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STABILITY OF A MIXED TYPE CUBIC AND QUARTIC FUNCTIONAL EQUATION IN FUZZY BANACH SPACES

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Abstract. In this paper, we generalized Ulam-Hyers stability of the mixed type cubic and quartic functional equation in fuzzy Banach space.

Keywords: fuzzy normed spaces; stability of quartic and cubic mapping; Banach space.

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1. Introduction

In 1940, Ulam [13] posed the first stability problem concerning group homomorphisms. In the next year, Hyers [7] gave an affirmative answer to the question of Ulam in Banach spaces. Aoki [14] generalized Hyers result for additive mappings. For additive mapping involving different powers of norms [18,20]. This stability is also investigates by Park [6]. In 1984, Katsaras [1] constructed a fuzzy vector topological structure on the linear space. Later, some mathematicians considered some other type fuzzy norms and some properties of fuzzy normed linear spaces [5,15]. Recently, several various fuzzy versions stability problem concerning quadratic,

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cubic and quartic function equation have been considered [2,3].

Hyers [8] was the first person to point out the direct method for studying the stability of functional equation. In 2003, Radu [19] proposed the fixed point alternative method to solve the Ulam problem. Subsequently, Mihet [9] applied the fixed point alternative method to solve fuzzy stability of Jensen functional equation in fuzzy normed space.

Jun and Kim [12] introduced cubic function equation and they established the solution of Hyers-Ulam-Rassias stability for the functional equation (1.1), and this equation is called the cubic function equation, if the cubic function $f(x) = cx^3$ satisfies (1.1). The quartic functional equation was introduced by Rassias [19] in 2000

$$f(2x+y)+f(2x-y) = 4f(x+y)+4f(x-y)+24f(x)-6f(y) \tag{1.2}$$

It is easy to show that the function $f(x) = cx^4$ satisfies the functional equation (1.2). In this paper, we establish a fuzzy version stability for following functional equation:

$$f(x+2y)+f(x-2y) = 4(f(x+y)+4f(x-y))-24f(y)-6f(x)+3f(2y) \tag{1.3}$$

in fuzzy Banach space and the function $f(x) = ax^3+bx^4$ is a solution of the functional equation (1.3). We using the fixed point alternative method to establish fuzzy stability.

2. Preliminaries

We start our works with basic definition using in this paper.

Definition 2.1. [16] *Let X be a real linear space. A fuzzy subset N of $X\times\mathbb{R}$ is called a* fuzzy norm *on X if and only if*

(N1) *For all $t\in\mathbb{R}$ with $t\le 0$, $N(x,t)=0$;*

(N2) *For all $t\in\mathbb{R}$ with $t>0$, $N(x,t)=1$ if and only if $x=0$;*

(N3) *For all $\lambda\in\mathbb{R}$ with $\lambda\neq 0$, $N(\lambda x,t)=N(x,t/|\lambda|)$;*

(N4) *For all $s,t\in\mathbb{R}$, $N(x+y,s+t)\ge \min\{(N(x,s),N(y,t)\}$;*

(N5) *$N(x,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$;*

(N6) *For $x\neq 0, N(x,\cdot)$ is (upper semi) continuous on $\mathbb{R}$.*

Then (X,N) is called a fuzzy normed linear space.

Example 2.2.[4] *Let* $(X, \|\cdot\|)$ *be a normed space. For every* $x \in X$*, we define*

$$N(x,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & when\ t > 0, \\ 0, & when\ t \leq 0. \end{cases}$$

Then (X,N) *is a fuzzy normed linear space.*

A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$. If every Cauchy sequence is convergent, then the fuzzy normed space space is called a fuzzy Banach space.

3.fuzzy stability of Cubic and Quartic Function Equation Using Direct Method

In this section, for given $f : X \to Y$, we define operator $Df : X \times X \to Y$ by

$$Df(x,y) = f(x+2y) + f(x-2y) - 4[f(x+y) + f(x-y)] - 3f(2y) + 24f(y) + 6f(x)$$

Theorem 3.1.(The fixed point alternative theorem,[17]) *Let* (Ω, d) *be a complete generalized metric space and* $T : \Omega \to \Omega$ *be a strictly contractive mapping with Lipschitz constant L, that is*

$$d(Tx,Ty) \leq Ld(x,y),\ \forall x,y \in \Omega.$$

Then for each given $x \in \Omega$*,either*

$$d(T^n x, T^{n+1} y) = \infty,\ \forall n \geq 0,$$

or there exists a natural number n_0 *such that*

(1)$d(T^n x, T^{n+1} y) < \infty,\ \forall n \geq 0,$

(2) *The sequence* T^n *is convergent to a fixed point* y^* *of* T.

(3) y^* *is the unique fixed point of* T *in the set* $\triangle = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$.

(4) $d(y,y^*) \leq \frac{1}{1-L} d(y.Ty)$ *for all* $y \in \triangle$.

Theorem 3.2. *Let X be a linear space,* (Y,N) *and* (Z,N') *be a fuzzy Banach space and a fuzzy normed linear space respectively. Suppose that* α *is a constant satisfies* $0 < |\alpha| < 16$*,* φ *is a* *mapping from* $X \times X \to Z$ *such that*

$$N^{'}(\varphi(2x,2y),t) \geq N^{'}(\alpha\varphi(x,y),t)$$

for all $x \in X, t > 0$*, and*

$$\lim_{k\to\infty} N^{'}(\varphi(2^k x, 2^k y), 16^k t) = 1$$

for all $x, y \in X, t > 0, k \geq 0$*. If* $f : X \to Y$ *be an even function and* $f(0) = 0$*. In the sense that*

$$N(Df(x,y),t) \geq N^{'}(\varphi(x,y),t) \tag{3.1}$$

for all $x, y \in X, t > 0$*. Then there exists a unique quartic mapping* $C : X \to Y$ *such that*

$$N(C(x) - f(x),t) \geq N^{'}(\varphi(0,x),(16-\alpha)t)$$

for all $x \in X, t > 0$*. Moreover,*

$$C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{16^n}$$

for all $x \in X$*.*

Proof. We assume that $0 < \alpha < 16$. Let

$$\Omega = \{g : g : X \to Y, g(0) = 0\}$$

and introduce the generalized metric d on Ω by

$$d(g,h) = inf\{\beta \in (0,\infty) : N(g(x) - h(x), \beta t) \geq N^{'}(\varphi(0,x), 16t)\}$$

We know that (Ω, d) is complete generalized metric on Ω. We now defined a mapping $T : \Omega \to \Omega$ by

$$Tg(x) = \frac{1}{16} g(2x)$$

We now prove T is a strictly contractive mapping with the Lipschitz constant $\frac{\alpha}{16}$. Given $g, h \in \Omega$, set $\varepsilon \in (0,\infty)$ be an arbitrary constant with $d(g.h) < \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq N^{'}(\varphi(0,x), 16t), \ \forall x \in X, t > 0.$$

Therefore

$$\begin{aligned} N(Tg-Th,\frac{\alpha\varepsilon t}{16}) &= N(\frac{1}{16}g(2x)-\frac{1}{16}h(2x),\frac{\alpha\varepsilon t}{16}) \\ &= N(g(2x)-h(2x),\alpha\varepsilon t) \\ &\geq N^{'}(\varphi(0,2x),16\alpha t) \\ &\geq N^{'}(\alpha\varphi(0,x),16\alpha t) \\ &= N^{'}(\varphi(0,x),16t) \end{aligned}$$

Hence ,we can conclude that

$$d(Tg,Th)\leq\frac{\alpha\varepsilon}{16}$$

Hence

$$d(g,h)<\varepsilon\Rightarrow d(Tg,Th)\leq\frac{\alpha\varepsilon}{16},\ g,h\in\Omega.$$

That is

$$d(Tg,Th)\leq\frac{\alpha}{16}d(g,h)$$

Put $x=0$ in (3.1), then replace y by x, we obtain

$$N(\frac{f(2x)}{16}-f(x),t)\geq N^{'}(\varphi(0,x),16t)$$

for all $x\in X,t>0$, it follows that $d(Tf,f)\leq 1$. From the fixed point alternative theorem, we can conclude that, there exists a fixed point C of T in Ω such that

$$C(2x)=16C(x),\forall x\in X.$$

Moreover, we have $\lim_{n\to\infty}d(T^nf,C)\to 0$, which implies

$$N(\lim_{n\to\infty}\frac{f(2^nx)}{16^n}-C(x),t)=0.$$

By the fixed point alternative, we conclude that

$$d(f,C)\leq\frac{1}{1-L}d(Tf,f)$$

Then

$$d(f,C)\leq\frac{16}{16-\alpha}$$

This means that

$$N(C(x)-f(x),t)\geq N^{'}(\varphi(0,x),(16-\alpha)t)$$

for all $x \in X, t > 0$. The uniqueness of C follows from the fact that C is the unique fixed point of T with the property that there exists $k \in (0,\infty)$ such that

$$N(C(x)-f(x),kt) \geq N^{'}(\varphi(0,x),t),\ \forall x \in X, t > 0.$$

This completes the proof of this theorem.

Corollary 3.3. *Let $(X,\|\cdot\|)$ be a normed space, (Y,N) be a fuzzy Banach space and $(Z,N^{'})$ be a fuzzy normed space, u,v,γ,s be non-negative real numbers satisfies $u+v,\gamma,s<4$.If $f: X \to Y$ be a mapping such that for some $u_0 \in Z$*

$$N(Df(x,y),t) \geq N^{'}((\|x\|^u\|y\|^v+\|x\|^\gamma+\|y\|^s)u_0,t)$$

for all $x,y \in X, t > 0$. Then there exists a unique quartic mapping $C: X \to Y$ such that

$$N(f(x)-C(x),t) \geq N^{'}(\|x\|^s u_0,(16-\alpha)t)$$

.

Proof. We define $\varphi: X \times X \to Z$ by

$$\varphi(x,y) = (\|x\|^u\|y\|^v+\|x\|^\gamma+\|y\|^s)u_0.$$

for all $x,y \in X$. It follows the conditions of Theorem 3.2, then completes the proof.

Theorem 3.4. *Let X be a linear space, (Y,N) and $(Z,N^{'})$ be a fuzzy Banach space and a fuzzy normed linear space respectively. Suppose that α is a constant satisfies $0<|\alpha|<8$, φ is a mapping from $X \times X \to Z$ such that*

$$N^{'}(\varphi(2x,2y),t) \geq N^{'}(\alpha\varphi(x,y),t)$$

for all $x \in X, t > 0$, and

$$\lim_{k\to\infty} N^{'}(\varphi(2^k x,2^k y),8^k t) = 1$$

for all $x,y \in X, t>0, k \geq 0$. If $f: X \to Y$ be an odd function and $f(0)=0$. In the sense that

$$N(Df(x,y),t) \geq N^{'}(\varphi(x,y),t) \tag{3.2}$$

for all $x,y \in X, t > 0$. Then there exists a unique cubic mapping $C: X \to Y$ such that

$$N(C(x)-f(x),t) \geq N^{'}(\varphi(0,x),3(8-\alpha)t)$$

for all $x \in X, t > 0$. *Moreover,*

$$C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{8^n}$$

for all $x \in X$.

Proof. Similar to the proof of Theorem 3.2. We can assume that $0 < \alpha < 8$. Let

$$\Omega = \{g : g : X \to Y, g(0) = 0\}$$

and introduce the generalized metric d on Ω by

$$d(g,h) = inf\{\beta \in (0,\infty) : N(g(x) - h(x), \beta t) \geq N^{'}(\varphi(0,x), 24t)\}$$

We know that (Ω, d) is complete generalized metric on Ω. We now defined a mapping $T : \Omega \to \Omega$ by

$$Tg(x) = \frac{1}{8} g(2x)$$

We now prove T is a strictly contractive mapping with the Lipschitz constant $\frac{\alpha}{8}$. Given $g, h \in \Omega$, set $\varepsilon \in (0,\infty)$ be an arbitrary constant with $d(g.h) < \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq N^{'}(\varphi(0,x), 24t),\ \forall x \in X, t > 0$$

Therefore

$$\begin{aligned} N(Tg - Th, \frac{\alpha\varepsilon t}{8}) &= N(\frac{1}{8} g(2x) - \frac{1}{8} h(2x), \frac{\alpha\varepsilon t}{8}) \\ &= N(g(2x) - h(2x), \alpha\varepsilon t) \\ &\geq N^{'}(\varphi(0,2x), 24\alpha t) \\ &\geq N^{'}(\alpha\varphi(0,x), 24\alpha t) \\ &= N^{'}(\varphi(0,x), 24t) \end{aligned}$$

Hence ,we can conclude that

$$d(Tg, Th) \leq \frac{\alpha\varepsilon}{8}$$

Hence

$$d(g,h) < \varepsilon \Rightarrow d(Tg, Th) \leq \frac{\alpha\varepsilon}{8},\ g,h \in \Omega.$$

That is

$$d(Tg, Th) \leq \frac{\alpha}{8} d(g,h)$$

Put $x = 0$ in (3.2), then replace y by x, we obtain

$$N(\frac{f(2x)}{8} - f(x), t) \geq N'(\varphi(0, x), 24t)$$

for all $x \in X, t > 0$, it follows that $d(Tf, f) \leq 1$. From the fixed point alternative theorem, we can conclude that, there exists a fixed point C of T in Ω such that

$$C(2x) = 8C(x), \forall x \in X$$

Moreover, we have $\lim_{n\to\infty} d(T^n f, C) \to 0$, which implies

$$N(\lim_{n\to\infty} \frac{f(2^n x)}{8^n} - C(x), t) = 0.$$

By the fixed point alternative, we can conclude that

$$d(f, C) \leq \frac{1}{1-L} d(Tf, f)$$

Then

$$d(f, C) \leq \frac{8}{8-\alpha}$$

This means that

$$N(C(x) - f(x), t) \geq N'(\varphi(0, x), 3(8-\alpha)t)$$

for all $x \in X, t > 0$. The uniqueness of C follows from the fact that C is the unique fixed point of T. This completes the proof of this theorem.

Corollary 3.5. *Let $(X, \|\cdot\|)$ be a normed space, (Y, N) be a fuzzy Banach space and (Z, N') be a fuzzy normed space,u, v, γ, s be non-negative real numbers satisfies $u + v, \gamma, s < 3$. If $f : X \to Y$ be a mapping such that for some $u_0 \in Z$*

$$N(Df(x, y), t) \geq N'((\|x\|^u \|y\|^v + \|x\|^\gamma + \|y\|^s) u_0, t)$$

for all $x, y \in X, t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$N(f(x) - C(x), t) \geq N'(\|x\|^s u_0, 3(8-\alpha)t)$$

Proof. Similar with the proof of Corollary 3.3.

Theorem 3.6. *Let X be a linear space, (Y,N) and $(Z,N^{'})$ be a fuzzy Banach space and a fuzzy normed linear space respectively. Suppose that α is a constant satisfies $0<|\alpha|<8$, φ is a mapping from $X\times X\to Z$ such that*

$$N^{'}(\varphi(2x,2y),t)\geq N^{'}(\alpha\varphi(x,y),t)$$

for all $x\in X, t>0$, and

$$\lim_{k\to\infty}N^{'}(\varphi(2^kx,2^ky),8^kt)=1$$

for all $x,y\in X,t>0,k\geq 0$. If $f:X\to Y$ be a function such that $f(0)=0$. In the sense that

$$N(Df(x,y),t)\geq N^{'}(\varphi(x,y),t)$$

for all $x,y\in X,t>0$. Then there exists a unique cubic mapping $C:X\to Y$ and a unique quartic mapping $Q:X\to Y$ such that

$$N(f(x)-C(x)-Q(x),t)\geq\begin{cases}N^{'}(\varphi(0,x),\dfrac{(16-\alpha)}{2}t),\ 0<\alpha\leq 4,\\ N^{'}(\varphi(0,x),\dfrac{3(8-\alpha)}{2}t),\ 4<\alpha<8.\end{cases}$$

for all $x\in X,t>0$. Moreover,

$$C(x)=\lim_{n\to\infty}\frac{f(2^nx)}{8^n},Q(x)=\lim_{n\to\infty}\frac{f(2^nx)}{16^n}$$

for all $x\in X$.

Proof. We assume that $0<\alpha<8$. Let $f_0(x)=\frac{1}{2}(f(x)-f(-x))$ for all $x\in X$. Then $f_0(0)=0, f_0(-x)=-f_0(x)$ and

$$N(D(f_0(x,y),t)\geq\min\{N^{'}(\varphi(x,y),t),N^{'}(\varphi(-x,-y),t)\}$$

Let $f_1(x)=\frac{1}{2}(f(x)+f(-x))$ for all $x\in X$. Then $f_1(0)=0, f_1(-x)=f_1(x)$ and

$$N(D(f_1(x,y),t)\geq\min\{N^{'}(\varphi(x,y),t),N^{'}(\varphi(-x,-y),t)\}$$

Using the proof Theorem 3.2 and 3.4, we get unique cubic mapping C and unique quartic mapping Q satisfying

$$N(f_0(x)-C(x))\geq N^{'}(\varphi(0,x),3(8-\alpha)t),N(f_1(x)-Q(x))\geq N^{'}(\varphi(0,x),(16-\alpha)t).$$

Therefore,

$$N(f(x)-C(x)-Q(x),t) \geq \min\{N(f_0(x)-C(x),\frac{t}{2}),N(f_1(x)-Q(x),\frac{t}{2})\}$$
$$\geq \min\{N^{'}(\varphi(0,x),\frac{3(8-\alpha)}{2}t),N^{'}(\varphi(0,x),\frac{(16-\alpha)}{2}t)\}.$$

This means that

$$N(f(x)-C(x)-Q(x),t) \geq \begin{cases} N^{'}(\varphi(0,x),\frac{(16-\alpha)}{2}t), & 0<\alpha\leq 4, \\ N^{'}(\varphi(0,x),\frac{3(8-\alpha)}{2}t), & 4<\alpha<8. \end{cases}$$

This completes the proof of this theorem.

Conflict of Interests

no conflict of interest.

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