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ON THE EİGENVALUES AND THE EIGENFUNCTIONS OF THE STURM-LIOUVILLE FUZZY BOUNDARY VALUE PROBLEM

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Abstract: In this paper, the fuzzy Sturm-Liouville equation is defined under the approach of Hukuhara differentiability. Several theorems are given for the eigenvalues and eigenfunctions of the worked Sturm-Liouville fuzzy boundary value problem and different examples are examined for these problems.

Keywords: fuzzy boundary value problems; Hukuhara differentiability; fuzzy numbers; eigenvalue; eigenfunction.

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1. Introduction

The concept of fuzzy numbers and fuzzy arithmetic were first introduced by Zadeh [9], Dubois and Prade [1]. [2,3] can be read for more information on fuzzy numbers and fuzzy arithmetic.

One of the major applications of fuzzy number arithmetic is treating fuzzy differential equations and fuzzy boundary value problems.

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In this paper, the fuzzy Sturm-Liouville equation is defined and the eigenvalues and the eigenfunctions of the Sturm-Liouville fuzzy boundary value problem are examined by using Hukuhara differentiability. To put it precisely, the Sturm-Liouville fuzzy boundary value problem is given as the form

$$Ly = p(x)y'' + q(x)y$$

$$Ly + \lambda y = 0,\ x \in (a,b), \quad (1.1)$$

$$B_1(y) := Ay(a) + By'(a) = 0, \quad (1.2)$$

$$B_2(y) := Cy(b) + Dy'(b) = 0, \quad (1.3)$$

where $p(x)$, $q(x)$, are continuous functions and are positive on $[a,b]$, $p'(x)=0$, $\lambda > 0$, $A,B,C,D \geq 0$, $A^2+B^2 \neq 0$ and $C^2+D^2 \neq 0$.

2. Preliminaries

In this section, we give some definitions and introduce the necessary notation which will be used throughout the paper.

Definition 2.1. [5] A fuzzy number is a function $u:\mathbb{R} \to [0,1]$ satisfying the following properties: u is normal, convex fuzzy set, upper semi-continuous on $\mathbb{R}$ and $cl\{x \in \mathbb{R} | u(x) > 0\}$ is compact, where cl denotes the closure of a subset.

Let $\mathbb{R}_F$ denote the space of fuzzy numbers.

Definition 2.2. [6] Let $u \in \mathbb{R}_F$. The α-level set of u, denoted $[u]^\alpha$, $0 < \alpha \leq 1$, is $[u]^\alpha = \{x \in \mathbb{R} | u(x) \geq \alpha\}$. If α=0, the support of u is defined $[u]^0 = cl\{x \in \mathbb{R} | u(x) > 0\}$. The notation, $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$ denotes explicitly the α-level set of u. We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u, respectively.

The following remark shows when $[\underline{u}_\alpha, \overline{u}_\alpha]$ is a valid α-level set.

Remark 2.1. [5] The sufficient and necessary conditions for $[\underline{u}_\alpha, \overline{u}_\alpha]$ to define the parametric form of a fuzzy number as follows:

1) $\underline{u}_\alpha$ is bounded monotonic increasing (nondecreasing) left-continuous function on (0,1] and right-continuous for $\alpha = 0$,

2) $\overline{u}_\alpha$ is bounded monotonic decreasing (nonincreasing) left-continuous function on (0,1] and right-continuous for $\alpha = 0$,

3) $\underline{u}_\alpha \leq \overline{u}_\alpha$, $0 \leq \alpha \leq 1$.

Definition 2.3. [7] If A is a symmetric triangular number with support $[\underline{a}, \overline{a}]$, the α-level set of A is $[A]^\alpha = \left[\underline{a} + \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha\right]$.

Definition 2.4. [6] For $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product λu are defined by $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda[u]^\alpha$, $\forall \alpha \in [0,1]$, where $[u]^\alpha + [v]^\alpha$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda[u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.

The metric structure is given by the Hausdorff distance

$$D: \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\},$$

by

$$D(u,v) = \sup_{\alpha \in [0,1]} \max\left\{\left|\underline{u}_\alpha - \underline{v}_\alpha\right|, \left|\overline{u}_\alpha - \overline{v}_\alpha\right|\right\} \quad [5].$$

Definition 2.5. [7] Let $u, v \in \mathbb{R}_F$. If there exist $w \in \mathbb{R}_F$ such that $u = v + w$, then w is called the H-difference of u and v and it is denoted $u \underset{H}{-} v$.

Definition 2.6. [5] Let I=(a,b), for $a, b \in \mathbb{R}$, and $F: I \to \mathbb{R}_F$ be a fuzzy function. We say F is Hukuhara differentiable at $t_0 \in I$ if there exists an element $F'(t_0) \in \mathbb{R}_F$ such that the limits

$$\lim_{h\to 0^+}\frac{F(t_0+h)-F(t_0)}{h}=\lim_{h\to 0^+}\frac{F(t_0)-F(t_0-h)}{h}$$

exist and equal $F'(t_0)$. Here the limits are taken in the metric space $(\mathbb{R}_F, D)$.

Theorem 2.1. [4] *Let* $f: I\to\mathbb{R}_F$ *be a function and denote* $[f(t)]^\alpha=[\underline{f}_\alpha(t),\overline{f}_\alpha(t)]$*, for each* $\alpha\in[0,1]$ *.If f is Hukuhara differentiable, then* $\underline{f}_\alpha$ *and* $\overline{f}_\alpha$ *are differentiable functions and* $[f'(t)]^\alpha=[\underline{f}'_\alpha(t),\overline{f}'_\alpha(t)]$.

Definition 2.7. [8] Let $f: A\subset\mathbb{R}\to\mathbb{R}$ be a function. f^+ and f^- are not the negative function defined as

$$f^+(x)=\begin{cases}f(x), & f(x)\ge 0\\ 0, & f(x)<0\end{cases}\quad,\quad f^-(x)=\begin{cases}-f(x), & f(x)\le 0\\ 0, & f(x)>0\end{cases}.$$

The function f^+ is called the positive piece of f , the function f^- is called the negative piece of f.

3. The Fuzzy Sturm-Liouville Equation

Consider the fuzzy differential equation

$$P_0(x)y''+P_1(x)y'+P_2(x)y+\lambda R(x)y=0,\quad x\in(a,b), \tag{3.1}$$

where $\lambda>0, P_0(x)$, $P_1(x)$, $P_2(x)$, $R(x)$ are continuous functions and are positive on $[a,b]$.

Theorem 3.1. *The fuzzy equation* (3.1) *can be rewritten as*

$$(p(x)y')'+q(x)y+\lambda r(x)y=0, \tag{3.2}$$

where $p(x), p'(x), q(x)$, $r(x)$ *are continuous functions and are positive on* $[a,b]$.

Proof. $[y(x)]^\alpha=[\underline{y}_\alpha(x),\overline{y}_\alpha(x)]$, the fuzzy equation (3.1)

$$P_0(x)[\underline{y}''_\alpha,\overline{y}''_\alpha]+P_1(x)[\underline{y}'_\alpha,\overline{y}'_\alpha]+P_2(x)[\underline{y}_\alpha,\overline{y}_\alpha]+\lambda R(x)[\underline{y}_\alpha,\overline{y}_\alpha]=0 \tag{3.3}$$

can be written as

$$[\underline{y}''_\alpha,\overline{y}''_\alpha]+u(x)[\underline{y}'_\alpha,\overline{y}'_\alpha]+v(x)[\underline{y}_\alpha,\overline{y}_\alpha]+\lambda R_1(x)[\underline{y}_\alpha,\overline{y}_\alpha]=0, \tag{3.4}$$

with $u=P_1/P_0$, $v=P_2/P_0$ and $R_1=R/P_0$. Now let $p(x)=e^{U(x)}$, where U is any antiderivative of u. Then p is positive on $[a,b]$ and since $U'=u$,

$$p'(x)=p(x)u(x) \tag{3.5}$$

is continuous on $[a,b]$. Multiplying (3.4) by $p(x)$ yields

$$\left[\underline{\ell}_\alpha(x),\overline{\ell}_\alpha(x)\right],$$

where

$$\underline{\ell}_\alpha(x)=p(x)\underline{y}''_\alpha+p(x)u(x)\underline{y}'_\alpha+p(x)v(x)\underline{y}_\alpha+p(x)\lambda R_1(x)\underline{y}_\alpha$$

$$\overline{\ell}_\alpha(x)=p(x)\overline{y}''_\alpha+p(x)u(x)\overline{y}'_\alpha+p(x)v(x)\overline{y}_\alpha+p(x)\lambda R_1(x)\overline{y}_\alpha$$

From (3.5)

$$\underline{\ell}_\alpha(x)=\left(p(x)\underline{y}'_\alpha\right)'+p(x)v(x)\underline{y}_\alpha+p(x)\lambda R_1(x)\underline{y}_\alpha$$

$$\overline{\ell}_\alpha(x)=\left(p(x)\overline{y}'_\alpha\right)'+p(x)v(x)\overline{y}_\alpha+p(x)\lambda R_1(x)\overline{y}_\alpha,$$

so (3.1) can be rewritten as in (3.2), with $p(x)v(x)=q(x)$ ve $p(x)R_1(x)=r(x)$. This completes the proof.

Example 3.1. Rewrite the fuzzy differential equation

$$y''+3y'+2y+\lambda y=0 \tag{3.6}$$

as the fuzzy differential equation (3.2).

Comparing (3.6) to (3.4) shows that $u(x)=3$, so we take $U(x)=3x$ and $p(x)=e^{3x}$.

Multiplying the fuzzy diferential equation (3.6) by e^{3x} yields

$$\left[e^{3x}\underline{y}''_\alpha+3e^{3x}\underline{y}'_\alpha+2e^{3x}\underline{y}_\alpha+\lambda e^{3x}\underline{y}_\alpha,e^{3x}\overline{y}''_\alpha+3e^{3x}\overline{y}'_\alpha+2e^{3x}\overline{y}_\alpha+\lambda e^{3x}\overline{y}_\alpha\right]=0.$$

Hence, yields

$$\left[\left(e^{3x}\underline{y}'_\alpha\right)'+2e^{3x}\underline{y}_\alpha+\lambda e^{3x}\underline{y}_\alpha,\left(e^{3x}\overline{y}'_\alpha\right)'+2e^{3x}\overline{y}_\alpha+\lambda e^{3x}\overline{y}_\alpha\right]=0.$$

Therefore, the fuzzy differential equation (3.6) can be rewritten

$$\left(e^{3x}y'\right)'+2e^{3x}y+\lambda e^{3x}y=0$$

as the form (3.2).

Example 3.2. Rewrite the fuzzy differential equation

$$x^2y''+xy'+\lambda y=0 \tag{3.7}$$

as the fuzzy differential equation (3.2).

Dividing the fuzzy differential equation (3.7) by x^2 yields

$$y''+\frac{1}{x}y'+\frac{\lambda}{x^2}y=0$$

Comparing this to (3.4) shows $u(x)=1/x$, so we take $U(x)=\ln x$ and $p(x)=e^{\ln x}=x$.

Multiplying the fuzzy diferential equation by x and using $xy''+y'=(xy')'$, the fuzzy differential equation (3.7) is equivalent to the fuzzy diferential equation

$$(xy')'+\frac{\lambda}{x}y=0.$$

Definition 3.1. If $p'(x)=0$, $r(x)=1$ and

$$Ly=p(x)y''+q(x)y$$

in the fuzzy differential equation (3.2), the fuzzy differential equation

$$Ly+\lambda y=0 \tag{3.8}$$

is called a fuzzy Sturm-Liouville equation.

Definition 3.2. $\left[y(x,\lambda_0)\right]^\alpha=\left[\underline{y}(x,\lambda_0),\overline{y}(x,\lambda_0)\right]\neq 0$, we say that $\lambda=\lambda_0$ is eigenvalue of (3.8) if the fuzzy differential equation (3.8) has the nontrivial solutions $\underline{y}(x,\lambda_0)\neq 0$, $\overline{y}(x,\lambda_0)\neq 0$.

4. The Eigenvalues and The Eigenfunctions of The Sturm-Liouville Fuzzy Boundary Value Problem

Consider the eigenvalues of the fuzzy boundary value problem (1.1)-(1.3).

Let $y_1(x,\lambda)$ and $y_2(x,\lambda)$ be linearly independent solutions of the classical differential equation $Ly+\lambda y=0$. Then, the general solution of the fuzzy differential equations (1.1) is

$$\left[y(x,\lambda)\right]^{\alpha}=\left[\underline{y}_{\alpha}(x,\lambda),\overline{y}_{\alpha}(x,\lambda)\right], \tag{4.1}$$

where

$$\underline{y}_{\alpha}(x,\lambda)=c_1(\alpha,\lambda)y_1(x,\lambda)+c_2(\alpha,\lambda)y_2(x,\lambda), \tag{4.2}$$

$$\overline{y}_{\alpha}(x,\lambda)=c_3(\alpha,\lambda)y_1(x,\lambda)+c_4(\alpha,\lambda)y_2(x,\lambda). \tag{4.3}$$

Again, let

$$\left[\phi(x,\lambda)\right]^{\alpha}=\left[\underline{\phi}_{\alpha}(x,\lambda),\overline{\phi}_{\alpha}(x,\lambda)\right]$$

be the solution of fuzzy differential equations (1.1) satisfying the conditions

$$y(a)=B,\quad y'(a)=-A, \tag{4.4}$$

and

$$\left[\psi(x,\lambda)\right]^{\alpha}=\left[\underline{\psi}_{\alpha}(x,\lambda),\overline{\psi}_{\alpha}(x,\lambda)\right]$$

be the solution of fuzzy differential equations (1.1) satisfying the conditions

$$y(b)=D,\quad y'(b)=-C, \tag{4.5}$$

where

$$\underline{\phi}_{\alpha}(x,\lambda)=\tilde{c}_1(\alpha,\lambda)y_1(x,\lambda)+\tilde{c}_2(\alpha,\lambda)y_2(x,\lambda),$$

$$\overline{\phi}_{\alpha}(x,\lambda)=\tilde{c}_3(\alpha,\lambda)y_1(x,\lambda)+\tilde{c}_4(\alpha,\lambda)y_2(x,\lambda),$$

$$\underline{\psi}_{\alpha}(x,\lambda)=\approx{c}_1(\alpha,\lambda)y_1(x,\lambda)+\approx{c}_2(\alpha,\lambda)y_2(x,\lambda),$$

$$\overline{\psi}_{\alpha}(x,\lambda)=\approx{c}_3(\alpha,\lambda)y_1(x,\lambda)+\approx{c}_4(\alpha,\lambda)y_2(x,\lambda).$$

Then, we have

$$W\left(\underline{\phi}_{\alpha},\underline{\psi}_{\alpha}\right)(x,\lambda)=\underline{\phi}_{\alpha}(x,\lambda)\underline{\psi}'_{\alpha}(x,\lambda)-\underline{\psi}_{\alpha}(x,\lambda)\underline{\phi}'_{\alpha}(x,\lambda)$$

$$=\left(\tilde{c}_1(\alpha,\lambda)\approx{c}_2(\alpha,\lambda)-\tilde{c}_2(\alpha,\lambda)\approx{c}_1(\alpha,\lambda)\right)\left(y_1(x,\lambda)y_2'(x,\lambda)-y_1'(x,\lambda)y_2(x,\lambda)\right)$$

$$W\left(\underline{\phi}_\alpha,\underline{\psi}_\alpha\right)(x,\lambda)=\left(\tilde{c}_1(\alpha,\lambda)\tilde{\tilde{c}}_2(\alpha,\lambda)-\tilde{c}_2(\alpha,\lambda)\tilde{\tilde{c}}_1(\alpha,\lambda)\right)W\left(y_1,y_2\right)(x,\lambda) \quad (4.6)$$

and similarly, we have

$$W\left(\overline{\phi}_\alpha,\overline{\psi}_\alpha\right)(x,\lambda)=\left(\tilde{c}_3(\alpha,\lambda)\tilde{\tilde{c}}_4(\alpha,\lambda)-\tilde{c}_4(\alpha,\lambda)\tilde{\tilde{c}}_3(\alpha,\lambda)\right)W\left(y_1,y_2\right)(x,\lambda). \quad (4.7)$$

On the other hand, let $\phi(x,\lambda)$ be the solution of the classical differential equation $Ly+\lambda y=0$ satisfying the conditions $y(a)=B$, $y'(a)=-A$. Then, the solution of the equation is

$$y(x,\lambda)=c_1(\lambda)y_1(x,\lambda)+c_2(\lambda)y_2(x,\lambda),$$

where $y_1(x,\lambda)$ and $y_2(x,\lambda)$ are linearly independent solutions of the differential equation $Ly+\lambda y=0$. Using boundary conditions,

$$y(a,\lambda)=c_1(\lambda)y_1(a,\lambda)+c_2(\lambda)y_2(a,\lambda)=B$$

$$y'(a,\lambda)=c_1(\lambda)y_1'(a,\lambda)+c_2(\lambda)y_2'(a,\lambda)=-A$$

are obtained. Solving for $c_1(\lambda)$, $c_2(\lambda)$ yields

$$c_1(\lambda)=\frac{\begin{vmatrix} B & y_2(a,\lambda) \\ -A & y_2'(a,\lambda)\end{vmatrix}}{W\left(y_1,y_2\right)(a,\lambda)}=\frac{By_2'(a,\lambda)+Ay_2(a,\lambda)}{W\left(y_1,y_2\right)(a,\lambda)},$$

$$c_2(\lambda)=\frac{\begin{vmatrix} y_1(a,\lambda) & B \\ y_1'(a,\lambda) & -A\end{vmatrix}}{W\left(y_1,y_2\right)(a,\lambda)}=\frac{-Ay_1(a,\lambda)-By_1'(a,\lambda)}{W\left(y_1,y_2\right)(a,\lambda)}.$$

Thus, yields

$$\phi(x,\lambda)=\frac{1}{W\left(y_1,y_2\right)(a,\lambda)}\left\{\left(Ay_2(a,\lambda)+By_2'(a,\lambda)\right)y_1(x,\lambda)-\right.$$

$$\left.-\left(Ay_1(a,\lambda)+By_1'(a,\lambda)\right)y_2(x,\lambda)\right\}.$$

Let $\psi(x,\lambda)$ be the solution of the classical differential equation $Ly+\lambda y=0$ satisfying the conditions $y(b)=D$, $y'(b)=-C$. Similary,

$$\psi(x,\lambda)=\frac{1}{W(y_1,y_2)(b,\lambda)}\{(Cy_2(b,\lambda)+Dy_2'(b,\lambda))y_1(x,\lambda)-$$

$$-(Cy_1(b,\lambda)+Dy_1'(b,\lambda))y_2(x,\lambda)\}$$

is obtained. Thus,

$$[\phi(x,\lambda)]^{\alpha}=[\underline{\phi}_{\alpha}(x,\lambda),\overline{\phi}_{\alpha}(x,\lambda)]=[c_1(\alpha),c_2(\alpha)]\phi(x,\lambda)$$

is the solution of the fuzzy differential equation (1.1) satisfying the conditions (4.4) and

$$[\psi(x,\lambda)]^{\alpha}=[\underline{\psi}_{\alpha}(x,\lambda),\overline{\psi}_{\alpha}(x,\lambda)]=[c_1(\alpha),c_2(\alpha)]\psi(x,\lambda)$$

is the solution satisfying the conditions (4.5), where

$$c'_1(\alpha)\geq 0,\ c'_2(\alpha)\leq 0,\ c_1(\alpha)\leq c_2(\alpha),\text{ for } \alpha=1\ \ c_1(\alpha)=c_2(\alpha)=1,$$

$$[1]^{\alpha}=[c_1(\alpha),c_2(\alpha)].$$

Specially, $[c_1(\alpha),c_2(\alpha)]=[\alpha,2-\alpha]$ can be taken. Then, Wronskian functions $\underline{W}_{\alpha}(x,\lambda)$ and $\overline{W}_{\alpha}(x,\lambda)$ are obtained as

$$\underline{W}_{\alpha}(x,\lambda)=\alpha^2(\phi(x,\lambda)\psi'(x,\lambda)-\psi(x,\lambda)\phi'(x,\lambda)) \tag{4.8}$$

$$\overline{W}_{\alpha}(x,\lambda)=(2-\alpha)^2(\phi(x,\lambda)\psi'(x,\lambda)-\psi(x,\lambda)\phi'(x,\lambda)). \tag{4.9}$$

Here, computing the value $\phi(x,\lambda)\psi'(x,\lambda)-\psi(x,\lambda)\phi'(x,\lambda)$ yields

$$\phi(x,\lambda)\psi'(x,\lambda)-\phi'(x,\lambda)\psi(x,\lambda)=\frac{W(y_1,y_2)(x,\lambda)}{W(y_1,y_2)(a,\lambda)W(y_1,y_2)(b,\lambda)}\{$$

$$\{(Ay_1(a,\lambda)+By_1'(a,\lambda))(Cy_2(b,\lambda)+Dy_2'(b,\lambda))-$$

$$-(Ay_2(a,\lambda)+By_2'(a,\lambda))(Cy_1(b,\lambda)+Dy_1'(b,\lambda))\}. \tag{4.10}$$

Hence, considering the equations (4.6) and (4.7) yields

$$\tilde{c}_1(\alpha,\lambda)\approx{c}_2(\alpha,\lambda)-\tilde{c}_2(\alpha,\lambda)\approx{c}_1(\alpha,\lambda)=\frac{\alpha^2}{W(y_1,y_2)(a,\lambda)W(y_1,y_2)(b,\lambda)}\{$$

$$\{(Ay_1(a,\lambda)+By_1'(a,\lambda))(Cy_2(b,\lambda)+Dy_2'(b,\lambda))-$$

$$-\left(Ay_2(a,\lambda)+By_2'(a,\lambda)\right)\left(Cy_1(b,\lambda)+Dy_1'(b,\lambda)\right)\Big\}$$

and

$$\tilde{c}_3(\alpha,\lambda)\tilde{\tilde{c}}_4(\alpha,\lambda)-\tilde{c}_4(\alpha,\lambda)\tilde{\tilde{c}}_3(\alpha,\lambda)=\frac{(2-\alpha)^2}{W(y_1,y_2)(a,\lambda)W(y_1,y_2)(b,\lambda)}\Big\{$$

$$\Big\{\left(Ay_1(a,\lambda)+By_1'(a,\lambda)\right)\left(Cy_2(b,\lambda)+Dy_2'(b,\lambda)\right)-$$

$$-\left(Ay_2(a,\lambda)+By_2'(a,\lambda)\right)\left(Cy_1(b,\lambda)+Dy_1'(b,\lambda)\right)\Big\}.$$

Consequently, the equation

$$W\left(\underline{\phi}_\alpha,\underline{\psi}_\alpha\right)(x,\lambda)=\frac{\alpha^2}{(2-\alpha)^2}W\left(\overline{\phi}_\alpha,\overline{\psi}_\alpha\right)(x,\lambda)$$

is obtained.

Theorem 4.1. *The Wronskian functions* $W\left(\underline{\phi}_\alpha,\underline{\psi}_\alpha\right)(x,\lambda)$ *and* $W\left(\overline{\phi}_\alpha,\overline{\psi}_\alpha\right)(x,\lambda)$ *are independent of variable x for* $x\in(a,b)$, *where functions* $\underline{\phi}_\alpha$, $\underline{\psi}_\alpha$, $\overline{\phi}_\alpha$, $\overline{\psi}_\alpha$ *are the solution of the fuzzy boundary value problem* (1.1)-(1.3).

Proof. Derivating of equations

$$W\left(\underline{\phi}_\alpha,\underline{\psi}_\alpha\right)(x,\lambda)=\underline{\phi}_\alpha(x,\lambda)\underline{\psi}'_\alpha(x,\lambda)-\underline{\psi}_\alpha(x,\lambda)\underline{\phi}'_\alpha(x,\lambda)$$

$$W\left(\overline{\phi}_\alpha,\overline{\psi}_\alpha\right)(x,\lambda)=\overline{\phi}_\alpha(x,\lambda)\overline{\psi}'_\alpha(x,\lambda)-\overline{\psi}_\alpha(x,\lambda)\overline{\phi}'_\alpha(x,\lambda)$$

according to variable x and using the functions $\left[\phi(x,\lambda)\right]^\alpha$, $\left[\psi(x,\lambda)\right]^\alpha$ are the solutions of (1.1) we have the equations

$$W'\left(\underline{\phi}_\alpha,\underline{\psi}_\alpha\right)(x,\lambda)=0 \text{ and } W'\left(\overline{\phi}_\alpha,\overline{\psi}_\alpha\right)(x,\lambda)=0.$$

Thus it is shown that the Wronskian functions $W\left(\underline{\phi}_\alpha,\underline{\psi}_\alpha\right)(x,\lambda)$ and $W\left(\overline{\phi}_\alpha,\overline{\psi}_\alpha\right)(x,\lambda)$ are independent of variable x. Hence the functions $W\left(\underline{\phi}_\alpha,\underline{\psi}_\alpha\right)(x,\lambda)$ and $W\left(\overline{\phi}_\alpha,\overline{\psi}_\alpha\right)(x,\lambda)$ can be shown as

$$\underline{W}_\alpha(\lambda)=W\left(\underline{\phi}_\alpha,\underline{\psi}_\alpha\right)(x,\lambda)=\underline{\phi}_\alpha(x,\lambda)\underline{\psi}'_\alpha(x,\lambda)-\underline{\psi}_\alpha(x,\lambda)\underline{\phi}'_\alpha(x,\lambda), \quad (4.11)$$

$$\overline{W}_{\alpha}(\lambda)=W\left(\overline{\phi}_{\alpha},\overline{\psi}_{\alpha}\right)(x,\lambda)=\overline{\phi}_{\alpha}(x,\lambda)\overline{\psi}'_{\alpha}(x,\lambda)-\overline{\psi}_{\alpha}(x,\lambda)\overline{\phi}'_{\alpha}(x,\lambda). \quad (4.12)$$

Theorem 4.2. *The eigenvalues of the fuzzy boundary value problem* (1.1)-(1.3) *if and only if are consist of the zeros of functions* $\underline{W}_{\alpha}(\lambda)$ *and* $\overline{W}_{\alpha}(\lambda)$.

Proof. Let λ_0 be zero of $\underline{W}_{\alpha}(\lambda)$ and $\overline{W}_{\alpha}(\lambda)$. Then the functions $\underline{\phi}_{\alpha}$, $\underline{\psi}_{\alpha}$ and $\overline{\phi}_{\alpha}$, $\overline{\psi}_{\alpha}$ are linearly dependent. That is,

$$\underline{\phi}_{\alpha}(x,\lambda_0)=k_1\underline{\psi}_{\alpha}(x,\lambda_0) \quad \text{and} \quad \overline{\phi}_{\alpha}(x,\lambda_0)=k_2\overline{\psi}_{\alpha}(x,\lambda_0), \qquad (4.13)$$

$k_1,k_2\neq 0$. In this case, since $\left[\psi(x,\lambda_0)\right]^{\alpha}=\left[\underline{\psi}_{\alpha}(x,\lambda_0),\overline{\psi}_{\alpha}(x,\lambda_0)\right]$ satisfies the boundary condition (1.3), $\underline{\psi}_{\alpha}(x,\lambda_0)$ and $\overline{\psi}_{\alpha}(x,\lambda_0)$ also satisfy the boundary condition (1.3). In addition, from (4.13) the functions $\underline{\phi}_{\alpha}(x,\lambda_0)$ and $\overline{\phi}_{\alpha}(x,\lambda_0)$ satisfy the boundary condition (1.3). So the fuzzy function $\left[\phi(x,\lambda_0)\right]^{\alpha}=\left[\underline{\phi}_{\alpha}(x,\lambda_0),\overline{\phi}_{\alpha}(x,\lambda_0)\right]$ satisfies the boundary condition (1.3). Hence, $\left[\phi(x,\lambda_0)\right]^{\alpha}$ is the solution of the boundary value problem (1.1)-(1.3) for $\lambda=\lambda_0$. Thus, $\lambda=\lambda_0$ is the eigenvalue.

Now, we show that if $\lambda=\lambda_0$ is the eigenvalue, $\underline{W}_{\alpha}(\lambda_0)=0$ and $\overline{W}_{\alpha}(\lambda_0)=0$. Let us assume the contrary. That is, $\underline{W}_{\alpha}(\lambda_0)\neq 0$ or $\overline{W}_{\alpha}(\lambda_0)\neq 0$. Let be $\underline{W}_{\alpha}(\lambda_0)\neq 0$. In this case, the functions $\underline{\phi}_{\alpha}(x,\lambda_0)$ and $\underline{\psi}_{\alpha}(x,\lambda_0)$ are linearly independent. Therefore, the general solution of the fuzzy differential equation (1.1) can be written in the form

$$\left[y(x,\lambda_0)\right]^{\alpha}=\left[\underline{y}_{\alpha}(x,\lambda_0),\overline{y}_{\alpha}(x,\lambda_0)\right], \qquad (4.14)$$

where

$$\underline{y}_{\alpha}(x,\lambda_0)=c_1(\alpha,\lambda_0)\underline{\phi}_{\alpha}(x,\lambda_0)+c_2(\alpha,\lambda_0)\underline{\psi}_{\alpha}(x,\lambda_0), \qquad (4.15)$$

$$\overline{y}_{\alpha}(x,\lambda_0)=c_3(\alpha,\lambda_0)\overline{\phi}_{\alpha}(x,\lambda_0)+c_4(\alpha,\lambda_0)\overline{\psi}_{\alpha}(x,\lambda_0). \qquad (4.16)$$

From the boundary condition (1.2)

$$A\left[\underline{y}_{\alpha}(a,\lambda_0),\overline{y}_{\alpha}(a,\lambda_0)\right]+B\left[\underline{y}_{\alpha}(a,\lambda_0),\overline{y}_{\alpha}(a,\lambda_0)\right]'=0.$$

Using Hukuhara differentiability and fuzzy arithmetic, the equalities

$$c_1(\alpha,\lambda_0)\left(A\underline{\phi}_\alpha(a,\lambda_0)+B\underline{\phi}'_\alpha(a,\lambda_0)\right)+c_2(\alpha,\lambda_0)\left(A\underline{\psi}_\alpha(a,\lambda_0)+B\underline{\psi}'_\alpha(a,\lambda_0)\right)=0$$

$$c_3(\alpha,\lambda_0)\left(A\overline{\phi}_\alpha(a,\lambda_0)+B\overline{\phi}'_\alpha(a,\lambda_0)\right)+c_4(\alpha,\lambda_0)\left(A\overline{\psi}_\alpha(a,\lambda_0)+B\overline{\psi}'_\alpha(a,\lambda_0)\right)=0$$

are obtained. Since the solution function $\left[\phi(x,\lambda_0)\right]^\alpha=\left[\underline{\phi}_\alpha(x,\lambda_0),\overline{\phi}_\alpha(x,\lambda_0)\right]$ satisfies the boundary condition (1.2), we have

$$c_2(\alpha,\lambda_0)\left(A\underline{\psi}_\alpha(a,\lambda_0)+B\underline{\psi}'_\alpha(a,\lambda_0)\right)=0,$$

$$c_4(\alpha,\lambda_0)\left(A\overline{\psi}_\alpha(a,\lambda_0)+B\overline{\psi}'_\alpha(a,\lambda_0)\right)=0.$$

From (4.4), we get

$$c_2(\alpha,\lambda_0)\left(\underline{\phi}_\alpha(a,\lambda_0)\underline{\psi}'_\alpha(a,\lambda_0)-\underline{\phi}'_\alpha(a,\lambda_0)\underline{\psi}_\alpha(a,\lambda_0)\right)=0,$$

$$c_4(\alpha,\lambda_0)\left(\overline{\phi}_\alpha(a,\lambda_0)\overline{\psi}'_\alpha(a,\lambda_0)-\overline{\phi}'_\alpha(x,\lambda_0)\overline{\psi}_\alpha(a,\lambda_0)\right)=0,$$

and from (4.11), (4.12), we have

$$c_2(\alpha,\lambda_0)\underline{W}_\alpha(\lambda_0)=0,\ \ c_4(\alpha,\lambda_0)\overline{W}_\alpha(\lambda_0)=0.$$

Since $\underline{W}_\alpha(\lambda_0)\neq 0$

$$c_2(\alpha,\lambda_0)=0$$

is obtained. In addition, from the boundary condition (1.3), (4.14)-(4.16) satisfies the equality

$$C\left[\underline{y}_\alpha(b,\lambda_0),\overline{y}_\alpha(b,\lambda_0)\right]+D\left[\underline{y}_\alpha(b,\lambda_0),\overline{y}_\alpha(b,\lambda_0)\right]'=0,$$

that is, satisfies the equalities

$$c_1(\alpha,\lambda_0)\left(C\underline{\phi}_\alpha(b,\lambda_0)+D\underline{\phi}'_\alpha(b,\lambda_0)\right)+c_2(\alpha,\lambda_0)\left(C\underline{\psi}_\alpha(b,\lambda_0)+D\underline{\psi}'_\alpha(b,\lambda_0)\right)=0,$$

$$c_3(\alpha,\lambda_0)\left(C\overline{\phi}_\alpha(b,\lambda_0)+D\overline{\phi}'_\alpha(b,\lambda_0)\right)+c_4(\alpha,\lambda_0)\left(C\overline{\psi}_\alpha(b,\lambda_0)+D\overline{\psi}'_\alpha(b,\lambda_0)\right)=0.$$

Similarly, since the function $\left[\psi(x,\lambda_0)\right]^\alpha=\left[\underline{\psi}_\alpha(x,\lambda_0),\overline{\psi}_\alpha(x,\lambda_0)\right]$ satisfies the boundary condition (1.3) and from (4.5), (4.11), (4.12) we obtained

$$c_1(\alpha,\lambda_0)=0$$

Substituting found this values in (4.15), we have

$$\underline{y}_\alpha(x,\lambda_0)=0.$$

Then λ_0 is not an eigenvalue. So we have a contradiction. Similarly, if $\overline{W}_\alpha(\lambda_0)\neq 0$, we have $\overline{y}_\alpha(x,\lambda_0)=0$. That is, λ_0 is not an eigenvalue. This completes the proof.

Example 4.1. Consider the fuzzy Sturm-Liouville problem

$$y''+\lambda y=0,\ \ y(0)=0,\ \ y(1)=0. \tag{4.17}$$

Let be $\lambda=k^2,\ k>0$,

$$\phi(x,\lambda)=\sin(kx)$$

be the solution of the classical differential equation $y''+\lambda y=0$ satisfying the condition $y(0)=0$ and

$$\psi(x,\lambda)=\sin(k)\cos(kx)-\cos(k)\sin(kx)$$

be the solution satisfying the condition $y(1)=0$. Then,

$$[\phi(x,\lambda)]^\alpha=\left[\underline{\phi}_\alpha(x,\lambda),\overline{\phi}_\alpha(x,\lambda)\right]=[\alpha,2-\alpha]\sin(kx) \tag{4.18}$$

is the solution of the fuzzy differential equation $y''+\lambda y=0$ satisfying the condition $y(0)=0$ and

$$[\psi(x,\lambda)]^\alpha=\left[\underline{\psi}_\alpha(x,\lambda),\overline{\psi}_\alpha(x,\lambda)\right] \tag{4.19}$$

$$=[\alpha,2-\alpha](\sin(k)\cos(kx)-\cos(k)\sin(kx))$$

is the solution satisfying the condition $y(1)=0$. Since the eigenvalues of the fuzzy Sturm-Liouville problem (4.17) are zeros the functions $\underline{W}_\alpha(\lambda)$ and $\overline{W}_\alpha(\lambda)$, $\underline{W}_\alpha(\lambda)$ is obtained as

$$\underline{W}_\alpha(\lambda)=-k\alpha^2\sin(k)\sin^2(kx)-k\alpha^2\cos(k)\cos(kx)\sin(kx)-$$

$$-k\alpha^2\sin(k)\cos^2(kx)+k\alpha^2\cos(k)\cos(kx)\sin(kx)$$

$$\Rightarrow \underline{W}_\alpha(\lambda)=-k\alpha^2\sin(k)$$

and similarly $\overline{W}_\alpha(\lambda)$ is obtained as

$$\overline{W}_\alpha(\lambda) = -k(2-\alpha)^2 \sin(k).$$

From here we have

$$\underline{W}_\alpha(\lambda) = -k\alpha^2 \sin(k) = 0 \Rightarrow \sin(k) = 0 \Rightarrow k_n = n\pi, \ n=1,2,..$$

$$\overline{W}_\alpha(\lambda) = -k(2-\alpha)^2 \sin(k) = 0 \Rightarrow \sin(k) = 0 \Rightarrow k_n = n\pi, \ n=1,2,..$$

Substituing this equalities in (4.18), (4.19), we obtain

$$[\phi_n(x)]^\alpha = [\underline{\phi_n}_\alpha(x), \overline{\phi_n}_\alpha(x)] = [\alpha, 2-\alpha]\sin(n\pi x)$$

$$[\psi_n(x)]^\alpha = [\underline{\psi_n}_\alpha(x), \overline{\psi_n}_\alpha(x)] = [\alpha, 2-\alpha](-\cos(n\pi)\sin(n\pi x)).$$

As $[\alpha, 2-\alpha](\sin(n\pi x))^+$ and $[\alpha, 2-\alpha](-\cos(n\pi)\sin(n\pi x))^+$, $[\phi_n(x)]^\alpha$ and $[\psi_n(x)]^\alpha$ are a valid $\alpha-$level set. Let be $n\pi x \in [(n-1)\pi, n\pi]$, n=1,2,....

i) If n is single, $\sin(n\pi x) \geq 0$. Then $[\phi_n(x)]^\alpha$ is a valid $\alpha-$level set.

ii) If n is double, $\sin(n\pi x) \leq 0$, $\cos(n\pi) = 1$ and $-\cos(n\pi)\sin(n\pi x) \geq 0$. Then $[\psi_n(x)]^\alpha$ is a valid $\alpha-$level set.

Consequently; $n\pi x \in [(n-1)\pi, n\pi]$, n=1,2,...

i) If n is single, the eigenvalues are $\lambda_n = n^2\pi^2$, with associated eigenfuctions

$$[y_{1n}(x)]^\alpha = [\alpha, 2-\alpha]\sin(n\pi x) = [\alpha, 2-\alpha](\sin(n\pi x))^+,$$

ii) If n is double, the eigenvalues are $\lambda_n = n^2\pi^2$, with associated eigenfuctions

$$[y_{2n}(x)]^\alpha = [\alpha, 2-\alpha](-\sin(n\pi x)) = [\alpha, 2-\alpha](\sin(n\pi x))^-,$$

iii) If $\alpha = 1$, the eigenvalues are $\lambda_n = n^2\pi^2$, with associated eigenfuctions

$$[y_n(x)]^\alpha = \sin(n\pi x).$$

Example 4.2. If we take

$$y(0)=0,\quad y'(1)=0 \tag{4.20}$$

as the boundary conditions of the fuzzy Sturm-Liouville problem (4.17),

$$[\phi(x,\lambda)]^{\alpha}=\left[\underline{\phi}_{\alpha}(x,\lambda),\overline{\phi}_{\alpha}(x,\lambda)\right]=[\alpha,2-\alpha]\sin(kx) \tag{4.21}$$

is the solution of the fuzzy differential equation $y''+\lambda y=0$ satisfying the condition $y(0)=0$ and

$$[\psi(x,\lambda)]^{\alpha}=\left[\underline{\psi}_{\alpha}(x,\lambda),\overline{\psi}_{\alpha}(x,\lambda)\right]$$

$$=[\alpha,2-\alpha]\left(\sin(k)\sin(kx)+\cos(k)\cos(kx)\right) \tag{4.22}$$

is the solution satisfying the condition $y'(1)=0$. Then, from (4.11), (4.12)

$$\underline{W}_{\alpha}(\lambda)=-k\alpha^{2}\cos(k)\Rightarrow\cos(k)=0\Rightarrow k_{n}=\frac{(2n-1)}{2}\pi,\ n=1,2,..$$

$$\overline{W}_{\alpha}(\lambda)=-k(2-\alpha)^{2}\cos(k)\Rightarrow\cos(k)=0\Rightarrow k_{n}=\frac{(2n-1)}{2}\pi,\ n=1,2,..$$

are obtained. Substituing this equalities in (4.21), (4.22), we have

$$[\phi_{n}(x)]^{\alpha}=\left[\underline{\phi_{n}}_{\alpha}(x),\overline{\phi_{n}}_{\alpha}(x)\right]=[\alpha,2-\alpha]\sin\left(\frac{(2n-1)}{2}\pi x\right)$$

$$[\psi_{n}(x)]^{\alpha}=\left[\underline{\psi_{n}}_{\alpha}(x),\overline{\psi_{n}}_{\alpha}(x)\right]=[\alpha,2-\alpha]\left(\sin\left(\frac{2n-1}{2}\pi\right)\sin\left(\frac{2n-1}{2}\pi x\right)\right).$$

As $[\alpha,2-\alpha]\left(\sin\left(\frac{(2n-1)}{2}\pi x\right)\right)^{+}$ and $[\alpha,2-\alpha]\left(\sin\left(\frac{2n-1}{2}\pi\right)\sin\left(\frac{2n-1}{2}\pi x\right)\right)^{+}$, $[\phi_{n}(x)]^{\alpha}$ and $[\psi_{n}(x)]^{\alpha}$ are a valid $\alpha-$ level set. Let be $\left(\frac{2n-1}{2}\right)\pi x\in[(n-1)\pi,n\pi],\ n=1,2,\dots$.

i) If n is single, $\sin\left(\left(\frac{2n-1}{2}\right)\pi x\right)\geq 0$. Then $[\phi_{n}(x)]^{\alpha}$ is a valid $\alpha-$level set.

ii) If n is double, $\sin\left(\left(\frac{2n-1}{2}\right)\pi x\right) \le 0$, $\sin\left(\frac{2n-1}{2}\pi\right) = -1$ and $\sin\left(\frac{2n-1}{2}\pi\right)\sin\left(\frac{2n-1}{2}\pi x\right) \ge 0$. Then $\left[\psi_n(x)\right]^\alpha$ is a valid $\alpha-$level set.

Consequently; $\left(\frac{2n-1}{2}\right)\pi x \in \left[(n-1)\pi, n\pi\right]$, n=1,2,...

i) If n is single, the eigenvalues are $\lambda_n = \frac{(2n-1)^2}{4}\pi^2$, with associated eigenfuctions

$$\left[y_{1n}(x)\right]^\alpha = [\alpha, 2-\alpha]\sin\left(\frac{(2n-1)}{2}\pi x\right) = [\alpha, 2-\alpha]\left(\sin\left(\frac{(2n-1)}{2}\pi x\right)\right)^+,$$

ii) If n is double, the eigenvalues are $\lambda_n = \frac{(2n-1)^2}{4}\pi^2$, with associated eigenfuctions

$$\left[y_{2n}(x)\right]^\alpha = [\alpha, 2-\alpha]\left(-\sin\left(\frac{2n-1}{2}\pi x\right)\right) = [\alpha, 2-\alpha]\left(\sin\left(\frac{(2n-1)}{2}\pi x\right)\right)^-,$$

iii) If $\alpha = 1$, the eigenvalues are $\lambda_n = \frac{(2n-1)^2}{4}\pi^2$, with associated eigenfuctions

$$\left[y_n(x)\right]^\alpha = \sin\left(\left(\frac{2n-1}{2}\right)\pi x\right).$$

Example 4.3. If we take

$$y'(0) = 0, \quad y(1) = 0 \tag{4.23}$$

as the boundary conditions of the fuzzy Sturm-Liouville problem (4.17),

$$\left[\phi(x,\lambda)\right]^\alpha = \left[\underline{\phi}_\alpha(x,\lambda), \overline{\phi}_\alpha(x,\lambda)\right] = [\alpha, 2-\alpha]\cos(kx) \tag{4.24}$$

is the solution of the fuzzy differential equation $y'' + \lambda y = 0$ satisfying the condition $y'(0) = 0$ and

$$\left[\psi(x,\lambda)\right]^\alpha = \left[\underline{\psi}_\alpha(x,\lambda), \overline{\psi}_\alpha(x,\lambda)\right]$$

$$= [\alpha, 2-\alpha]\left(\sin(k)\cos(kx) - \cos(k)\sin(kx)\right) \tag{4.25}$$

is the solution satisfying the condition $y(1)=0$. Then,

$$\underline{W}_{\alpha}(\lambda)=-k\alpha^2\cos(k)\Rightarrow\cos(k)=0\Rightarrow k_n=\frac{(2n-1)}{2}\pi,\ n=1,2,..,$$

$$\overline{W}_{\alpha}(\lambda)=-k(2-\alpha)^2\cos(k)\Rightarrow\cos(k)=0\Rightarrow k_n=\frac{(2n-1)}{2}\pi,\ n=1,2,...$$

Substituing this equalities in (4.24), (4.25), we have

$$[\phi_n(x)]^{\alpha}=\left[\underline{\phi_n}_{\alpha}(x),\overline{\phi_n}_{\alpha}(x)\right]=[\alpha,2-\alpha]\cos\left(\frac{(2n-1)}{2}\pi x\right)$$

$$[\psi_n(x)]^{\alpha}=\left[\underline{\psi_n}_{\alpha}(x),\overline{\psi_n}_{\alpha}(x)\right]=[\alpha,2-\alpha]\left(\sin\left(\frac{2n-1}{2}\pi\right)\cos\left(\frac{2n-1}{2}\pi x\right)\right).$$

i) If $\left(\frac{2n-1}{2}\right)\pi x\in\left(\frac{(2(n-1)-1)}{2}\pi,\frac{(2(n-1)+1)}{2}\pi\right)$, n=1,3,5,... , $\cos\left(\left(\frac{2n-1}{2}\right)\pi x\right)>0$.

Then $[\phi_n(x)]^{\alpha}$ is a valid $\alpha-$level set.

ii) If $\left(\frac{2n-1}{2}\right)\pi x\in\left(\frac{(2(n-2)+1)}{2}\pi,\frac{(2(n-2)+3)}{2}\pi\right)$, n=2,4,6,... , $\cos\left(\left(\frac{2n-1}{2}\right)\pi x\right)<0$,

$\sin\left(\frac{2n-1}{2}\pi\right)=-1$ and $\sin\left(\frac{2n-1}{2}\pi\right)\cos\left(\frac{2n-1}{2}\pi x\right)>0$. Then $[\psi_n(x)]^{\alpha}$ is a valid $\alpha-$level set.

Consequently;

i) If $\left(\frac{2n-1}{2}\right)\pi x\in\left(\frac{(2(n-1)-1)}{2}\pi,\frac{(2(n-1)+1)}{2}\pi\right)$, n=1,3,5,... , the eigenvalues are $\lambda_n=\frac{(2n-1)^2}{4}\pi^2$, with associated eigenfuctions

$$[y_{1n}(x)]^{\alpha}=[\alpha,2-\alpha]\cos\left(\frac{(2n-1)}{2}\pi x\right)=[\alpha,2-\alpha]\left(\cos\left(\frac{(2n-1)}{2}\pi x\right)\right)^{+},$$

ii) If $\left(\frac{2n-1}{2}\right)\pi x \in \left(\frac{(2(n-2)+1)}{2}\pi, \frac{(2(n-2)+3)}{2}\pi\right)$, n=2,4,6,... , the eigenvalues are $\lambda_n = \frac{(2n-1)^2}{4}\pi^2$, with associated eigenfuctions

$$[y_{2n}(x)]^\alpha = [\alpha, 2-\alpha]\left(-\cos\left(\frac{2n-1}{2}\pi x\right)\right) = [\alpha, 2-\alpha]\left(\cos\left(\frac{(2n-1)}{2}\pi x\right)\right)^-,$$

iii) If $\alpha = 1$, the eigenvalues are $\lambda_n = \frac{(2n-1)^2}{4}\pi^2$, with associated eigenfuctions

$$[y_n(x)]^\alpha = \cos\left(\left(\frac{2n-1}{2}\right)\pi x\right).$$

Example 4.4. If we take

$$y'(0) = 0, \;\; y'(1) = 0 \qquad (4.26)$$

as the boundary conditions of the fuzzy Sturm-Liouville problem (4.17),

$$[\phi(x,\lambda)]^\alpha = [\underline{\phi}_\alpha(x,\lambda), \overline{\phi}_\alpha(x,\lambda)] = [\alpha, 2-\alpha]\cos(kx) \qquad (4.27)$$

is the solution of the fuzzy differential equation $y'' + \lambda y = 0$ satisfying the condition $y'(0) = 0$ and

$$[\psi(x,\lambda)]^\alpha = [\underline{\psi}_\alpha(x,\lambda), \overline{\psi}_\alpha(x,\lambda)]$$
$$= [\alpha, 2-\alpha](\sin(k)\sin(kx) + \cos(k)\cos(kx)) \qquad (4.28)$$

is the solution satisfying the condition $y'(1) = 0$. Then,

$$\underline{W}_\alpha(\lambda) = k\alpha^2\sin(k) \Rightarrow \sin(k) = 0 \Rightarrow k_n = n\pi, \text{ n=1,2,..,}$$

$$\overline{W}_\alpha(\lambda) = k(2-\alpha)^2\sin(k) \Rightarrow \sin(k) = 0 \Rightarrow k_n = n\pi, \text{ n=1,2,...}$$

Substituing this equalities in (4.27), (4.28), we have

$$[\phi_n(x)]^\alpha = [\underline{\phi_n}_\alpha(x), \overline{\phi_n}_\alpha(x)] = [\alpha, 2-\alpha]\cos(n\pi x),$$

$$[\psi_n(x)]^\alpha = [\underline{\psi_n}_\alpha(x), \overline{\psi_n}_\alpha(x)] = [\alpha, 2-\alpha](\cos(n\pi)\cos(n\pi x)).$$

i) If $n\pi x \in \left(\frac{(2(n-2)-1)}{2}\pi, \frac{(2(n-2)+1)}{2}\pi \right)$, n=2,4,6,..., $\cos(n\pi x) > 0$. Then $[\phi_n(x)]^\alpha$ is a valid $\alpha-$level set.

ii) If $n\pi x \in \left(\frac{(2(n-1)+1)}{2}\pi, \frac{(2(n-1)+3)}{2}\pi \right)$, n=1,3,5,..., $\cos(n\pi x) < 0$, $\cos(n\pi) = -1$ and $\cos(n\pi)\cos(n\pi x) > 0$. Then $[\psi_n(x)]^\alpha$ is a valid $\alpha-$level set.

Consequently;

i) If $n\pi x \in \left(\frac{(2(n-2)-1)}{2}\pi, \frac{(2(n-2)+1)}{2}\pi \right)$, n=2,4,6,..., the eigenvalues are $\lambda_n = n^2\pi^2$, with associated eigenfuctions

$$[y_{1n}(x)]^\alpha = [\alpha, 2-\alpha]\cos(n\pi x) = [\alpha, 2-\alpha](\cos(n\pi x))^+,$$

ii) If $n\pi x \in \left(\frac{(2(n-1)+1)}{2}\pi, \frac{(2(n-1)+3)}{2}\pi \right)$, n=1,3,5,..., the eigenvalues are $\lambda_n = n^2\pi^2$, with associated eigenfuctions

$$[y_{2n}(x)]^\alpha = [\alpha, 2-\alpha](-\cos(n\pi x)) = [\alpha, 2-\alpha](\cos(n\pi x))^-,$$

iii) If $\alpha = 1$, the eigenvalues are $\lambda_n = n^2\pi^2$, with associated eigenfuctions

$$[y_n(x)]^\alpha = \cos(n\pi x).$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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