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BERTRAND CURVES OF AW(K)-TYPE IN THREE DIMENSIONAL LIE GROUPS

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Abstract. In this paper, we consider curves of AW(k)-type $(1 \leq k \leq 3)$ in Three Dimensional Lie Groups. We give harmonic curvature conditions of AW(k)-type curves. Furthermore, we investigate that under what conditions AW(k)-type curves are helix. Besides, considering AW(k)-type curves, we investigate Bertrand curves and we show that there are Bertrand curves of AW(2), AW(3) and weak AW$(2)-$types.

Keywords: Lie Groups; Aw(k)-type curve; Bertrand curves; helix.

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1 Introduction

In the curve theory of Euclidean space, the most important subject is to obtain a characterization for a regular curves. These characterizations can be given for a single curve or for a curve pair. Helix, slant helix, plane curve, spherical curve, etc. especially the helices, are used in many applications $[2,3,19]$. Similarly, by considering two curves, some special curve pairs such as involute evolute curves, Bertrand curves, Mannheim curves have been defined and studied so

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far [10, 11, 14]. Accordingly, Bertrand mates represent particular examples of offset curves which are used in computer-aided design (CAD) and computer-aided manufacture (CAM). The distance between a Bertrand curve and its mate measured along the principal normal is known to be constant. We can see helical structures in nature and mechanic tools.

As a matter of fact, it is the simplest of the three-dimensional spirals. One of the most interesting spirals is referred to as the k-Fibonacci spirals which appears naturally from studying the k-Fibonacci numbers and the related hyperbolic k-Fibonacci function. Fibonacci numbers and the related Golden Mean or Golden Section appear very often in theoretical physics and physics of the high energy particles (see [7, 8, 9]). Besides, in the field of computer aided design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion or design of highways [18]. Also we can see the helix curve or helical structure in fractal geometry, for instance hyperhelices. In differential geometry; a curve of constant slope or general helix in Euclidean 3-space E^3 is defined by the property that its tangent vector field makes a constant angle with a fixed straight line (the axis of the general helix).

Çöken and Çiftçi have studied the degenarete semi-Riemannian geometry of Lie group [6]. They obtained a naturally reductive homogeneous semi-Riemannian space using the Lie group. Later, some of subjects given above have been considered in three dimensional Lie groups and some characterizations for these curves have been obtained in a three dimensional Lie group [15, 16]. Also, Çiftçi[5] defined general helices in three dimensional Lie groups with a bi-invariant metric and obtained a generalization of Lancret's theorem and gave a relation between the geodesics of the so-called cylinders and general helices.

Recently, many interesting results on curves of AW(k)-type have been obtained by many mathematicians (see [12, 13, 17]). For example, Özgür and Gezgin studied a Bertrand curve of AW(k)-type and they showed that there was no such Bertrand curve of AW(1)-type and α was of AW(3)-type if and only if it was a right circular helix. In addition they studied weak AW(2)-type and AW(3)-type conical geodesic curves in E^3. Külahci, Bektaş and Ergüt give curvature conditions of a AW(k)-type Frenet curve in Lorentzian space.

In this paper, we have done a study on Bertrand curves of AW(k)-type. However, to the best of author's knowledge, Bertrand curves of AW(k)-type have not been presented in Three Dimensional Lie Groups. Thus, the study is proposed to serve such a need.

2 Preliminaries

Let G be a Lie group with a bi-invariant metric $\langle , \rangle$ and D be the Levi-Civita connection of Lie group G. İf g denotes the Lie algebra of G then we know that g is isomorphic to T_eG where e is neutral element of G. If $\langle , \rangle$ is a bi-invariant metric on G then we have

$$\langle X,[Y,Z]\rangle = \langle [X,Y],Z\rangle \tag{2.1}$$

and

$$D_xY = \frac{1}{2}[X,Y] \tag{2.2}$$

for all X,Y and $Z \in g$.

Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lenghted curve and $\{X_1,X_2,...,X_n\}$ be an orthonormal basis of g. In this case, we write that any two vector fields W and Z along the curve α as $W = \sum_{i=1}^{n} w_iX_i$ and $Z = \sum_{i=1}^{n} z_iX_i$ where $w_i : I \to \mathbb{R}$ and $z_i : I \to \mathbb{R}$ are smooth functions. Also the Lie bracket of two vector fields W and Z is given

$$[W,Z] = \sum_{i=1}^{n} w_iz_i\left[X_i,X_j\right]$$

and the covariant derivative of W along the curve α with the notation $D_{\alpha^{\shortmid}}W$ is given as follows

$$D_{\alpha^{\shortmid}}W = \dot{W} + \frac{1}{2}[T,W] \tag{2.3}$$

where $T = \alpha^{\shortmid}$ and $\dot{W} = \sum_{i=1}^{n} \dot{w}_iX_i$ or $\dot{W} = \sum_{i=1}^{n} \frac{dw}{dt}X_i$. Note that if W is the left-invariant vector field to the curve α then $\dot{W} = 0$ (For see detail [4]).

Let G be a three dimensional Lie group and (T,N,B,κ,τ) denote the Frenet apparatus of the curve α, and calculate $\kappa = \left\|\dot{T}\right\|$.

Definition 1. *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with the Frenet apparatus (T,N,B,κ,τ) then*

$$\tau_G = \frac{1}{2}\left\langle [T,N], B\right\rangle \tag{2.4}$$

or

$$\tau_G = \frac{1}{2\kappa^2\tau}\left\langle \ddot{T}, \left[T,\dot{T}\right]\right\rangle + \frac{1}{4\kappa^2\tau}\left\|\left[T,\dot{T}\right]\right\|^2$$

(see [4]).

Definition 2. *Let $\alpha : I \subset \mathbb{R} \to G$ be an arc length parametrized curve with the Frenet apparatus (T,N,B,κ,τ). Then the harmonic curvature function of the curve α is defined by*

$$H = \frac{\tau - \tau_G}{\kappa}$$

where $\tau_G = \frac{1}{2}\left\langle [T,N], B\right\rangle$.

Theorem 3. *Let $\alpha : I \subset \mathbb{R} \to G$ be an arc length parametrized curve with the Frenet apparatus (T,N,B,κ,τ). If the curve α is a general helix if and only if*

$$H = const.$$

$(see$ [5]$)$.

Theorem 4. *Let $\alpha : I \subset \mathbb{R} \to G$ be an arc length parametrized curve with the Frenet apparatus (T,N,B,κ,τ). Then α is a slant helix if and only if*

$$\sigma = \frac{\kappa\left(1+H^2\right)^{\frac{3}{2}}}{H'} = \tan\theta$$

is a constant where H is a harmonic curvature function of the curve α and $\theta \neq \frac{\pi}{2}$ is a constant [16].

Proposition 5. *Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-length parametrized curve with the Frenet apparatus $\{T,N,B\}$. Then the following equalities*

$$[T,N] = \langle [T,N],B\rangle B = 2\tau_G B$$

$$[T,B] = \langle [T,B],N\rangle N = -2\tau_G N$$

hold [16].

Remark 6. *Let G be a Lie group with a bi-invariant metric* $\langle , \rangle$*.Then the following equalities can be given in different lie groups* [4].

$i)$ *If G is abelian group then* $\tau_G = 0$

$ii)$ *If G is* SO^3 *then* $\tau_G = \frac{1}{2}$

$iii)$ *If G is* SU^2 *then* $\tau_G = 1$

3 Aw(k)-type curves in Three Dimensional Lie Groups

In this section, harmonic curvature of curves of AW(k)-type are considered. We give some theorems and corollaries.

Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-length parametrized unit speed curve in three dimensional Lie groups. The curve α is called a Frenet curve of osculating order 3 if its derivatives $\alpha^{'}(s), \alpha^{''}(s), \alpha^{'''}(s), \alpha^{''''}(s)$ are linearly dependent and $\alpha^{'}(s), \alpha^{''}(s), \alpha^{'''}(s), \alpha^{''''}(s)$ are no longer linearly independent for all $s \in I$. To each Frenet curve of order 3 one can associate an orthonormal $3-$frame $\{T(s), N(s), B(s)\}$ along α such that $(\alpha^{'}(s) = T(s))$ called the Frenet frame and functions $\kappa, \tau : I \to \mathbb{R}$ called the Frenet curvatures, such that the Frenet formulas in three dimensional Lie groups are defined

$$D_T T(s) = \kappa(s)N(s) \tag{3.1}$$

$$D_T N(s) = -\kappa(s)T(s) + (\tau - \tau_G)(s)B(s)$$

$$D_T B(s) = (\tau_G - \tau)(s)N(s)$$

where D is the Levi-Civita connections of Lie group G and $\tau_G = \frac{1}{2}\langle [T,N],B\rangle$ [16].

Proposition 7. *Let $\alpha : I \subset \mathbb{R} \to G$ be a Frenet curve in three dimensional Lie groups, then we have*

$$\alpha^{'}(s) = T(s)$$

$$\alpha^{''}(s) = \kappa(s)N(s)$$

$$\alpha^{'''}(s) = -\kappa^2(s)T(s) + \kappa^{'}(s)N(s) + \kappa^2(s)H(s)B(s)$$

$$\alpha^{''''}(s) = (-3\kappa(s)\kappa^{'}(s))T(s) + (\kappa^{''}(s) - \kappa^3(s)(1-H^2(s)))N(s) + (2\kappa^{'}(s)\kappa(s)H(s) + (\kappa(s)H(s))^{'})B(s).$$

Proof. From Frenet formulas in three dimensional Lie groups (3.1) and by using $H = \frac{\tau-\tau_G}{\kappa}$, we have the results. □

Notation. Let us write

$$N_1(s) = \kappa(s)N(s) \tag{3.2}$$

$$N_2(s) = \kappa^{'}(s)N(s) + \kappa^2(s)H(s)B(s) \tag{3.3}$$

$$N_3(s) = (\kappa^{''}(s) - \kappa^3(s)(1-H^2(s)))N(s) + (3\kappa^{'}(s)\kappa(s)H(s) + \kappa^2(s)H^{'}(s))B(s) \tag{3.4}$$

Remark 8. *$\alpha^{'}(s), \alpha^{''}(s), \alpha^{'''}(s), \alpha^{''''}(s)$ are linearly dependent if and only if $N_1(s), N_2(s), N_3(s)$ are linearly dependent.*

As the definition of Aw(k) type curves in $[1]$, we have

Definition 9. *Frenet curves $($of osculating order$3)$ in three dimensional Lie groups are*

(i) of type weak Aw(2) if they satisfy

$$N_3(s) = \langle N_3(s), N_2^*(s)\rangle N_2^*(s), \tag{3.5}$$

(ii) of type weak Aw(3) if they satisfy

$$N_3(s) = \langle N_3(s), N_1^*(s)\rangle N_1^*(s) \tag{3.6}$$

where

$$N_1^*(s) = \frac{N_1(s)}{\|N_1(s)\|}, N_2^*(s) = \frac{N_2(s) - \langle N_2(s), N_1^*(s)\rangle N_1^*(s)}{\left\|N_2(s) - \langle N_2(s), N_1^*(s)\rangle N_1^*(s)\right\|}.$$

Proposition 10. *Let α be a Frenet curve$($of osculating order$3)$ in three dimensional Lie groups. If α is of type weak $Aw(2)$ then*

$$\kappa^{''}(s)-\kappa^{3}(s)(1-H^{2}(s))=0. \tag{3.7}$$

Corollary 11. *Let α be a Frenet curve of type weak $Aw(2)$. If α is plane curve then*

$$\kappa(s)=\pm\frac{\sqrt{2}}{s+c} \tag{3.8}$$

where c is constant.

Proof. Suppose that α is a Frenet curve of type weak Aw(2). Then the Eq. (3.7) hold on α. Since α is a plane curve, we have

$$H(s)=0. \tag{3.9}$$

Substituting (3.9) in (3.7), we get

$$\kappa^{''}(s)-\kappa^{3}(s)=0.$$

So the solution of the last equation gives us (3.8). Hence, the proof is completed. □

Proposition 12. *Let α be a Frenet curve $($of osculating order$3)$ in three dimensional Lie groups. If α is of type weak $Aw(3)$ then*

$$3\kappa^{'}(s)\kappa(s)H(s)+\kappa^{2}(s)H^{'}(s)=0. \tag{3.10}$$

Definition 13. *Frenet curves $($of osculating order$3)$ in three dimensional Lie groups are*

(i) *of type $Aw(1)$ if they satisfy $N_3(s)=0$,*

(ii) *of type $Aw(2)$ if they satisfy*

$$\|N_2(s)\|^2 N_3(s)=\langle N_3(s),N_2(s)\rangle N_2(s). \tag{3.11}$$

(iii) *of type $Aw(3)$ if they satisfy*

$$\|N_1(s)\|^2 N_3(s)=\langle N_3(s),N_1(s)\rangle N_1(s). \tag{3.12}$$

Theorem 14. *Let α be a Frenet curve $($of osculating order$3)$ in three dimensional Lie groups. Then α is of type $Aw(1)$ if and only if*

$$\kappa^{''}(s)-\kappa^{3}(s)(1-H^{2}(s))=0 \tag{3.13}$$

and

$$3\kappa^{'}(s)\kappa(s)H(s)+\kappa^{2}(s)H^{'}(s)=0 \tag{3.14}$$

Proof. Since α is a curve of type Aw(1), we have $N_3(s)=0$. Then from Eq. (3.4), we have

$$(\kappa^{''}(s)-\kappa^{3}(s)(1-H^{2}(s)))N(s)+(3\kappa^{'}(s)\kappa(s)H(s)+\kappa^{2}(s)H^{'}(s))B(s)=0.$$

Furthermore, since N and B are linearly independent, we get

$$\kappa^{''}(s)-\kappa^{3}(s)(1-H^{2}(s))=0 \text{ and } 3\kappa^{'}(s)\kappa(s)H(s)+\kappa^{2}(s)H^{'}(s)=0.$$

The converse statement is trivial. Hence our theorem is proved. □

Corollary 15. *Let α be a Frenet curve $($of osculating order$3)$ in three dimensional Lie groups. Then there is no $($circular or general$)$ helix of type $Aw(1)$.*

Proof. Assume that α be a helix. Then by the Theorem (3) $H(s)$ is constant. So, $H^{'}(s)=0$. Therefore the equations (3.13) and (3.14) can be written as follows:

$$\kappa^{''}(s)-\kappa^{3}(s)(1-H^{2}(s))=0$$

and

$$3\kappa^{'}(s)\kappa(s)H(s)=0.$$

Since the solution of above differential equations does not exist, there are not circular and general helix of type Aw(1). □

Theorem 16. *Let α be a Frenet curve $($of osculating order$3)$ in three dimensional Lie groups. Then α is of type $Aw(2)$ if and only if*

$$3(\kappa^{'}(s))^{2}\kappa(s)H(s)+\kappa^{'}(s)\kappa^{2}(s)H^{'}(s)-\kappa^{''}(s)\kappa^{2}(s)H(s)+\kappa^{5}(s)H(s)(1-H^{2}(s))=0. \tag{3.15}$$

Proof. Suppose that α is a Frenet curve of order 3, then from (3.3) and (3.4), we can write

$$N_2(s) = \gamma(s)N(s) + \beta(s)B(s), \tag{3.16}$$

$$N_3(s) = \eta(s)N(s) + \delta(s)B(s), \tag{3.17}$$

where γ, β, η and δ are differentiable functions. Since $N_2(s)$ and $N_3(s)$ are linearly dependent, coefficients determinant is equal to zero and hence one can write

$$\begin{vmatrix} \gamma(s) & \beta(s) \\ \eta(s) & \delta(s) \end{vmatrix} = 0. \tag{3.18}$$

Here,

$$\gamma(s) = \kappa'(s), \beta(s) = \kappa^2(s)H(s)$$

and

$$\eta(s) = \kappa''(s) - \kappa^3(s)(1 - H^2(s)),$$

$$\delta(s) = 3\kappa'(s)\kappa(s)H(s) + \kappa^2(s)H'(s).$$

Substituting these into (3.18), we obtain (3.15).

Conversely if the equation (3.15) holds it is easy to show that α is of type Aw(2). This completes the proof. □

Corollary 17. *If a Frenet curve of order 3 is a general helix of type Aw(2), then one can have*

$$3(\kappa'(s))^2 - \kappa''(s)\kappa(s) + \kappa^4(s)(1 - H^2(s)) = 0. \tag{3.19}$$

Theorem 18. *Let α be a general helix in three dimensional Lie groups. If α is of type Aw(2), then*

$$\kappa(s) = \frac{1}{\sqrt{-As^2 + Bs + C}} \text{ and } (\tau - \tau_G)(s) = \sqrt{1 - A}\,\kappa(s) \tag{3.20}$$

where $A = 1 - H^2(s)$, B and C are real constants.

Proof. Suppose that α is a general helix of type Aw(2). Then Eq.(3.19) holds. If we substitute $\kappa(s) = x$ in (3.19), we get

$$x\frac{d^2x}{ds^2} - 3\left(\frac{dx}{ds}\right)^2 = Ax^4, \; A = 1 - H^2(s). \tag{3.21}$$

Let us take $x = y^p$ and differentiating it twice we obtain

$$\frac{dx}{ds} = py^{p-1}\frac{dy}{ds}, \tag{3.22}$$

$$\frac{d^2x}{ds^2} = p(p-1)y^{p-2}\left(\frac{dy}{ds}\right)^2 + py^{p-1}\frac{d^2y}{ds^2}. \tag{3.23}$$

Now, the substitution of (3.22) and (3.23) into (3.21), we get

$$y^p\left[py^{p-1}\frac{d^2y}{ds^2} + p(p-1)y^{p-2}\left(\frac{dy}{ds}\right)^2\right] - 3p^2y^{2p-2}\left(\frac{dy}{ds}\right)^2 = Ay^{4p},$$

$$py^{2p-1}\frac{d^2y}{ds^2} + p(p-1)y^{2p-2}\left(\frac{dy}{ds}\right)^2 - 3p^2y^{2p-2}\left(\frac{dy}{ds}\right)^2 = Ay^{4p}.$$

Putting $p(p-1) = 3p^2$ (i.e. $p = -\frac{1}{2}$) into the last equation we get

$$py^{2p-1}\frac{d^2y}{ds^2} = Ay^{4p}.$$

So,

$$\frac{d^2y}{ds^2} = -2A.$$

Now, we solve this last equation. Since $\frac{dy}{ds} = -2As + B$, we get

$$y = -As^2 + Bs + C.$$

Furthermore, use of $x = y^{\frac{-1}{2}}$ we obtain

$$x = (-As^2 + Bs + C)^{\frac{1}{2}}.$$

Since $H(s) = \frac{(\tau - \tau_G)(s)}{\kappa(s)}$, we have the result. □

Theorem 19. *Let α be a Frenet curve(of osculating order 3) in three dimensional Lie groups. Then α is of type Aw(3) if and only if*

$$3\kappa'(s)\kappa(s)H(s) + \kappa^2(s)H'(s) = 0. \tag{3.24}$$

Proof. Suppose that α is a Frenet curve of order 3 which is of type Aw(3). If substituting (3.2) and (3.4) in (3.12), we get (3.24).

The converse statement is trivial. Hence our proposition is proved. □

Theorem 20. *Let be α a general helix of osculating order 3. Then α is of type $Aw(3)$ if and only if α is a circular helix.*

Proof. Suppose that α is a general helix, then by the Theorem (3) $H^{'}(s) = 0$. So, the equation (3.24) becomes $\kappa^{'}(s)\kappa(s)H(s) = 0$. Since $H(s)$ is none zero, $\kappa^{'}(s) = 0$. By the general helix $(\tau - \tau_G)(s)$ must be constant. So, α is a circular helix. The converse statement is trivial. Hence our theorem is proved. □

4 AW(k)-type Bertrand Curves in Three Dimensional Lie Groups G

This section characteries the curvatures of AW(k)-type Bertrand curves in G. We obtain some theorems and results about these curves in three dimensional Lie groups.

Definition 21. *A curve $\alpha : I \subset \mathbb{R} \to G$ with $\kappa(s) \neq 0$ is called a Bertrand curve if there exist a curve $\tilde{\alpha} : I \subset \mathbb{R} \to G$ such that the principal normal lines of α and $\tilde{\alpha}$ at $s \in I$ are equal. In this case $\tilde{\alpha}$ is called a Bertrand mate of α* [15].

Theorem 22. *Let $\alpha \subset G$ be a Bertrand curve. A Bertrand mate of α is as follows:*

$$\tilde{\alpha}(s) = \alpha(s) + \lambda N(s) \tag{4.1}$$

where λ is constant [15].

Corollary 23. *If $\tilde{\alpha}$ is a Bertrand mate of α, then*

$$(\tilde{\alpha}(s))^{'} = (1 - \lambda\kappa(s))T(s) + (\lambda\kappa(s)H(s))B(s). \tag{4.2}$$

Proof. Since $(\alpha, \tilde{\alpha})$ is a Bertrand mate, then the Eq.(4.1) hold on α. Differentiating (4.1) with respect to s , by using Frenet formulas in three dimensional Lie groups (3.1) and $H = \frac{\tau - \tau_G}{\kappa}$, then (4.2) is obtained. □

Theorem 24. *Let $\alpha : I \subset \mathbb{R} \to G$ be unit speed curve. If $\tilde{\alpha}$ is a Bertrand mate of α, then angle measurement of this curve between tangent vectors at corresponding points is constant.*

Proof. If $\left\langle \tilde{T}(s), T(s) \right\rangle^{'} = 0$, then the proof is complete.

$$\left\langle \tilde{T}(s), T(s) \right\rangle^{'} = \left\langle \left(\tilde{T}(s)\right)^{'}, T(s) \right\rangle + \left\langle \tilde{T}(s), (T(s))^{'} \right\rangle \tag{4.3}$$

$$= \left\langle \tilde{\kappa}(s)\tilde{N}(s), T(s) \right\rangle + \left\langle \tilde{T}(s), \kappa(s)N(s) \right\rangle \tag{4.4}$$

$$= \tilde{\kappa}(s) \left\langle \tilde{N}(s), T(s) \right\rangle + \kappa(s) \left\langle \tilde{T}(s), N(s) \right\rangle \tag{4.5}$$

Since $\tilde{N}(s)$ is parallel to $N(s)$ and $N(s) \perp T(s)$, then

$$\left\langle \tilde{N}(s), T(s) \right\rangle = 0. \tag{4.6}$$

Since $\tilde{N}(s)$ is parallel to $N(s)$ and $\tilde{T}(s) \perp \tilde{N}(s)$, then

$$\left\langle \tilde{T}(s), N(s) \right\rangle = 0. \tag{4.7}$$

Substituting (4.6) and (4.7) in (4.5), we have

$$\left\langle \tilde{T}(s), T(s) \right\rangle^{'} = 0.$$

Hence, the proof is completed. □

Proposition 25. *Let α be a Frenet curve (of osculating order 3) in three dimensional Lie groups. For $\kappa(s) \neq 0$, α is a Bertrand curve if and only if there exists a linear relation*

$$\lambda \kappa(s) + \mu \kappa(s) H(s) = 1. \tag{4.8}$$

where λ, μ are non-zero constants and H is the harmonic curvature function of the curve α[13].

Corollary 26. *Suppose that $\kappa(s) \neq 0$ and $(\tau - \tau_G)(s) \neq 0$. Then α is a Bertrand curve if and only if there exist a nonzero real number λ such that*

$$\lambda \left(\kappa^{'}(s)\kappa(s)H(s) - \kappa(s)\left(\kappa(s)H(s)\right)^{'}\right) - \left(\kappa(s)H(s)\right)^{'} = 0. \tag{4.9}$$

Proof. By the proposition(25), α is a Bertrand curve if and only if there exist real numbers $\lambda \neq 0$ and μ such that $\lambda \kappa(s) + \mu \kappa(s) H(s) = 1$. This is equivalent to the condition that there exists a real number $\lambda \neq 0$ such that $\frac{1 - \lambda \kappa(s)}{\kappa H(s)}$ is constant. Differentiating both sides of the last equality, we get (4.9). The converse assertion is also true. □

Proposition 27. *Let $\alpha : I \subset \mathbb{R} \to G$ be a Bertrand curve with $\kappa(s) \neq 0$ and $(\tau - \tau_G)(s) \neq 0$. Then α is of $AW(2)$-type if and only if there is a non zero real number λ such that*

$$3(\kappa^{'}(s))^2 H(s) + \kappa^2(s)\frac{\lambda\kappa^{'}(s)H(s)}{\lambda\kappa(s)-1} - \kappa^2(s)H(s)(3\kappa^{'}(s)H(s)+\kappa(s)H^{'}(s)) = 0. \tag{4.10}$$

Proof. Since α is of Aw(2)-type, Eq.(3.15) holds and since α is a Bertrand curve, Eq.(4.9) holds. If both of these equations are considered, (4.10) is obtained. □

Theorem 28. *Let $\alpha : I \subset \mathbb{R} \to G$ be a Bertrand curve with $\kappa(s) \neq 0$ and $(\tau - \tau_G)(s) \neq 0$. If α is of type $Aw(3)$, then α is a circular helix.*

Proof. Suppose that $\alpha : I \subset \mathbb{R} \to G$ is a Bertrand curve of AW(3)-type with $\kappa(s) \neq 0$ and $(\tau - \tau_G)(s) \neq 0$. Then the Eqs.(3.24) and (4.9) hold on α, we get

$$H^{'}(s)(2\lambda\kappa^3(s) - \kappa^2(s)) = 0. \tag{4.11}$$

Since $\kappa(s) \neq 0$, from Eq.(4.11) $H^{'}(s) = 0$. Thus, $H(s)$ is constant, then α is a circular helix. Hence our theorem is proved. □

Proposition 29. *Let $\alpha : I \subset \mathbb{R} \to G$ be a Bertrand curve with $\kappa(s) \neq 0$ and $(\tau - \tau_G)(s) \neq 0$. If α is of weak $AW(2)$-type, then*

$$H^{'}(s)(\lambda\kappa^2(s) - \kappa(s)) + H^{'}(s)(2\lambda\kappa(s)\kappa^{'}(s) - 2\kappa^{'}(s)) - \kappa^3(s)H(s)(1 - H^2(s)) = 0. \tag{4.12}$$

Proof. Since α is of weak Aw(2)-type, From Eq.(3.7) we have

$$\kappa^{''}(s) - \kappa^3(s)\left(1 - H^2(s)\right) = 0. \tag{4.13}$$

Since α is a Bertrand curve, Eq.(4.9) holds

$$H^{'}(s)\left(\lambda\kappa^2(s) - \kappa(s)\right) = \kappa^{'}(s)H(s). \tag{4.14}$$

Differentiating above equation(4.14), we get

$$\kappa^{''}(s) = \frac{H^{''}(s)(\lambda\kappa^2(s) - \kappa(s)) + H^{'}(s)(2\lambda\kappa(s)\kappa^{'}(s) - 2\kappa^{'}(s))}{H(s)} \tag{4.15}$$

If equation (4.13) is substituted in (4.15), then (4.12) is obtained. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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