is an eventually nonincreasing function of positive numbers. In view of this, and taking into account that* $\tau(t) \leq h(t) \leq t$ *and* $h(t)$ *is nondecreasing, (E) gives*

$$x^{\Delta}(t) + p(t)x(h(t)) \leq 0, \quad t \geq t_1. \tag{2.4}$$

Integrating (2.4) from $h(t)$ *to* $\sigma(t)$ *and using Lemma* 2.1, *we obtain*

$$x(\sigma(t)) - x(h(t)) + \int_{h(t)}^{\sigma(t)} p(s)x(h(s))\Delta s \leq 0$$

and

$$-x(h(t)) + x(h(t))\int_{h(t)}^{\sigma(t)} p(s)\Delta s \leq 0$$

or

$$x(h(t))\left[\int_{h(t)}^{\sigma(t)} p(s)\Delta s - 1\right] \leq 0.$$

Consequently,

$$\limsup_{t\to\infty}\int_{h(t)}^{\sigma(t)} p(s)\Delta s \leq 1,$$

which contradicts (2.3). The proof of the theorem is complete.

We remark that if $\tau(t)$ *is nondecreasing, then we have* $\tau(t) = h(t)$ *for all* $t \geq 0$, *and the condition (2.3) reduce to*

$$\limsup_{t\to\infty}\int_{\tau(t)}^{\sigma(t)} p(s)\Delta s > 1,$$

which implies that it is condition (1.17).

Lemma 2.3. *Assume that (2.1) holds and $m > 0$. Then, we have*

$$m = \liminf_{t\to\infty} \int_{h(t)}^{t} p(s)\Delta s = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s, \tag{2.5}$$

where $h(t)$ is defined (2.1).

Proof. *Clearly $h(t) \geq \tau(t)$ and so*

$$\int_{h(t)}^{t} p(s)\Delta s \leq \int_{\tau(t)}^{t} p(s)\Delta s.$$

Hence

$$\liminf_{t\to\infty} \int_{h(t)}^{t} p(s)\Delta s \leq \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s.$$

If (2.5) does not hold, then there exists a $m' > 0$ and a sequence $\{t_n\}$ ($t_n \in \mathbb{T}$, $n \in \mathbb{N}$) such that $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \int_{h(t_n)}^{t_n} p(s)\Delta s \leq m' < m.$$

By definition, $h(t_n) = \sup_{s\leq t_n} \tau(s)$, and hence there exists a $t_n' \leq t_n$ such that $h(t_n) = \tau(t_n')$. Hence

$$\int_{h(t_n)}^{t_n} p(s)\Delta s = \int_{\tau(t_n')}^{t_n} p(s)\Delta s \geq \int_{\tau(t_n')}^{t_n'} p(s)\Delta s.$$

It follows that $\left\{ \int_{\tau(t_n')}^{t_n'} p(s)\Delta s \right\}_{n=1}^{\infty}$ is a bounded sequence having a convergent subsequence, say

$$\int_{\tau(t_{n_k}')}^{t_{n_k}'} p(s)\Delta s \to c \leq m', \quad \text{as } k \to \infty$$

which implies that

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s \leq m' < m$$

contradicting (2.5).

Theorem 2.4. *Assume that* (1.1) *holds. If* $\tau(t)$ *is non-monotone or nondecreasing and*

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)\Delta s > \frac{1}{e}, \tag{2.6}$$

then all solutions of (E) *oscillate.*

Proof. *Assume, for the sake of contradiction, that there exists a nonoscillatory solution* $x(t)$ *of (E). Since* $-x(t)$ *is also a solution of (E), we can confine our discussion only to the case where the solution* $x(t)$ *is eventually positive. Then there exists a* $t_1 > t_0$ *such that* $x(t),\ x(\tau(t)) > 0$, *for all* $t \geq t_1$. *Thus, from (E) we have*

$$x^{\Delta}(t) = -p(t)x(\tau(t)) \leq 0, \quad \text{for all } t \geq t_1,$$

which means that $x(t)$ *is an eventually nonincreasing function of positive numbers.*

Since $\tau(t) \leq h(t) \leq t$ *and* $h(t)$ *is nondecreasing for all* $t \geq 0$, *from Eq. (E), we have*

$$x^{\Delta}(t) + p(t)x(h(t)) \leq 0, \quad t \geq t_1. \tag{2.7}$$

Integrating (2.7) from $h(t)$ *to* t, *we have*

$$x(t) - x(h(t)) + \int_{h(t)}^{t} p(s)x(h(s))\Delta s \leq 0, \quad \text{for all } t \geq t_1$$

or

$$x(t) - x(h(t)) + x(h(t))\int_{h(t)}^{t} p(s)\Delta s \leq 0, \quad \text{for all } t \geq t_1 \tag{2.8}$$

From (2.8) dividing by $x(h(t))$, *we have*

$$\frac{x(t)}{x(h(t))} - 1 + \int_{h(t)}^{t} p(s)\Delta s \leq 0 \tag{2.9}$$

Using by Lemma 2.3 and from (2.5) it follows that there exists a constant $c > 0$ *such that*

$$\int_{h(t)}^{t} p(s)\Delta s \geq c > \frac{1}{e}, \quad t \geq t_2 > t_1. \tag{2.10}$$

Combining the inequalities (2.9) and (2.10), we obtain

$$\frac{x(t)}{x(h(t))} - 1 + c \leq 0, \quad t \geq t_2$$

or

$$\frac{x(t)}{x(h(t))} \leq 1-c, \;\; t \geq t_2$$

Thus, we have $c < 1$ *and*

$$\frac{x(h(t))}{x(t)} \geq \frac{1}{1-c}, \quad t \geq t_2,$$

Repeating the above procedure, it follows by induction that for any positive integer k,

$$\frac{x(h(t))}{x(t)} \geq \left(\frac{1}{1-c}\right)^k, \quad \text{for sufficiently large } t, \tag{2.11}$$

where $c < 1$.

Now, in view of (2.10), and for all large t*, there exists a real number* $t^* \in [h(t),t]$, $t^* \in \mathbb{T}$*, such that*

$$\int_{h(t)}^{t^*} p(s)\Delta s \geq \frac{c}{2} \text{ and } \int_{t^*}^{t} p(s)\Delta s \geq \frac{c}{2}. \tag{2.12}$$

Integrating (2.7) from t^* *to* t*, and using the fact that the function* $x(t)$ *is nonincreasing and the function* $h(t)$ *is nondecreasing, we obtain*

$$x(t) - x(t^*) + \int_{t^*}^{t} p(s)x(h(s))\,\Delta s \leq 0,$$

and using (2.12), we obtain

$$-x(t^*) + x(h(t)) \int_{t^*}^{t} p(s)\Delta s \leq 0$$

or

$$x(t^*) - x(h(t))\frac{c}{2} \geq 0. \tag{2.13}$$

Integrating (2.7) from $h(t)$ *to* t^**, and using the same arguments we have*

$$x(t^*) - x(h(t)) + \int_{h(t)}^{t^*} p(s)x(h(s))\,\Delta s \leq 0,$$

or

$$-x(h(t)) + x(h(t^*)) \int_{h(t)}^{t^*} p(s)\Delta s \leq 0$$

and

$$x(h(t)) - x(h(t^*))\frac{c}{2} \geq 0. \tag{2.14}$$

Combining the inequalities (2.13) and (2.14), we obtain

$$x(t^*) \geq x(h(t))\frac{c}{2} \geq x(h(t^*))\left(\frac{c}{2}\right)^2,$$

or

$$\frac{x(h(t^*))}{x(t^*)} \leq \left(\frac{2}{c}\right)^2 < +\infty$$

i.e., $\liminf_{t\to\infty} \frac{x(h(t))}{x(t)}$ *exists. This contradicts (2.11).*

The proof of the theorem is complete.

Example 2.5. *For* $\mathbb{T} = \mathbb{R}$, *consider the retarded differential equation*

$$x'(t) + (0.37)x(\tau(t)) = 0, \quad t \geq 0, \tag{2.15}$$

where

$$\tau(t) = \begin{cases} t-1, & \text{if } t \in [3k, 3k+1] \\ -3t+12k+3, & \text{if } t \in [3k+1, 3k+2] \\ 5t-12k-13, & \text{if } t \in [3k+2, 3k+3] \end{cases}, \quad k \in \mathbb{N}_0.$$

By (2.1), *we see that*

$$h(t) := \sup_{s\leq t} \tau(s) = \begin{cases} t-1, & \text{if } t \in [3k, 3k+1] \\ 3k, & \text{if } t \in [3k+1, 3k+2.6] \\ 5t-12k-13, & \text{if } t \in [3k+2.6, 3k+3] \end{cases}, \quad k \in \mathbb{N}_0.$$

(For figure of $\tau(t)$ *and* $h(t)$, *see Example 1 in [5]). Computing, we get*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = 0.37 > \frac{1}{e},$$

that is, condition (2.6) *of Theorem 2.4 is satisfied and therefore all solutions of* (2.15) *oscillate.*

Observe, however, that

$$\begin{aligned}\int_{h(t)}^{\sigma(t)} p(s)ds &= \int_{h(t)}^{t} p(s)ds \\ &= \int_{h(3k+2.6)}^{3k+2.6} p(s)ds = \int_{3k}^{3k+2.6} p(s)ds = 0,962\end{aligned}$$

and therefore

$$\limsup_{t\to\infty} \int_{h(t)}^{\sigma(t)} p(s)ds = 0,962 < 1,$$

that is, condition (2.3) *of Theorem 2.2 is not satisfied.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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