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REGULAR NONLINEAR DYNAMICS OF A PIECEWISE MAP

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Abstract. A discrete time dynamical systems represented by the iteration of nonlinear piecewise functions is introduced in this paper. The dynamical behaviors, multiple basins with fractal boundary, attractors, route to chaos via bifurcations are investigated. Moreover, we point out an fascinating form of complex basin structure in the presence of multistability and coexistence of several attractors.

Keywords: fractal basin, contact bifuraction, piecewise smooth systems.

2000 AMS Subject Classification: 34C23; 37C28; 34C35; 70H33

1. Introduction

Application of nonlinear dynamics and bifurcation theory in practice often leads to the analysis of piecewise systems, which has attracted significant research attention because such systems arise very naturally in many physical systems. In the studies of chaos theory, it is usually assumed that the dynamical system arises from a differentiable process. Systems that are not differentiable on a surface in phase space occur in variety of applications in several areas of engineering and applied sciences, e.g. grazing impacting systems in mechanical oscillators, piecewise linear electronic circuits, cardiac dynamics, and so on [2, 9, 12]. Piecewise smooth maps constitute a class of maps that are smooth everywhere except along borders separating regions of smooth behavior and for which critical curves can be defined. Critical curves play a fundamental role for determining the dynamical

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properties of attractors, and define bounded domains, called areas, containing attractors of the map [1, 3, 7,9]. We are interested by the bifurcations phenomena, particularly we emphasize the role played by the contact bifurcations of these connected bounded domains which are subsets of attraction basins of attractors. Till now, we know that a chaotic area ceases to exist when a critical curve belonging to the area's boundary has a contact with the basin boundary [10]. Despite some resemblances, the identification of different aspects of bifurcations is far from complete for piecewise-smooth maps, and certainly very preliminary in comparison to the results obtained in the smooth case. In the following the study will be essentially centred on the contact bifurcation in nonlinear piecewise maps, and to determine chaotic areas, to characterize their properties via critical singularities, and to discover the bifurcations involving these sets.

In such systems, as a parameter is varied through a critical value, atypical transitions may exhibit like period orbit with period one bifurcating a chaotic orbit or a period orbit vanishing as it collides the border, which are commonly used to analyse these kinds of maps. These bifurcation phenomena lead to a specific class of bifurcations, called as border collision bifurcations. This new concept of bifurcation was coined by Nusse and collaborators [11]. Other important works corroborated by Feigin [5,6], who was first to define these nouvel notions in the Russian literature, and given under the so-called C-bifurcation. In addition to these, this class can also exhibit some new bifurcation phenomena which are unique to piecewise smooth systems called discontinuous bifurcations by Di Bernardo *et al.* [4]. These atypical degenerate bifurcations will not be studied in this paper.

To be more specific we consider a mathematical model given by a planar nonlinear piecewise map expressed by

$$T \ :(x,y) \to \begin{cases} T_1(x,y) & ,(x,y) \in R_1 \\ T_2(x,y) & ,(x,y) \in R_2 \end{cases} \tag{1}$$

$$T_1 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ ay - x^2 + c \end{pmatrix}$$

$$R_1 = \{(x, y) : x > 0\}$$

$$T_2 : \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ y - bx + x^2 \end{pmatrix}$$

$$R_2 = \{(x, y) : x \leq 0\}$$

Where a, b and c are real parameters.

We analyze bifurcations of different attractors and modifications of attraction basins, i.e. basin bifurcations which can generate complex dynamics with respect to parameter change. The non-invertible property of the system displays a complicated basin structure. The situation can be more complex, when the boundary between two basins becomes fractal. Our results are based on numerical evidence. Aiming at a better understanding of the dynamics of the map (1), this present work is motivated in the identification of chaotic areas.

2. Dynamical properties

The analysis of the bifurcations occurring in the piecewise smooth map T given above, depending on three parameters a, b, and c,can be performed in few steps. First, we study the conditions that guarantee the existence of fixed points of this nonlinear piecewise map. Second, we introduce the critical singularities [3, 7, 9]. By bifurcation we denote change in the system behavior; for example, switching from one stable to another different stable state, from stable to unstable state, change in the geometrical shape of a basin, and so on. By means of bifurcation diagrams, with varying parameters, we determine the behavior of existing solutions and how they evolve.

In our investigations of map (1), a meaningful characterization of T consists in the identification of its singularities, and our analysis focuses on showing bifurcations sets, attractors, and so on.

First, representative bifurcation diagrams of this system are shown in Figs. $1(a, b, c, d)$, which provide information on stability region for the fixed point (blue domain), and the existence region for attracting cycles of order k. Each different color defines a region of admissibility of a k-periodic cycle ($k \leq 14$), delimited by a bifurcation curve. The black regions ($k = 15$) corresponds to the existence of bounded iterated sequences including chaotic behavior. The white area corresponds to divergent iterated sequences in the phase space.

Second, Using these graphical illustrations we notice that we have a succession of domains of attracting k-cycles of increasing period k from 3 to 12 in Fig.1d or from 6 to10 in Fig.1c which may interpret the existence of border collision bifurcations, but more explicit investigations remain to be done, and more mathematical details and informations must be given.We have visualized a phenomenon of period-doubling bifurcations in Figs.1b and 1c.

2.1. **Existence of fixed points.** In this section, we start the study of Eq. (1) with a catalogue of its fixed points, including existence conditions of fixed points, and critical sets. The analysis is standard and may be pursued for more k-cycles.

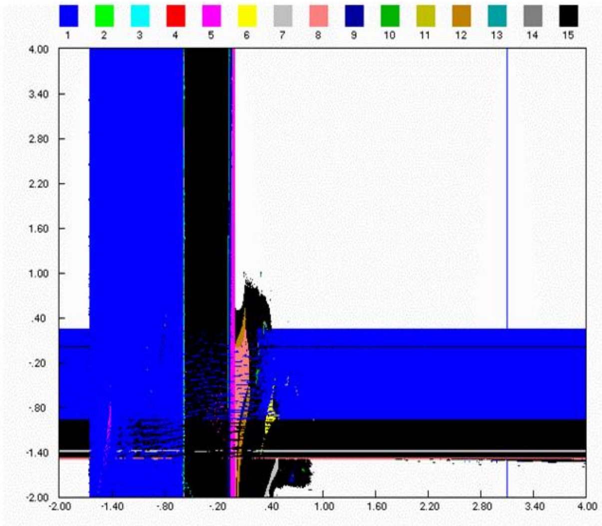
Proposition 1.

1) For $(1-a)^2 + 4c \geq 0$ and $b < 0$. T admits at most three fixed points:
$P_1 = (\frac{(a-1)-\sqrt{(1-a)^2+4c}}{2}, \frac{(a-1)-\sqrt{(1-a)^2+4c}}{2})$, $Q_1 = (0, 0)$ and $Q_2 = (b, b)$ if $a > 1$, and $c > 0$.
$P_2 = (\frac{(a-1)+\sqrt{(1-a)^2+4c}}{2}, \frac{(a-1)+\sqrt{(1-a)^2+4c}}{2})$, $Q_1 = (0, 0)$ and $Q_2 = (b, b)$ if $a < 1$, and $c > 0$.

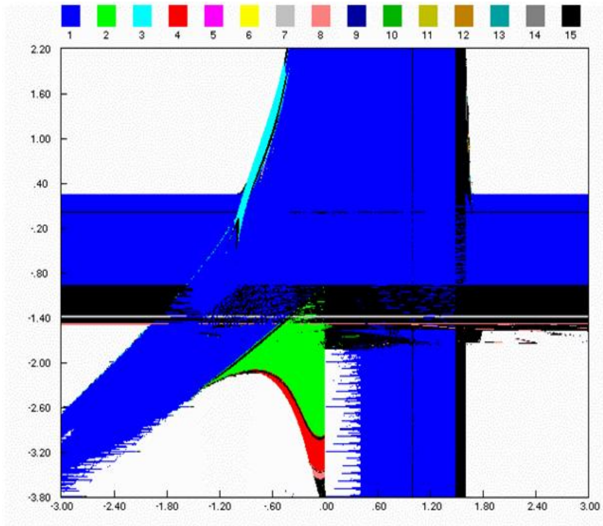
2) For $(1-a)^2 + 4c \geq 0$ if we keep $b \geq 0$, T has two fixed points:
$P_1 = (\frac{(a-1)-\sqrt{(1-a)^2+4c}}{2}, \frac{(a-1)-\sqrt{(1-a)^2+4c}}{2})$, and $Q_1 = (0, 0)$ if $a > 1$, and $c > 0$.
$P_2 = (\frac{(a-1)+\sqrt{(1-a)^2+4c}}{2}, \frac{(a-1)+\sqrt{(1-a)^2+4c}}{2})$, and $Q_1 = (0, 0)$ if $a < 1$, and $c > 0$.

3) For $(1-a)^2 + 4c < 0$, T has two fixed points $Q_1 = (0, 0)$, and $Q_2 = (b, b)$, if $b < 0$. And a unique fixed point $Q_1 = (0, 0)$ if $b \geq 0$.

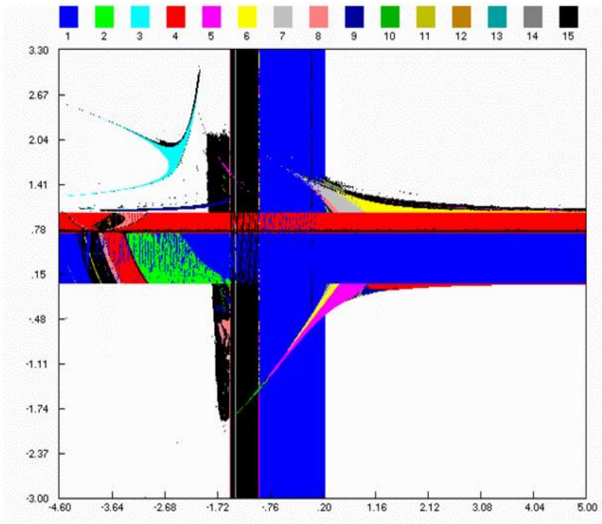
Proof:



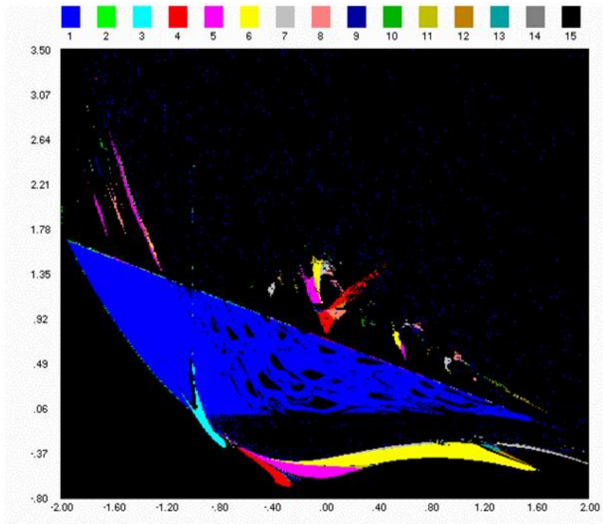
(A) Bifurcation diagram in (a,b) plane c=1



(B) Bifurcation diagram in (a,b) plane c=0



(C) Bifurcation diagram in (b,c) plane a=0



(D) Bifurcation diagram in (a,b) plane b=0

FIGURE 1.

1) In the region R_1 $(x > 0)$, fixed points of T are those of T_1, These fixed points are determined by

$$\begin{cases} x^* = y^* \\ y^* = ay^* - (x^*)^2 + c \end{cases}$$

$$\Rightarrow \begin{cases} x^* = y^* \\ (x^*)^2 + (1-a)x^* - c = 0 \end{cases}$$

Fixed points are located in the line $x^* = y^*$. The fact that $(x^*)^2 + (1-a)x^* - c = 0$ conveys several cases:

• For $(1-a)^2 + 4c \geq 0$, T has two fixed points P_1 and P_2,
The fixed point $P_1 = (\frac{(a-1)-\sqrt{(1-a)^2+4c}}{2}, \frac{(a-1)-\sqrt{(1-a)^2+4c}}{2})$ exists for $\frac{(a-1)-\sqrt{(1-a)^2+4c}}{2} > 0$
For the fixed point P_1 to exist, we must have $a > 1$, and $c < 0$
Similarly $P_2 = (\frac{(a-1)+\sqrt{(1-a)^2+4c}}{2}, \frac{(a-1)+\sqrt{(1-a)^2+4c}}{2})$ exists for $\frac{(a-1)+\sqrt{(1-a)^2+4c}}{2} > 0$
Then the fixed point P_2 exists for the two following cases: $i)$ It is a simple exercise to see that P_2 exists for $a < 1$, and $c > 0$, $ii)$ For $a > 1$, P_2 exists $\forall c \in IR$
• For $(1-a)^2 + 4c < 0$, T has no fixed points.

2) In the region R_2 $(x \leq 0)$, fixed points are those of T_2, and verify

$$\begin{cases} x^* = y^* \\ y^* = y^* - bx^* + (x^*)^2 \end{cases}$$

$$\Rightarrow \begin{cases} x^* = y^* \\ x^*(-b + x^*) = 0 \end{cases}$$

We assume that T has two fixed points $Q_1 = (0,0)$, and $Q_2 = (b,b)$ for $b < 0$, and only one fixed point $Q_1 = (0,0)$, for $b \geq 0$.

2.2. **Critical sets.** The critical curve LC is considered as the geometrical locus of points x having at least two coincident inverses. The locus of these coincident first rank preimages is a curve denoted by LC_{-1}.The critical line $LC = T(LC_{-1})$ is constituted of one or several branches, and $LC_{-1} \sqsubseteq T^{-1}(LC)$.

We determine the curve LC_{-1}, when T is differentiable, by taking the Jacobian of T equal to zero $(J = \det(DF(x,y)) = 0)$. If F is non differentiable, then LC_{-1} is the set of discontinuity.

Till now, arcs of critical curves are used to determine the boundary of invariant bounded areas, and to characterize their bifurcations due to contacts between the boundary of an absorbant or chaotic area (see for more detail [10]) and the boundary of its basin of attraction.

Proposition 2 : For $b \leq 0$, LC_{-1} is constituted by two branches LC^1_{-1} and LC^2_{-1},($LC_{-1} = LC^1_{-1} \cup LC^2_{-1}$) , where LC^1_{-1} is the line of equation $x = 0$, and the respective equation of the line LC^2_{-1} is $x = \dfrac{b}{2}$, which designate the lines of discontinuity.The critical curve $LC = T(LC_{-1})$ is constituted of three branches L, L', and L'', $(LC = L \cup L' \cup L'')$ with $L: y = ax + c$, $L' : y = x - \frac{b^2}{4}$, and L'': $y = x$. For $b > 0$, LC_{-1} is the line $x = 0$. LC is

made up only of two branches L and L'', $(LC = L \cup L'')$.

Proof:

1) For $x \succeq 0$, Jacobian matrix of T_1 is equal to:

$$DT_1(x, y) = \begin{bmatrix} 0 & 1 \\ -2x & a \end{bmatrix}$$

The determinant $\det(DT(x, y)) = 2x$ vanishes on the straight line whose respective equation is $x = 0$ (the line of non differentiability).

LC^1_{-1} is the line $x = 0$

The calculation of inverse determinations of T gives us:

$$T_1^{-1}(x', y') = \begin{cases} x = \pm\sqrt{ax' - y' + c} \\ y = x' \end{cases}$$

As $x > 0$, one inverse determination appears then

$$T_1^{-1}(x', y') = \begin{cases} x = +\sqrt{ax' - y' + c} \\ y = x' \end{cases}$$

This determination emanates from the condition $ax' - y' + c \geq 0$. This leads to consider that the first critical curve is $L : y' = ax' + c$.

The critical line L separates the plane in two regions. The region Z_1 verifies $y' \leq ax' + c$ and is constituted of the set of points having one antecedent of rank one by the inverse determination T_1^{-1}, and a region Z_0 such that $y' \geq ax' + c$ in which points have no antecedents.

2) For $x < 0$, The Jacobian matrix of T_2 is equal to

$$DT_1(x, y) = \begin{bmatrix} 0 & 1 \\ -b + 2x & a \end{bmatrix}$$

The Jacobian $\det(DT(x, y)) = b - 2x$ vanishes on $x = \frac{b}{2}$, which is mentioned by LC^2_{-1} curve. This curve exists only for $b < 0$.

The search of the inverse determinations of T for the case $x < 0$ yields to the following result

$$\begin{cases} x^2 - bx + (x' - y') = 0 \\ y = x' \end{cases}$$

By solving $x^2 - bx + (x' - y') = 0$ with respect to x. We have two solutions : $x_1 = \dfrac{b - \sqrt{b^2 - 4(x' - y')}}{2}$ and $x_2 = \dfrac{b + \sqrt{b^2 - 4(x' - y')}}{2}$

Consequently, in the region $y' \geqslant x' - \frac{b^2}{4}$, we obtain two inverse determinations of T :

$$T_{2,1}^{-1}(x', y') = \begin{cases} x = \dfrac{b + \sqrt{b^2 - 4(x' - y')}}{2} \\ y = x' \end{cases}$$

$$T_{2,2}^{-1}(x', y') = \begin{cases} x = \dfrac{b - \sqrt{b^2 - 4(x' - y')}}{2} \\ y = x' \end{cases}$$

These inverse determinations merge on the line $y' = x' - \frac{b^2}{4}$, which is the image of LC^2_{-1} by T. Namely, the line $y' = x' - \frac{b^2}{4}$ is a critical curve denoted by L'.

For $x < 0$, LC^2_{-1} exists only for $b < 0$. We have $x_1 \leq \frac{b}{2} = LC^2_{-1} \leq x_2$, from this, we get: $y' < x'$.

As $x' - \frac{b^2}{4} \leq y' < x'$, we obtain another critical curve given by : $y' = x'$ and denoted by L''.

For these cases $y' > x'$ and $y' < x' - \frac{b^2}{4}$ no inverse exist.

Branches of critical curves may change for any value of x according to the number of inverses of T , and to the parameters values of a, b, and c. For $b < 0$, $c = 0.5$, $a = 0.4$, we have the configuration illustrated by Fig. 2.

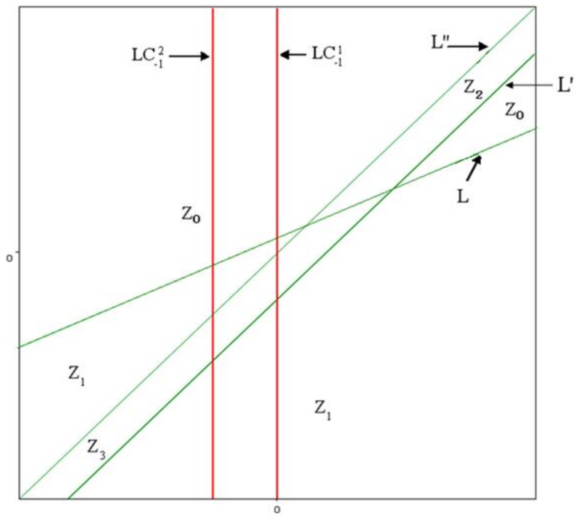


FIGURE 2. *Critical curves of T $(b < 0)$.*

3. Studies of the phase plane

As a parameter is varied, the attractor may move in phase space and collides with the boundary of its basin. When this happens, the Jacobian matrix can change abruptly, leading to a special class of nonlinear dynamic phenomena known as bifurcations. The study of the different attractors and their basins reveals extremely interesting phenomena associated with the piecewise nonlinear structure which exhibits different types of bifurcations found in a class of two-dimensional quadratic smooth maps.

We choose parameter values $b = -1.5$, $c = 0.5$, and by variation of value of a, we plot the different attractors and their basins. The map (1) has a fixed point, and a 8-cycle. Each initial condition plotted in pink corresponds to trajectories converging towards the fixed point $P_1(0.36, 0.37)$, and each initial condition plotted in green corresponds to trajectories converging towards the period 8-cycle. The basin of cycle of period 8 is spread in a countable number of nonconnected components all over the basin of the fixed point which is multiply connected (connected with holes). After variation of the parameter a the structure of basin of 8-cycle has changed. An attracting closed invariant curve softly arises from the fixed point as the parameter a crosses the Neimark-Sacker bifurcation point at $a = 0.5$. These numerical basin results strongly suggest that basins of attraction are intermingled. Further evidence of intermingling is provided by examining the dynamics of map (1). Each basin gives rise to holes inside the other one. The closed curve disappears, and we have apparition of a chaotic attractor which coexists with a cycle of period 8.

For b and c fixed ($b = -1.5$ and $c = 0.5$), and varying a, the map (1) permits to make appear such a situation:

1) For $a = 0.4$, T has three fixed points:

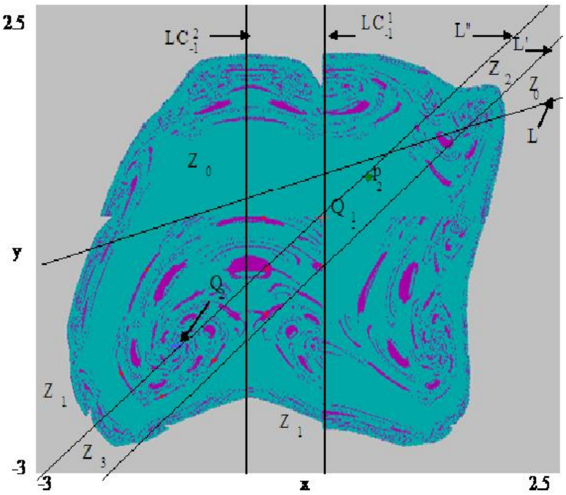
$P_2 = (\frac{(a-1)+\sqrt{(1-a)^2+4c}}{2}, \frac{(a-1)+\sqrt{(1-a)^2+4c}}{2})$, $Q_1 = (0,0)$, and $Q_2 = (b,b)$, all belong to the line (bisectrix) $y = x$.

Two of them have an important role in the basin structure D, the first point $P_2 = (0.46, 0.46)$ which is a stable focus, and the second point $Q_2 = (-1.5, -1.5)$ unstable focus. This latter plays a primordial role in the heteroclinic bifurcation.

Figure 5(a) shows the coexistence of two attractors, a cycle stable of order 8 (denoted by A_8 and the 8 points constituting by A_8^i, $i = 1, ..., 8$). Each point A_8^i is a fixed point of T^8 and possesses its own basin by T^8. The basin $D(A_8)$ of 8-cycle is nonconnected. The second attractor is the fixed point $P_2 = (0.46, 0.46)$, its basin is $D(P_2)$ is multiply connected. Many holes constitute islands non connected in the basin $D(A_8)$.

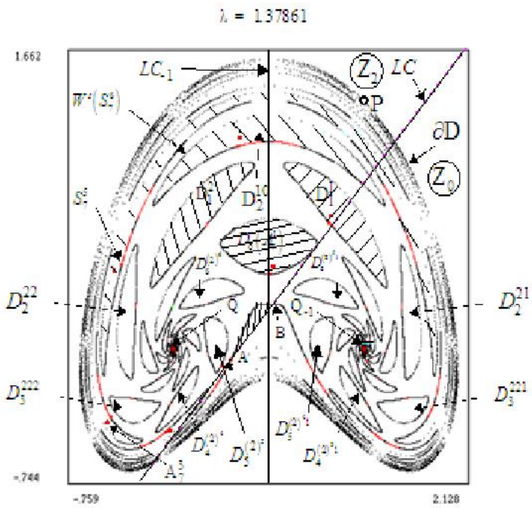
The global basin D is simply connected, it results from the union of the two basins associated with A_8 and P_2.

This map has properties equivalent to those of the map (2) cited in Millérioux and Mira [8,9] with some differences due to the discontinuity of the map (1), and exhibits a special pattern. From zooming in toward an area in the basin, Fig. 3a illustrates the intricate structure of a part of the basin D which looks similar to the basin structure given in [8] , and this structure is repeated in several zones of the basin. The map (2) considered in [8] is continuous, and defined by



(A) Coexistence of two attractors.

(B) Enlargement of a portion of D.



(C) The total basin of two attractors.

FIGURE 3.

$$(x, y) \to \begin{cases} x = y \\ y = y - \lambda x + x^2 \end{cases} \tag{2}$$

The map (2) has an interesting property, that of basin symmetries with respect to LC_{-1}(here LC^2_{-1} in Fig.3b). For $\lambda = 1.37861$, This map has two attractors: a stable

7-cycle, and a closed curve Γ. Millérioux has shown that the map (2) has its proper total basin illustrated in in Fig3c.

For $a = 0.5$, a Hopf bifurcation gives rise to a closed curve Γ, the fixed point $P_2 = (0.5, 0.5)$ of focus type is destabilized. When $a = 0.74$, the curve Γ has a contact with LC^1_{-1} in a point (Fig. 4). the image of this point by T gives two contact points between Γ and (L and L''). In the continuous case [9], the image of the point of contact of Γ with LC_{-1} gives only one contact point with LC, but in this piecewise case we have two contact with LC which are the images of the point of contact with LC_{-1} by the two determinations of T. This bifurcation creates a shifting in the critical curves.

For $a = 0.85$, the closed curve disappears and a chaotic attractor appears, of Lyapunov dimension equal to 1.23 (Fig.5). Fix a and c ($a = 0.2$ and $c = 1$).If we vary b taking

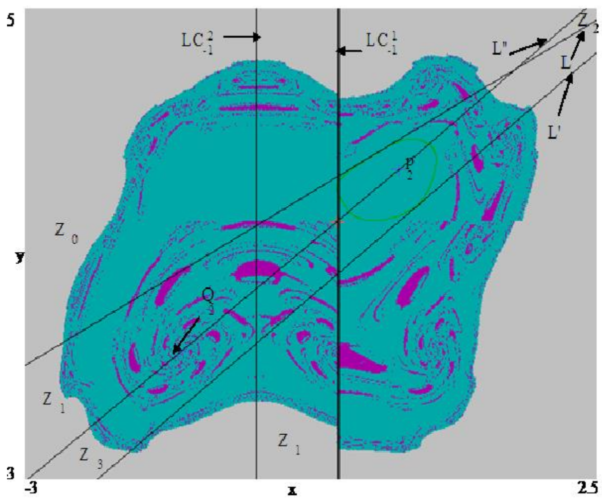


FIGURE 4. *The curve* Γ *has a contact with* LC^1_{-1}.

positive and negative values, the map (1) undergoes a series of changes in the basin structure.

a) For $b > 0$, the map$T$ has two fixed points :$Q_1 = (0, 0)$ and

$P_2 = (\frac{(a-1)+\sqrt{(1-a)^2+4c}}{2}, \frac{(a-1)+\sqrt{(1-a)^2+4c}}{2})$ two unstable focus. T is of type $(Z_0 - Z_1 - Z_2 - Z_1)$.

1) For $b = 0.3$ (Fig. 6a), we have a chaotic area with a simply connected basin of attraction D , when $b = 0.7$ (Fig. 6b), a contact bifurcation between the chaotic area and

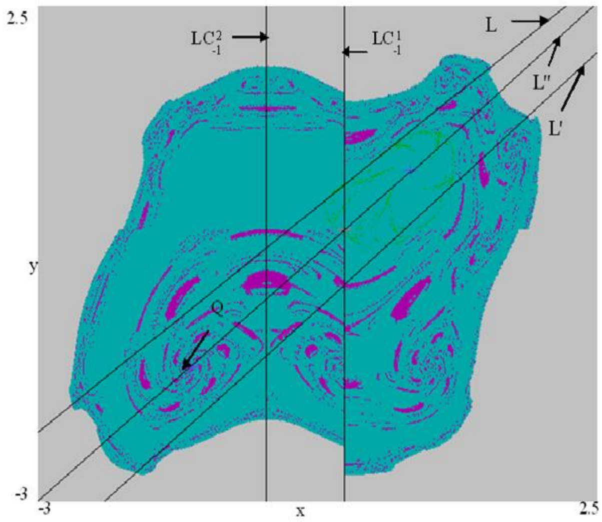
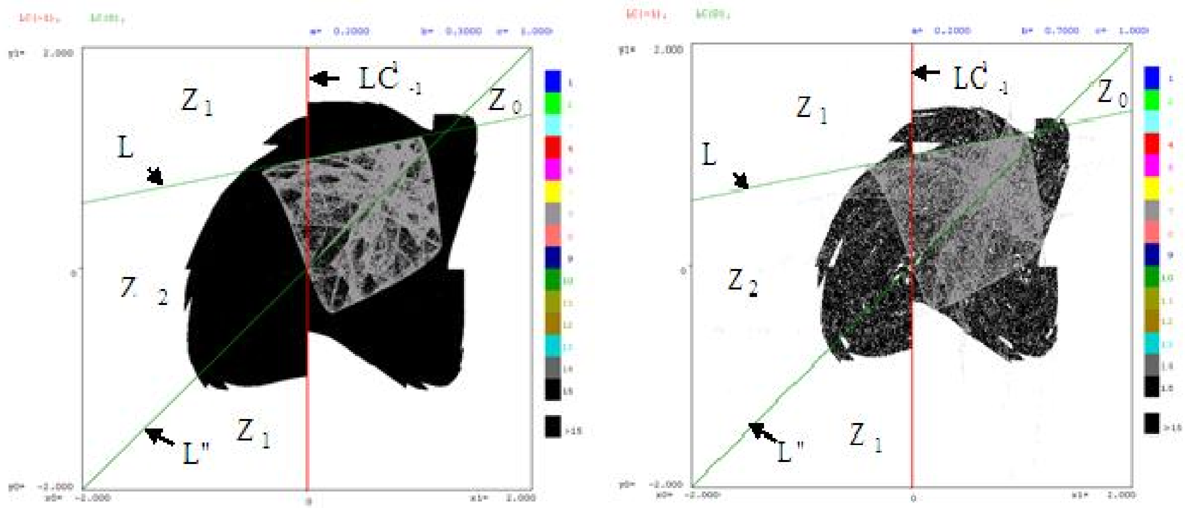


FIGURE 5. *Apparition of a chaotic attractor.*

the boundary of the basin occurs leading to the disparition of both of them for $b = 0.8$ 2)



(A) Connected Basin D of a chaotic area.

(B) Contact Bifurcation and disparition of basin.

FIGURE 6.

<u>For $b < 0$</u>, T has three fixed points: $Q_1 = (0, 0)$, $Q_2 = (b, b)$,

and $P_2 = (\dfrac{(a-1)+\sqrt{(1-a)^2+4c}}{2}, \dfrac{(a-1)+\sqrt{(1-a)^2+4c}}{2})$.

P_2 and Q_1 are two unstable focus , but the point Q_2 changes of stability according to the value of b. T is of type $(Z_0 - Z_1 - Z_3 - Z_1 - Z_0 - Z_2)$.

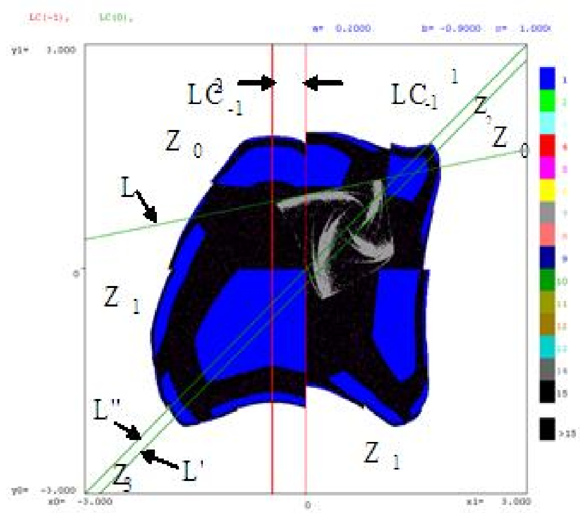
Varying b from -0.9 to -2.5, we obtain the following situations :

For $b = -0.9$ (Fig.7a), the fixed point $Q_2 = (-0.9, -0.9)$ is a stable focus, its basin is nonconnected (in blue color), it coexists with a chaotic area having a multiply connected basin (black color). The total basin is simply connected.

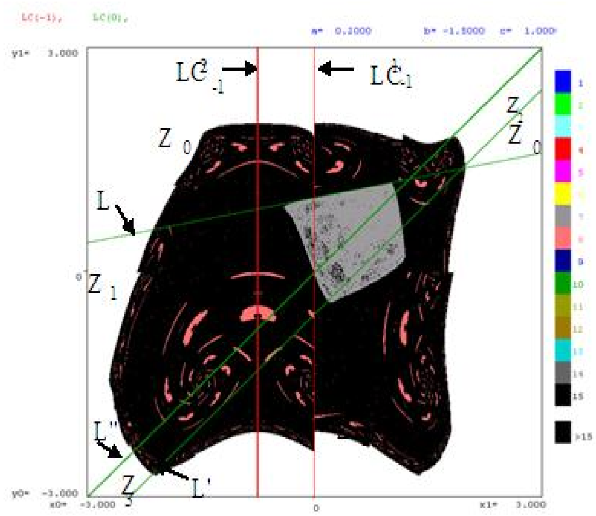
When $b = -1.5$ (Fig.7b), the chaotic area still exists with a 8-cycle. For $b = -1.9$ (Fig.7c), the cycle 8 disappears when one of its points has a contact with L'. The basin of the chaotic area becomes multiply connected via a bifurcation of type "simply connected basin $\leftrightarrow$multiply connected basin " the critical curve L' has a contact with the boundary of the basin . A bay H_0 is created, limited by segments of L' and those of the boundary of the basin.

The successive preimages of H_0 'lakes' (in means of Mira [9]) are created inside ($Z_0-Z_1-Z_2-Z_3$) which generate the fractalization of the frontier of the basin.

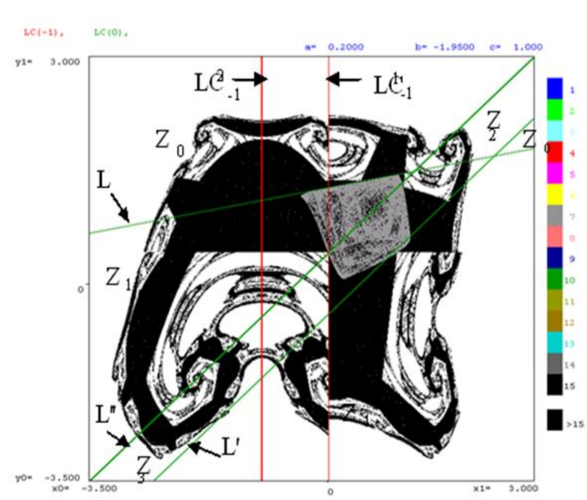
Varying b, lakes (holes) increase and change drastically the aspect of the basin which becomes connected with a particular form, and its boundary has a fractal structure (see Figs.7c and 7d) Similar bifurcation has been shown in the case $b > 0$, the basin undergoes successive bifurcations creating more holes. These complex phenomena arise in very small regions of parameter variations.



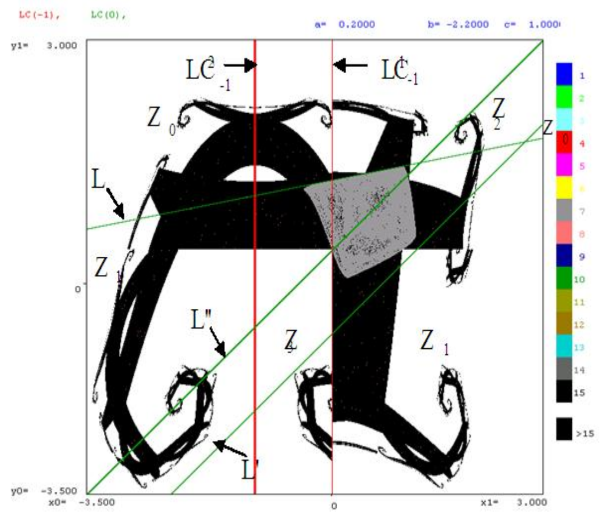
(A) Simply total basin.



(B) Coexistence of a chaotic area and 8-cycle.



(C) Disappearance of 8-cycle basin.



(D) Fractal basin of the chaotic area.

FIGURE 7.

4. CONCLUSION

In this paper we investigate bifurcations associated with two-dimensional piecewise nonlinear map in the parameter and the phase spaces. We focus our attention on the consequences of a contact bifurcation occurring between the boundary of a chaotic area and the frontier of its basin of attraction.

References

[1] Barugola, A., Cathala, J.C., Mira, C.:Extensions of the notion of chaotic area in second order endomorphisms. Int. J. of Bifurc.Chaos 5(3), 751-777 (1995).

[2] Behnia, S., Akhshani, A., Ahadpour, S., Mahmodi, H., akhavan, A.: A fast chaotic encryption scheme based on piecewise nonlinear chaotic maps, Phys. lett. 366, 391-396 (2007)

[3] Cathala, J.C.:On some Properties of Absorptive Areas in 2nd Order Endomorphisms. ECIT Batschuns, World Scientific (1989).

[4] Di Bernardo, M., Feigin, M. I., Hogan, S. J., Homer, M.E.: Local analysis of C-bifurcations in n-dimensional piecewise smooth dynamical systems,. Chaos Solitons Fractals **10**, 1881–1908 (1999).

[5] Feigin, M. I.: Doubling of the oscillation period with C-bifurcations in piecewise continuous systems. Prikladnaya Matematika i Mechanika 34, 861–869(1970).

[6] Feigin, M. I.:The increasingly complex structure of the bifurcation tree of a piecewise-smooth system. Journal of Applied Mathematics and Mechanics 59(6), 853–863(1995).

[7] Gumowski, I., Mira, C.: Bifurcation déstabilisant une solution chaotique d'un endomorphisme du second ordre. C.R.Acad. Sci. A 285, 427-431 (1978).

[8] Millerioux, G.: Modèles par Transformations Non Inversible et Application à la Sécurité des Communications. Synchronisation du Chaos. Thèse de doctorat de INSA Toulouse (1997).

[9] Mira, C., Gardini, L., Barugola, A., Cathala, J.C.: Chaotic dynamics in two dimensional noninvertible maps. World Scientific Series on NonlinearSciences, Series A(20) .

[10] Mira, C.: About Two-Dimensional Piecewise Continuous Noninversible Maps. Int J. of Bifurc. Chaos 6(5), 893-918(1996).

[11] Nusse, H. E., Yorke, J. A.: Border-collision bifurcations including "period two to period three for piecewise smooth maps. Physica D, vol. 57, 39–57(1992).

[12] Zhusubaliyev, Z.T., Mosekilde, E.: Bifurcations and chaos in piecewise-smooth dynamical systems. SinpagoreWorld scientific (2003).