Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 1, 15-22
ISSN: 1927-5307

BRIGHT AND DARK SOLITON SOLUTIONS TO THE OSTROVSKY-BENJAMIN-BONA-MAHONY (OS-BBM) EQUATION

MARWAN ALQURAN*

Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan

Abstract. In this paper, we apply two solitary wave ansatzes in terms of sech ^{p} and $\tanh^p$ to obtain bright and dark soliton solutions of the Ostrovsky-Benjamin-Bona-Mahony (OS-BBM) which describes the motion of ocean currents.

Keywords: Solitary wave ansatze, Bright soliton, Dark soliton.

2000 AMS Subject Classification: 74J35; 35C08

1. Introduction

The Ostrovsky equation [6] is a model of ocean currents motion,

$$(u_t + (u^2)_x - \beta u_{xxx})_x = \gamma u, \quad x \in \Re, \tag{1}$$

where, β, γ are constants. Parameter β determines the type of dispersion, namely, $\beta = -1$ (negative-dispersion) for surface and internal waves in the ocean and surface waves in a shallow channel with an uneven bottom; $\beta = 1$ (positive dispersion) for capillary waves on the surface of liquid or for oblique magneto-acoustic waves. Parameter $\gamma > 0$ measures the effect of rotation. When $\gamma = 0$, integrating once with respect to x and letting the integral constant be zero, the Ostrovsky equation becomes the well-known KdV equation.

*Corresponding author

E-mail addresses: marwan04@just.edu.jo

Received December 5, 2011

Existence and non-existence of localized solitary waves of the Ostrovsky equation is classified according to the sign of the dispersion parameter (which can be either positive or negative). Yue Liu and Vladimir Varlamov [7] proved that for the case of positive dispersion the set of solitary waves is stable with respect to perturbations. The issue of passing to the limit as the rotation parameter tends to zero for solutions of the Cauchy problem is investigated on a bounded time interval. V. Varlamov and Yue Liu [8] studied initial-value problems that arises in modeling the unidirectional propagation of long waves in a rotating homogeneous incompressible fluid. Local- and global-in-time solvability is investigated. For the case of positive dispersion a fundamental solution of the Cauchy problem for the linear equation is constructed, and its asymptotic is calculated as $t \longrightarrow \infty$ and $\frac{x}{t} = constant$. For the nonlinear problem solutions are constructed in the form of a series and the analogous long$-$time asymptotic is obtained. The Benjamin-Bona-Mahony (BBM) equation [9, 16]

$$u_t + u_x - a(u^2)_x - bu_{xxt} = 0, \tag{2}$$

is a well-known equation in physics. The equation has dispersion effect and exists solitary wave behavior.

The experimental realization of bright and dark solitons for nonlinear PDEs lead to interest in their formation and dynamics. Having analytical solutions to nonlinear PDEs will thus be of great importance as it will help in understanding the dynamical behavior of solitons. The mathematical physics literature has appeared a large amount of new powerful methods to calculate exact solutions to nonlinear wave equations, among these methods, we cite [10, 12, 13, 14, 15, 16].

In this work, the following OS-BBM equation will be studied

$$(u_t + u_x - \alpha(u^2)_x - \beta u_{xxt})_x = \gamma(u + u^2). \tag{3}$$

In what follows we highlight the main features of the solitary wave ansatzes that will be used in this work where more details and examples can be found in [1, 2].

2. Solitary Wave Ansatzes; bright and dark solitons

In this section, we will highlight briefly the main steps of the methods that will be used in this paper. We first unite the independent variables x and t into one wave variable $\zeta = x - \lambda t$ to convert the PDE

$$P(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, ...) = 0, \tag{4}$$

into an ODE

$$Q(u, -\lambda u', u', u'', \lambda u'', \lambda^2 u'', u''', ...) = 0. \tag{5}$$

Eq. (5) is then integrated as long as all terms contain derivatives.

Bright solitons 2.1.

In order to obtain the bright soliton solution of (4), the solitary wave ansatze is assumed as [1, 3, 4, 11]

$$u(x,t) = \frac{A}{\cosh^q(\mu\zeta)}, \qquad q > 0, \tag{6}$$

where $\zeta = x - \lambda t$. Here A is the soliton amplitude, μ is the inverse width of the soliton and λ is the soliton velocity. The unknown index q as well as A, μ and λ are to be determined during the course of derivation of the solution of (5). Based on this ansatze, we have

$$u' = \frac{du(\zeta)}{d\zeta} = -Aq\mu\cosh^{-1-q}(\mu\zeta)\sinh(\mu\zeta) \tag{7}$$

$$u'' = \frac{d^2u(\zeta)}{d\zeta^2} = A\left(-q\mu^2\cosh^{-q}(\mu\zeta) + q(1+q)\mu^2\cosh^{-2-q}(\mu\zeta)\sinh^2(\mu\zeta)\right) \tag{8}$$

Substituting (6)−(8) into the reduced ODE (5) gives a polynomial equation of cosh terms after using the identity $\sinh^2 = \cosh^2 - 1$. To determine the parameters we first balancing the exponents of each pair of cosh to determine q. Then we collect the coefficients of the same power in cosh and setting them to zeros, to get a system of algebraic equations among the unknowns A, λ and μ. The problem is now completely reduced to an algebraic one. Having determined A, λ and μ by algebraic calculations or by using Mathematica, the solutions proposed in (6) follows immediately.

Dark solitons 2.2.

In order to obtain the bright soliton solution of (4), the solitary wave ansatze is assumed as [1, 11]

$$u(x,t) = A\tanh^q(\mu\zeta), \qquad q > 0, \tag{9}$$

where $\zeta = x - \lambda t$. Here A and μ are free parameters and λ is the soliton velocity. The unknown index q as well as A, μ and λ are to be determined during the course of derivation of the solution of (5). Based on this ansatze, we have

$$u' = Aq\mu\tanh^{-1+q}(\mu\zeta)\,\mathrm{sech}^2(\mu\zeta) \tag{10}$$

$$u'' = A\left(q(-1+q)\mu^2\mathrm{sech}^4(\mu\zeta)\tanh^{-2+q}(\mu\zeta) - 2q\mu^2\mathrm{sech}^2(\mu\zeta)\tanh^q(\mu\zeta)\right) \tag{11}$$

Substituting (9)−(11) into the reduced ODE (5) gives a polynomial equation of tanh terms after using the identity $\mathrm{sech}^2 = 1 - \tanh^2$. To determine the parameters we first balancing the exponents of each pair of tanh to determine q and then we proceed as the above analysis stated in the bright soliton case.

3. Ostrovsky-Benjamin-Bona-Mahony (OS-BBM)

In this section we consider the OS−BBM

$$(u_t + u_x - \alpha(u^2)_x - \beta u_{xxt})_x = \gamma(u + u^2). \tag{12}$$

Using the wave variable $\zeta = x - \lambda t$ transforms (12) into the ODE

$$(1-\lambda)u'' - 2\alpha(uu'' + (u')^2) + \beta u'''' = \gamma(u + u^2). \tag{13}$$

3.1. Bright soliton solution

Substituting (6)−(8) into (13) gives

$$\begin{aligned} 0 &= 2Aq(1+2q)\alpha\mu^2\cosh^2(z\mu) - A(\gamma + 4q^2\alpha\mu^2)\cosh^4(z\mu) + q(6+11q+6q^2+q^3)\beta\lambda\mu^4\cosh^q(z\mu) \\ &- q(1+q)\mu^2(1+\lambda(-1+2(2+2q+q^2)\beta\mu^2))\cosh^{2+q}(z\mu) \\ &+ \left(-\gamma + q^2\mu^2(1+\lambda(-1+q^2\beta\mu^2))\right)\cosh^{4+q}(z\mu) \end{aligned} \tag{14}$$

By equating the exponents and the coefficients of each pair of the cosh function we obtain the following algebraic system:

$$\begin{aligned} 2 &= q \\ 0 &= 2Aq(1+2q)\alpha\mu^2 + q(6+11q+6q^2+q^3)\beta\lambda\mu^4 \\ 0 &= -A(\gamma+4q^2\alpha\mu^2) - q(1+q)\mu^2(1+\lambda(-1+2(2+2q+q^2)\beta\mu^2)) \\ 0 &= -\gamma + q^2\mu^2(1+\lambda(-1+q^2\beta\mu^2)). \end{aligned} \tag{15}$$

Solving the above system yields

$$\begin{aligned} A &= \frac{-3}{2} \\ \lambda &= \frac{\alpha+\alpha^2}{\alpha+\beta\gamma} \\ \mu &= \mp\frac{\sqrt{\alpha+\beta\gamma}}{2\sqrt{\beta+\alpha\beta}}. \end{aligned} \tag{16}$$

Based on the obtained results, a bright soliton solution of (12) exists if $\alpha + \beta\gamma > 0$ and $\beta + \alpha\beta > 0$, and thus the solution is

$$u(x,t) = \frac{-3}{2}\operatorname{sech}^2\left(\frac{\sqrt{\alpha+\beta\gamma}}{2\sqrt{\beta+\alpha\beta}}(x - \frac{\alpha+\alpha^2}{\alpha+\beta\gamma}t)\right). \tag{17}$$

Figure $1, 2$ shows the plots of the above obtained solution when $\beta = 1, -1$ respectively.

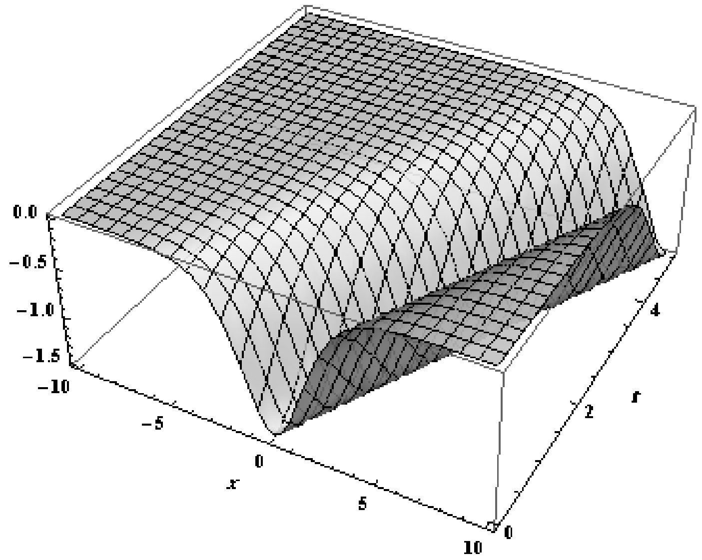

FIGURE 1. The obtained bright soliton solution of OS-BBM: $\beta = 1$; $\alpha = 2$; $\gamma = 1$; $-10 \le x \le 10$; $0 \le t \le 5$

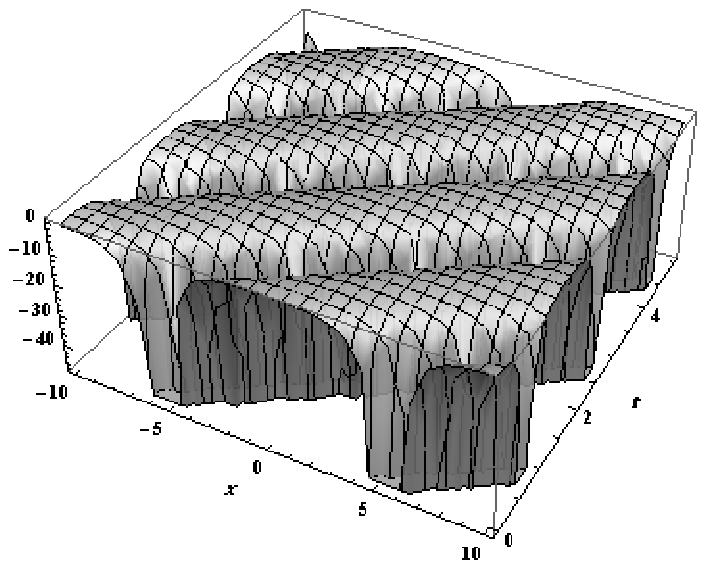

FIGURE 2. The obtained bright soliton solution of OS-BBM: $\beta = -1$; $\alpha = 2$; $\gamma = 1$; $-10 \le x \le 10$; $0 \le t \le 5$

3.2. Dark soliton solution

Substituting (9)−(11) into (13) gives

$$
\begin{aligned}
2 &= q \\
0 &= Aq(-6+11q-6q^2+q^3)\beta\lambda\mu^4 \\
0 &= -A(-1+q)q\mu^2(-1+\lambda(1+4(2-2q+q^2)\beta\mu^2)) \\
0 &= A(-\gamma+2q\mu^2(A\alpha+3q^3\beta]\lambda\mu^2+q(-1-2A\alpha+\lambda+5\beta\lambda\mu^2))) \\
0 &= -A(A(\gamma-8q^2\alpha\mu^2)+q(1+q)\mu^2(-1+\lambda(1+4(2+2q+q^2)\beta\mu^2))) \\
0 &= Aq\mu^2(-2A(\alpha+2q\alpha)+(6+11q+6q^2+q^3)\beta\lambda\mu^2). \qquad (18)
\end{aligned}
$$

For the case $\beta = 1$, solving the above system yields

$$
\begin{aligned}
\lambda &= \frac{1}{1+8\mu^2} \\
A &= \pm\frac{6\mu^2}{\alpha(1+8\mu^2)} \\
\gamma &= 0. \qquad (19)
\end{aligned}
$$

Therefore, a dark soliton solution exists for OS-BBM equation if the coefficient γ is zero and accordingly, the solution is

$$u(x,t) = \frac{6\mu^2}{\alpha(1+8\mu^2)}\tanh^2(x-\frac{t}{1+8\mu^2}), \qquad (20)$$

where μ is a free parameter.

Where as $\beta = -1$, solving the above system yields

$$
\begin{aligned}
\lambda &= \frac{1}{1-8\mu^2} \\
A &= \pm\frac{-6\mu^2}{\alpha(1-8\mu^2)} \\
\gamma &= 0. \qquad (21)
\end{aligned}
$$

Therefore, a dark soliton solution exists for OS-BBM equation if the coefficient γ is zero and accordingly, the solution is

$$u(x,t) = \frac{-6\mu^2}{\alpha(1-8\mu^2)}\tanh^2(x-\frac{t}{1-8\mu^2}). \qquad (22)$$

4. Conclusions

In this work we used solitary wave ansatzes in terms of sech ^{p} and $\tanh^p$ to obtain bright and dark soliton solutions of the Ostrovsky−Benjamin−Bona−Mahony (OS−BBM). We have stated the conditions on the parameters that guarantees the existence of bright and dark soliton solutions of the OS−BBM.

References

[1] Triki, H., Wazwaz, A. M. Bright and dark soliton solution for a K(m,n) equation with t−dependent coefficients. *Phys. Lett. A.*, Volume 373 (2009) 2162-5.

[2] Triki, H., Wazwaz, A. M. Sub-ODE method and soliton solutions for the variable-coefficient mKdV equation. *Applied Mathematics and Computation*, Volume 214 (2009).

[3] Biswas, A., Solitary wave solution for the generalized Kawahara equation. *Appl. Math. Lett.*, Volume 22(2009) 208-10.

[4] Biswas, A., 1−soliton solution of the B(m,n) equation with generalized evolution. *Nonlinear Sci. Numer. Simul.*, Volume 14(2009) 3226-29.

[5] A.H.A. Ali, A.A. Soliman, K.R. Raslan. Soliton solution for nonlinear partial differential equations by cosine-function method. *Physics Letters A*, Volume 368(2007) 299-304.

[6] L. A Ostrovsky. Nonlinear internal waves in a rotating ocean. *Okeanologia 18.*

[7] Y. Liu, V. Varlamov, Stability of solitary waves and weak rotation limit for the ostrovsky equation. *J. Differ. Equ.*, Volume 203(2004) 159-83.

[8] V. Varlamov, Y. Liu, Cauchy problem for the ostrovsky equation. *Discrete Dynam Systems*, Volume 10(3)(2004) 731-53.

[9] M. Sorion, On the controllability of the linearized Benjamin-Bona-Mahony equation. *SIAM J Control Optim*, Volume 39(6)(2004) 1677-96.

[10] S. L. Palacios, Two simple ansatze for obtaining exact solutions of high dispersive nonlinear schrodinger equations. *Chaos, Solitons and Fractals.*, Volume 19(2004) 203-7.

[11] K. Nakkeeran, Bright and dark optical solitons in fiber media with higher order effects. *Chaos, Solitons and Fractals.*, Volume 13, (2002) 673-9.

[12] X. Li and M. Wang, A sub-ODE method for finding exact solutions of a generalized KdV-mKdV equation with high order nonlinear terms. *Physics Letters A.*, Volume 361, (2007) 115-8.

[13] A. M. Wazwaz, Integrable (2+1)-dimentional and (3+1)-dimentional breaking soliton equations. *Physica Scripta*, Volume 81, (2010).

[14] Marwan Alquran, Kamel Al-Khaled, Sinc and solitary wave solutions to the generalized Benjamin-Bona-Mahony-Burgers equations. *Physica Scripta*, Volume 83, (2011).

[15] Marwan Alquran, Kamel Al-Khaled, The tanh and sine-cosine methods for higher order equations of Korteweg-de Vries type. *Physica Scripta*, Volume 84, (2011).

[16] Marwan Alquran, Solitons and periodic solutions to nonlinear partial differential equations by the Sine-Cosine method . *Appl. Math. Inf. Sci.*, Volume 6(1), (2012) pp. 85-8