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NUMERICAL TREATMENT OF THE MOST GENERAL LINEAR VOLTERRA INTEGRO-FRACTIONAL DIFFERENTIAL EQUATIONS WITH CAPUTO DERIVATIVES BY QUADRATURE METHODS

SHAZAD SHAWKI AHMED*, AND SHOKHAN AHMED HAMA SALIH

Department of Mathematics, School of Science, Sulaimani University, Kurdistan Region, Iraq

Abstract: A quadrature method for numerically solving multi-order α_n fractional linear integro-differential equations of Volterra type with variable coefficients (VIFDE) for $0 < \alpha_n \leq 1$ and $n \in \mathbb{N}$ is presented. The fractional derivative is described in the Caputo sense. The method is based on first evaluate the Caputo derivative at any fixed points by finite difference approximation and then apply quadrature method including Trapezoidal and Simpson rules to obtain a finite difference expression for our fractional equation.

Algorithm for treating linear VIFDEs using above process have been developed, in order to express these solutions, program is written in MatLab (V7.6). In addition, some numerical examples are presented to illustrate the accuracy of the method and the results of study are discussed.

Keywords: Integro-Fractional Differential Equation, Caputo Fractional Derivative, Finite difference approximation, Trapezoidal and Simpson's Rules.

2000 AMS Subject Classification: 65L05, 65L06

1. Introduction:

In this paper, we consider the multi-order linear Volterra Integro-Fractional Differential Equations (VIFDE) of order α_n for $0 < \alpha_n \leq 1$ and $n \in \mathbb{N}$, with variable coefficients in the general form:

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*Corresponding author

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$$ {}_{a}^{C}D_{x}^{\alpha_n} y(x) + \sum_{i=0}^{n-1} \mathcal{P}_i(x)\, {}_{a}^{C}D_{x}^{\alpha_{n-i}} y(x) + \mathcal{P}_n(x) y(x) $$
$$ = f(x) + \lambda \sum_{\ell=0}^{m} \int_a^x \mathcal{K}_\ell(x,t)\, {}_{a}^{C}D_{t}^{\beta_{m-\ell}} y(t)\, dt, \quad x \in [a,b] = I \qquad \dots (1) $$

with the initial condition : $y(a) = y_0$ ... (2)

where $0 < \max_{i,j}\{\alpha_i, \beta_j\} \le 1$ for all $i = \overline{0{:}n}$ and $j = \overline{0{:}m}$ with property that $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n \le 1$ and $0 = \beta_0 < \beta_1 < \cdots < \beta_m \le 1$ as well as $\mathcal{K}_\ell : S \times \mathbb{R} \to \mathbb{R}$, (with $S = \{(x,t): a \le t \le x \le b\}$); $\ell = 0,1,2,\dots,m$, and $f, \mathcal{P}_i : I \to \mathbb{R}$ denotes the given continuous functions, $y(x)$ is the unknown function, which is the solution of equation (1) and λ is a scalar parameter.

Differential equations involving these fractional derivatives have proved to be valuable tools in the modeling of many physical phenomena, engineering, bioengineering and Free Electron Laser [1, 5, 11].

Al-Nasir [1] used quadrature methods to solve Volterra integral equation of second kind and Al-Rawi [2] applied it to solve the first kind of integral equation of convolution type while Saadati and Shakeri [15] used trapezoidal rule for solving linear IDEs although Al-Timeme [12] used quadrature technique for the first order of VIDEs and Oras,K.A [14] solved the VIE using quadrature. In this paper, we extend this method to further deal with our consider equation (1-2), summarized it in two good algorithms with computer package program MatLab (V 7.6).

2. Preliminaries and Propositions

Definition (1): [11]

A real valued function y defined on $[a,b]$ be in the space $C_\gamma[a,b], \gamma \in \mathbb{R}$, if there exists a real number $p > \gamma$, such that $y(x) = (x-a)^p y_*(x)$, where $y_* \in C[a,b]$, and it is said to be in the space $C_\gamma^n[a,b]$ iff $y^{(n)} \in C_\gamma[a,b], n \in \mathbb{N}_0$.

Definition (2): [1, 3]

Let $y \in C_\gamma[a,b], \gamma \ge -1$ and $\alpha \in \mathbb{R}^+$. Then the Riemann-Liouville fractional integral operator ${}_aJ_x^\alpha$ of order α of a function y, is defined as:

$$ \left. \begin{aligned} {}_aJ_x^\alpha y(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} y(t)\, dt \quad , \quad \alpha > 0 \\ {}_aJ_x^0\, y(x) &= I y(x) = y(x) \qquad\qquad , \quad \alpha = 0 \end{aligned} \right\} $$

Definition (3): [1, 3]

Let $\alpha \geq 0$ and $m = \lceil \alpha \rceil$.The Riemann-Liouville fractional derivative operator ${}_{a}^{R}D_{x}^{\alpha}$, of order α and $y \in C_{-1}^{m}[a,b]$, is defined as:

$$ {}_{a}^{R}D_{x}^{\alpha} y(x) = D_{x}^{m} \ {}_{a}J_{x}^{m-\alpha} y(x) $$

If $\alpha = m$, $m \in \mathbb{N}_0$, and $y \in C^m\,[a,b]$ we have

$$ {}_{a}^{R}D_{x}^{0} y(x) = y(x) \quad ; \quad {}_{a}^{R}D_{x}^{m} y(x) = y^{(m)}(x) $$

Definition (4): [6, 13]

The Caputo fractional derivative operator ${}_{a}^{C}D_{x}^{\alpha}$ of order $\alpha \in \mathbb{R}^{+}$ of a function $y \in C_{-1}^{m}[a,b]$ and $m-1 < \alpha \leq m, m \in \mathbb{N}$ is defined as:

$$ {}_{a}^{C}D_{x}^{\alpha} y(x) = \ {}_{a}J_{x}^{m-\alpha} D_{x}^{m} y(x) $$

$$ = \frac{1}{\Gamma(m-\alpha)} \int_{a}^{x} (x-t)^{m-\alpha-1} \left(\frac{d}{dt}\right)^{m} y(t)dt\,, \qquad x \in [a,b] $$

Thus for $\alpha = m, m \in \mathbb{N}_0$, and $y \in C^m[a,b]$, we have for all $a \leq x \leq b$

$$ {}_{a}^{C}D_{x}^{0} y(x) = y(x) \quad ; \quad {}_{a}^{C}D_{x}^{m} y(x) = D_{x}^{m} y(x) = \frac{d^m y(x)}{dx^m} $$

Note that: [3, 4, 6, 9]

(i) For $\alpha \geq 0$ and $\beta > 0$, then ${}_{a}J_{x}^{\alpha}(x-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}$.

(ii) ${}_{a}^{R}D_{x}^{\alpha}\mathcal{A} = \mathcal{A}\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$ and ${}_{a}^{C}D_{x}^{\alpha}\mathcal{A} = 0$; $\mathcal{A}$ is any constant;($\alpha \geq 0, \alpha \notin \mathbb{N}$).

(iii) ${}_{a}^{C}D_{x}^{\alpha} y(x) = D_{x}^{m}\ {}_{a}J_{x}^{m-\alpha} y(x) \neq \ {}_{a}J_{x}^{m-\alpha} D_{x}^{m} y(x) = {}_{a}^{R}D_{x}^{\alpha} y(x)\,; m = \lceil \alpha \rceil$.

(iv) Let $\alpha \geq 0$, $m = \lceil \alpha \rceil$ and $y \in C^m[a,b]$,then, the relation between the Caputo derivative and the R-L integral are formed:

$$ {}_{a}^{C}D_{x}^{\alpha}\ {}_{a}J_{x}^{\alpha} y(x) = y(x) \quad ; \quad a \leq x \leq b $$

$$ {}_{a}J_{x}^{\alpha}\, {}_{a}^{C}D_{x}^{\alpha} y(x) = y(x) - \sum_{k=0}^{m-1} \frac{y^{k}(x)}{k!}(x-a)^{k} $$

(v) ${}_{a}^{C}D_{x}^{\alpha} y(x) = {}_{a}^{R}D_{x}^{\alpha}\big[y(x) - T_{m-1}[y;a]\big]$; $(m-1 < \alpha \leq m)$ and $T_{m-1}[y;a]$ denotes the Taylor polynomial of degree $m-1$ for the function y , centered at a.

(vi) Let $\alpha \geq 0; m = \lceil \alpha \rceil$and for $y(x) = (x-a)^{\beta}$ for some $\beta \geq 0$.Then:

$$ {}_{a}^{C}D_{x}^{\alpha} y(x) = \begin{cases} 0 & \text{if} \quad \beta \in \{0,1,2,\dots,m-1\} \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(x-a)^{\beta-\alpha} & \text{if} \quad \beta \in \mathbb{N} \text{ and } \beta \geq m \\ & \text{or} \quad \beta \notin \mathbb{N} \text{ and } \beta > m-1 \end{cases} $$

Proposition (1): [18]

The finite difference approximation of Caputo derivative for $0 < \alpha \leq 1$ at define points $x = x_{r+1}; r = 0,1,\dots,N-1$ and $h = (b-a)/N$, is formed

$$ {}_{a}^{C}D_{x}^{\alpha}\, y(x_{r+1}) = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{r} \left[y(x_{r-j+1}) - y(x_{r-j}) \right] b_j^{\alpha} \qquad \dots (3) $$

where $b_j^{\alpha} = (j+1)^{1-\alpha} - j^{1-\alpha}$.

Proposition (2): [3, 13]

Let $\alpha \geq 0$, $\alpha \notin \mathbb{N}$ and $m = \lceil \alpha \rceil$ .Moreover, assume that $y \in C_{-1}^{m}[a,b]$. Then the Caputo fractional derivative ${}_{a}^{C}D_{x}^{\alpha} y(x)$ is continuous on $[a,b]$ and $[{}_{a}^{C}D_{x}^{\alpha} y(x)]_{x=a} = 0$.

3. Quadrature Rule

Quadrature rule is weighted sum of finite number of sample values of integrand function. Let $f(x)$ be a real-valued function defined on a finite interval $a \leq x \leq b$. we seek to compute the value of the integral $\int_a^b f(x)dx$, by $\sum_{j=1}^{N} w_j f(x_j) + R[f]$,where $R[f]$ is the remainder and the quadrature rule $\{w_j, x_j\}_{j=1}^{N}$ is a available in tabulated form: the real numbers x_j are the integration nodes and w_j are constants which are called quadrature weights, [5, 10, 12, 17].

We present here two algorithms for generating the quadrature rule defined by the weight function and numbers of nodes:

For $[a,b]$, we subdivide the interval from a to b into N-subintervals of size h , $h = \frac{b-a}{N}; N \geq 1$ with grid points $x_i = a + ih \ (i = 0,1,2,\dots N)$, then we can written the integration by Trapezoidal rule as:

$$ \int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{N-1} f(x_i) + f(b) \right] = h \sum_{k=0}^{N} w_k^t f(x_k) \qquad \dots (4) $$

Hence w_k^t is weights for trapezoidal rule, where $w_0^t = w_N^t = \frac{1}{2}$; $w_i^t = 1$; $(0 < i < n)$.

Also we can written the integration by Simpson's rule as in the following generalization formula:

(N- is even):

$$\int_a^b f(x)dx = h\sum_{i=1}^{N/2}\sum_{k=0}^{2} w_k^s\, f(x_{2i-k}) \qquad \dots (5)$$

(N- is odd):

$$\int_a^b f(x)dx = h\sum_{i=1}^{(N-1)/2}\sum_{k=0}^{2} w_k^s f(x_{2i-k}) + h\sum_{k=0}^{1} w_k^t f(x_{N-k}) \qquad \dots (6)$$

While $w_k^s and\ w_k^t$ are the weights for Simpson's and trapezoidal rules respectively; where $w_0^s = w_2^s = \frac{1}{3}$, $w_1^s = \frac{4}{3}$ $and\ w_0^t = w_1^t = \frac{1}{2}$.

4. The Method

In this section, new two algorithms are presented for treating linear VIFDE with variable coefficients using quadrature methods with aid of finite difference approximation. Recall equation (1) for $0 < \max_{i,j}\{\alpha_i, \beta_j\} \leq 1$ with strictly decreasing for α_i and β_j for all $i = 0,1,\dots,n$ and $j = 0,1,..,m$. The quadrature method can be extended to give a class of Volterra quadrature methods for finite difference solution of (1) and setting $x = x_r$ with take $\mathcal{P}_0(x) = 1$ for all $x \in [a,b]$, thus for all $r = 1,2,3,\dots,N (N \in \mathbb{N})$:

$$\left[\sum_{i=0}^{n-1} \mathcal{P}_i(x)\, {}_a^C D_x^{\alpha_{n-i}} y(x) + \mathcal{P}_n(x) y(x)\right]_{x=x_r} = f(x_r) +$$

$$\lambda \sum_{\ell=0}^{m-1} \int_a^{x_r} \mathcal{K}_\ell(x_r,t)\, {}_a^C D_t^{\beta_{m-\ell}} y(t)dt + \lambda \int_a^{x_r} \mathcal{K}_m(x_r,t)\ y(t)dt \qquad \dots (7)$$

Thus, we approximating the fractional differential by using forward difference in proposition (1) and the integrals in (7) by quadrature methods; so:

4.1 Using Trapezoidal Rule

Applying quadrature rule (4) to evaluate each integral term in equation (7) and take into account the formula (3) with proposition (2) ,that $\left[{}_a^C D_t^{\beta_{m-\ell}} y(t)\right]_{t=t_0}$ equal to zero, we obtain the following equation :

$$\sum_{i=0}^{n-1} \mathcal{P}_i(x_r) \frac{h^{-\alpha_{n-i}}}{\Gamma(2-\alpha_{n-i})} \left\{ [y_r - y_{r-1}] + \sum_{j=1}^{r-1} [y_{r-j} - y_{r-j-1}] b_j^{\alpha_{n-i}} \right\} + \mathcal{P}_n(x_r) y_r =$$

$$f(x_r) + \lambda h \sum_{\ell=0}^{m-1} \sum_{p=1}^{r-1} \mathcal{K}_\ell(x_r, t_p) \left\{ \frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})} \sum_{q=0}^{p-1} [y_{p-q} - y_{p-q-1}] b_q^{\beta_{m-\ell}} \right\} +$$

$$\frac{\lambda h}{2} \sum_{\ell=0}^{m-1} \mathcal{K}_\ell(x_r, t_r) \left\{ \frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})} \left([y_r - y_{r-1}] + \sum_{k=1}^{r-1} [y_{r-k} - y_{r-k-1}] b_k^{\beta_{m-\ell}} \right) \right\} +$$

$$\lambda \left\{ \frac{h}{2} \mathcal{K}_m(x_r, t_0) y_0 + h \sum_{p=1}^{r-1} \mathcal{K}_m(x_r, t_p) y_p + \frac{h}{2} \mathcal{K}_m(x_r, t_r) y_r \right\} \qquad \dots (8)$$

After some simple manipulation we obtain the following equation for determining an approximation y_r to $y(x_r)$ for all $r = 1,2,3, \dots, N$, and y_0 is given from initial condition:

$$\left[\sum_{i=0}^{n-1} \mathcal{P}_{ir} \frac{h^{-\alpha_{n-i}}}{\Gamma(2-\alpha_{n-i})} + \mathcal{P}_{nr} - \frac{\lambda h}{2} \sum_{\ell=0}^{m} \mathcal{K}_\ell(x_r, t_r) \frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})} \right] y_r =$$

$$f_r + \left[\sum_{i=0}^{n-1} \mathcal{P}_{ir} \frac{h^{-\alpha_{n-i}}}{\Gamma(2-\alpha_{n-i})} - \frac{\lambda h}{2} \sum_{\ell=0}^{m-1} \mathcal{K}_\ell(x_r, t_r) \frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})} \right] y_{r-1} +$$

$$\lambda h \sum_{\ell=0}^{m-1} \sum_{p=1}^{r-1} \mathcal{K}_\ell(x_r, t_p) \frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})} \sum_{q=0}^{p-1} [y_{p-q} - y_{p-q-1}] b_q^{\beta_{m-\ell}} +$$

$$\frac{\lambda h}{2} \sum_{\ell=0}^{m-1} \mathcal{K}_\ell(x_r, t_r) \frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})} \sum_{k=1}^{r-1} [y_{r-k} - y_{r-k-1}] b_k^{\beta_{m-\ell}} - \sum_{i=0}^{n-1} \mathcal{P}_{ir} \frac{h^{-\alpha_{n-i}}}{\Gamma(2-\alpha_{n-i})} *$$

$$* \sum_{j=1}^{r-1} [y_{r-j} - y_{r-j-1}] b_j^{\alpha_{n-i}} + \lambda \left[\frac{h}{2} \mathcal{K}_m(x_r, t_0) y_0 + h \sum_{p=1}^{r-1} \mathcal{K}_m(x_r, t_p) y_p \right] \qquad \dots (9)$$

Assuming that:

$$A_k^\sigma(s) = \frac{h^{-\sigma_{k-s}}}{\Gamma(2-\sigma_{k-s})}, \qquad \text{for each } s = 0,1,2, \dots, k-1; k \in \mathbb{Z}^+ \qquad \dots (10)$$

and all σ's $(\sigma_n > \sigma_{n-1} > \cdots > \sigma_1 > \sigma_0 = 0)$ are lies in (0,1) , with letting

$$\mathcal{H}_k^\sigma(r) = \mathcal{P}_{kr} + \sum_{s=0}^{n-1} \mathcal{P}_{sr} A_k^\sigma(s) \quad ; \text{ for each } \quad r \in \mathbb{Z}^+ \qquad \dots (11)$$

clearly here , $A_k^\sigma(k) = 1$, then the equation (9) can be written as a form:

$$\left[\mathcal{H}_n^\alpha(r) - \frac{\lambda h}{2} \sum_{\ell=0}^{m} \mathcal{K}_{rr}^\ell A_m^\beta(\ell) \right] y_r = f_r + \left[\mathcal{H}_n^\alpha(r) - \mathcal{P}_{nr} - \frac{\lambda h}{2} \sum_{\ell=0}^{m-1} \mathcal{K}_{rr}^\ell A_m^\beta(\ell) \right] y_{r-1}$$

$$+\lambda h\sum_{p=1}^{r-1}\sum_{q=0}^{p-1}\left[y_{p-q}-y_{p-q-1}\right]*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{rp}^{\ell}\,A_m^{\beta}(\ell)b_q^{\beta_{m-\ell}}\right)$$

$$+\frac{\lambda h}{2}\sum_{k=1}^{r-1}\left[y_{r-k}-y_{r-k-1}\right]*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{rr}^{\ell}\,A_m^{\beta}(\ell)b_k^{\beta_{m-\ell}}\right)$$

$$-\sum_{j=1}^{r-1}\left[y_{r-j}-y_{r-j-1}\right]*\left(\sum_{i=0}^{n-1}\mathcal{P}_{ir}\,A_n^{\alpha}(i)b_j^{\alpha_{n-i}}\right)+\lambda h\sum_{p=0}^{r-1}w_p^t\mathcal{K}_{rp}^m y_p \quad \dots (12)$$

where $\mathcal{K}_{rp}^{\ell}=\mathcal{K}_{\ell}(x_r,t_p)$ for all ℓ, r and p .

The Algorithm [*AVIFT*]:

The finite difference approximation solution for linear VIFDE using Trapezoidal rule is computed as follows:

Step 1: a. Take $h=\frac{b-a}{N}$; $N\in\mathbb{N}$.

b. Set $y_0=y(x_0)$. [from initial condition (2) is given].

Step 2: a. To compute $A_k^{\sigma}(s)$, for each $s=0,1,2,\dots,k-1$, $k\in\mathbb{Z}^+$ and all $\sigma's$ (fractional order lines in $(0,1)$ and $\sigma_k>\sigma_{k-1}>\cdots>\sigma_1>\sigma_0=0$) here σ is (α or β) and k is (n or m) respectively, we use step 1 and equation (10) .

b. Using equation (11) and step $(2,a)$ for $\alpha_k (k=0,1,2,\dots,n)$, to evaluate $H_n^{\alpha}(r)$, $r\in\mathbb{Z}^+$.

Step 3: for each $r=1,2,3,\dots,N$:

a. Put $T[r-1]=y(x_{r-1})=y_{r-1}$

b. Evaluate: $R_m[r]=\lambda h\sum_{p=0}^{r-1}w(p)\,\mathcal{K}_m(x_r,x_p)T[p]$

$$Q_m[r,k]=\frac{\lambda h}{2}\sum_{\ell=0}^{k}\mathcal{K}_{\ell}(x_r,x_r)\,A_m^{\beta}(\ell)$$

with $w(p)=\begin{cases}1/2 & if\ p=0\ \ or\ \ p=r\\ 1 & otherwise\end{cases}$

c. Calculate $B_{n,m}^{\alpha,\beta}[r]=B_n^{\alpha}[r]+B_m^{\beta}[r]$, take into account $B_{n,m}^{\alpha,\beta}[1]=0$, where

$$B_n^{\alpha}[r]=-\sum_{j=1}^{r-1}(T[r-j]-T[r-j-1])*\left(\sum_{i=0}^{n-1}p_i(x_r)\,A_n^{\alpha}(i)b_j^{\alpha_{n-i}}\right) \text{ and}$$

$$B_m^{\beta}[r]=\lambda h\sum_{p=1}^{r-1}\left\{\sum_{q=0}^{p-1}(T[p-q]-T[p-q-1])*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{\ell}(x_r,x_p)\,A_m^{\beta}(\ell)b_q^{\beta_{m-\ell}}\right)\right.$$

$$+\frac{1}{2}(T[r-p]-T[r-p-1]) * \left(\sum_{\ell=0}^{m-1} \mathcal{K}_{\ell}(x_r, x_r)\, A_m^{\beta}(\ell) b_p^{\beta_{m-\ell}}\right)\Bigg\}$$

Step 4: Compute $y_r = y(x_r)$ for each $r = 1,2,3,\dots,N$, from:

$$(H_n^{\alpha}(r) - Q_m[r,m])y_r$$
$$= (H_n^{\alpha}(r) - \mathcal{P}_n(x_r) - Q_m[r,m-1])T[r-1] + B_{n,m}^{\alpha,\beta}[r] + R_m[r] + f(x_r).$$

where $b_\theta^\sigma = (1+\theta)^{1-\sigma} - \theta^{1-\sigma}$; $0 < \sigma \le 1$ and $\theta = 0,1,2,\dots$

4.2 Using Simpson's Rule

Now, the Simpson's rule is used to approximate each integral term of equation (7), at the points $x = x_r; r = 0,1,2,\dots,N$ which are equation (5 and 6) for even and odd value of r, then we have the following classification:

- For r is odd :

$$\left[\sum_{i=0}^{n-1} \mathcal{P}_i(x)\, {}_a^C D_x^{\alpha_{n-i}} y(x) + \mathcal{P}_n(x) y(x)\right]_{x=x_r} =$$

$$f(x_r) + \lambda h \sum_{\ell=0}^{m-1} \left\{ \sum_{p=1}^{(r-1)/2} \sum_{k=0}^{2} w_k^s\, \mathcal{K}_\ell\big(x_r, t_{2p-k}\big) \left[{}_a^C D_t^{\beta_{m-\ell}} y(t)\right]_{t=t_{2p-k}} \right.$$

$$\left. + \sum_{k=0}^{1} w_k^t\, \mathcal{K}_\ell(x_r, t_{r-k}) \left[{}_a^C D_t^{\beta_{m-\ell}} y(t)\right]_{t=t_{r-k}} \right\} +$$

$$\lambda h \sum_{p=1}^{(r-1)/2} \sum_{k=0}^{2} w_k^s\, \mathcal{K}_m\big(x_r, t_{2p-k}\big) y\big(t_{2p-k}\big) + \lambda h \sum_{k=0}^{1} w_k^t \mathcal{K}_m(x_r, t_{r-k})\, y(t_{r-k}) \quad \dots (13)$$

where $w_k^s and\, w_k^t$ are the weights for simpson's and trapezoidal rules, $w_0^s = w_2^s = \frac{1}{3}$, $w_1^s = \frac{4}{3}$ and $w_0^t = w_1^t = \frac{1}{2}$ respectively .

Take into account the proposition (2) i.e., $\left[{}_a^C D_t^{\beta_{m-\ell}} y(t)\right]_{t=t_0}$ is equal to zero for each β's, and using the proposition (1), then the equation (13) becomes:

$$\sum_{i=0}^{n-1} \mathcal{P}_{ir} \frac{h^{-\alpha_{n-i}}}{\Gamma(2-\alpha_{n-i})} \left\{ [y_r - y_{r-1}] + \sum_{j=1}^{r-1} [y_{r-j} - y_{r-j-1}] b_j^{\alpha_{n-i}} \right\} + \mathcal{P}_{nr} y_r =$$

$$f_r + \lambda h \sum_{\ell=0}^{m-1} \left\{ \frac{1}{3} \sum_{p=1}^{(r-1)/2} \mathcal{K}_{r,2p}^{\ell} \frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})} \sum_{j=0}^{2p-1} [y_{2p-j} - y_{2p-j-1}] b_j^{\beta_{m-\ell}} \right.$$

$$+\frac{4}{3}\sum_{p=1}^{(r-1)/2}\mathcal{K}_{r,2p-1}^{\ell}\frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})}\sum_{j=0}^{2p-2}\left[y_{2p-j-1}-y_{2p-j-2}\right]b_j^{\beta_{m-\ell}}$$

$$+\frac{1}{3}\sum_{p=2}^{(r-1)/2}\mathcal{K}_{r,2p-2}^{\ell}\frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})}\sum_{j=0}^{2p-3}\left[y_{2p-j-2}-y_{2p-j-3}\right]b_j^{\beta_{m-\ell}}$$

$$+\frac{1}{2}\mathcal{K}_{r,r}^{\ell}\,\frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})}\sum_{j=0}^{r-1}\left[y_{r-j}-y_{r-j-1}\right]b_j^{\beta_{m-\ell}}+\frac{1}{2}\mathcal{K}_{r,r-1}^{\ell}\,\frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})}*$$

$$\left.\sum_{j=0}^{r-2}\left[y_{r-j-1}-y_{r-j-2}\right]b_j^{\beta_{m-\ell}}+\right\}+\frac{\lambda h}{3}\sum_{p=1}^{(r-1)/2}\mathcal{K}_{r,2p}^{m}y_{2p}+\frac{4\lambda h}{3}\sum_{p=1}^{(r-1)/2}\mathcal{K}_{r,2p-1}^{m}y_{2p-1}$$

$$+\frac{\lambda h}{3}\sum_{p=1}^{(r-1)/2}\mathcal{K}_{r,2p-2}^{\ell}\,y_{2p-2}+\frac{\lambda h}{2}\mathcal{K}_{r,r}^{m}y_r+\frac{\lambda h}{2}\mathcal{K}_{r,r-1}^{m}y_{r-1} \qquad \ldots(14)$$

where $\mathcal{K}_{rp}^{\ell}=\mathcal{K}_{\ell}(x_r,t_p)$ for all ℓ, r and p .

Since r is odd so there exist $\bar{r}\in\mathbb{Z}^{+}$ such that $r=2\bar{r}+1$ thus $r-1$ is even. Then from equation (14) calculating it to find $y_{2\bar{r}+1}$ so:

$$\left[\sum_{i=0}^{n-1}\mathcal{P}_{i(2\bar{r}+1)}\frac{h^{-\alpha_{n-i}}}{\Gamma(2-\alpha_{n-i})}+\mathcal{P}_{n(2\bar{r}+1)}-\frac{\lambda h}{2}\mathcal{K}_{2\bar{r}+1,2\bar{r}+1}^{m}\right.$$

$$\left.-\frac{\lambda h}{2}\sum_{\ell=0}^{m-1}\mathcal{K}_{2\bar{r}+1,2\bar{r}+1}^{\ell}\frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})}\right]y_{2\bar{r}+1}=$$

$$f_{2\bar{r}+1}+\left[\sum_{i=0}^{n-1}\mathcal{P}_{i(2\bar{r}+1)}\frac{h^{-\alpha_{n-i}}}{\Gamma(2-\alpha_{n-i})}-\frac{\lambda h}{2}\sum_{\ell=0}^{m-1}\mathcal{K}_{2\bar{r}+1,2\bar{r}+1}^{\ell}\frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})}\right.$$

$$\left.+\frac{\lambda h}{2}\mathcal{K}_{2\bar{r}+1,2\bar{r}}^{m}\right]y_{2\bar{r}}-\sum_{i=0}^{n-1}\mathcal{P}_{i(2\bar{r}+1)}\frac{h^{-\alpha_{n-i}}}{\Gamma(2-\alpha_{n-i})}\sum_{j=1}^{2\bar{r}}\left[y_{2\bar{r}+1-j}-y_{2\bar{r}-j}\right]b_j^{\alpha_{n-i}}$$

$$+\frac{\lambda h}{3}\sum_{p=1}^{\bar{r}}\left\{\sum_{j=0}^{2p-1}\left[y_{2p-j}-y_{2p-j-1}\right]*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{2\bar{r}+1,2p}^{\ell}\frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})}b_j^{\beta_{m-\ell}}\right)\right\}$$

$$+\frac{4\lambda h}{3}\sum_{p=1}^{\bar{r}}\left\{\sum_{j=0}^{2p-2}\left[y_{2p-j-1}-y_{2p-j-2}\right]*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{2\bar{r}+1,2p-1}^{\ell}\frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})}b_j^{\beta_{m-\ell}}\right)\right\}$$

$$+\frac{\lambda h}{3}\sum_{p=2}^{\bar{r}}\left\{\sum_{j=0}^{2p-3}\left[y_{2p-j-2}-y_{2p-j-3}\right]*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{2\bar{r}+1,2p-2}^{\ell}\frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})}b_j^{\beta_{m-\ell}}\right)\right\}$$

$$+\frac{\lambda h}{2}\sum_{j=1}^{2\bar{r}}\left[y_{2\bar{r}+1-j}-y_{2\bar{r}-j}\right]*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{2\bar{r}+1,2\bar{r}+1}^{\ell}\frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})}b_j^{\beta_{m-\ell}}\right)$$

$$+\frac{\lambda h}{2}\sum_{j=0}^{2\bar{r}-1}\left[y_{2\bar{r}-j}-y_{2\bar{r}-j-1}\right]*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{2\bar{r}+1,2\bar{r}}^{\ell}\frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})}b_j^{\beta_{m-\ell}}\right)+$$

$$\frac{\lambda h}{3}\sum_{p=1}^{\bar{r}}\mathcal{K}_{2\bar{r}+1,2p}^{m}y_{2p}+\frac{4\lambda h}{3}\sum_{p=1}^{\bar{r}}\mathcal{K}_{2\bar{r}+1,2p-1}^{m}y_{2p-1}+\frac{\lambda h}{3}\sum_{p=1}^{\bar{r}}\mathcal{K}_{2\bar{r}+1,2p-2}^{m}y_{2p-2}\quad \ldots(15)$$

After some simple manipulation and rechange every $\bar{r}$ to $(r-1)/2$ we obtain the approximation y_r to $y(x_r)$ for all $r=1,2,\ldots,N$, for there more using the assuming equation (10 and 11) we get:

$$\left[\mathcal{H}_n^{\alpha}(r)-\frac{\lambda h}{2}\sum_{\ell=0}^{m}\mathcal{K}_{r,r}^{\ell}A_m^{\beta}(\ell)\right]y_r$$

$$=f_r+\left[\mathcal{H}_n^{\alpha}(r)-\mathcal{P}_{nr}+\frac{\lambda h}{2}\mathcal{K}_{r,r-1}^{m}-\frac{\lambda h}{2}\sum_{\ell=0}^{m-1}\mathcal{K}_{r,r}^{\ell}A_m^{\beta}(\ell)\right]y_{r-1}$$

$$-\sum_{j=1}^{r-1}\left[y_{r-j}-y_{r-j-1}\right]*\left(\sum_{i=0}^{n-1}\mathcal{P}_{ir}A_n^{\alpha}(i)b_j^{\alpha_{n-i}}\right)$$

$$+\frac{\lambda h}{3}\sum_{p=1}^{(r-1)/2}\left(\sum_{j=0}^{2p-1}\left[y_{2p-j}-y_{2p-j-1}\right]\right)*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{r,2p}^{\ell}A_m^{\beta}(\ell)b_j^{\beta_{m-\ell}}\right)$$

$$+\frac{4h}{3}\sum_{p=1}^{(r-1)/2}\left(\sum_{j=0}^{2p-2}\left[y_{2p-j-1}-y_{2p-j-2}\right]\right)*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{r,2p-1}^{\ell}A_m^{\beta}(\ell)b_j^{\beta_{m-\ell}}\right)$$

$$+\frac{\lambda h}{3}\sum_{p=2}^{(r-1)/2}\left(\sum_{j=0}^{2p-3}\left[y_{2p-j-2}-y_{2p-j-3}\right]\right)*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{r,2p-2}^{\ell}A_m^{\beta}(\ell)b_j^{\beta_{m-\ell}}\right)$$

$$+\frac{\lambda h}{2}\sum_{j=1}^{r-1}\left[y_{r-j}-y_{r-j-1}\right]*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{r,r}^{\ell}A_m^{\beta}(\ell)b_j^{\beta_{m-\ell}}\right)$$

$$+\frac{\lambda h}{2}\sum_{j=0}^{r-2}\left[y_{r-j-1}-y_{r-j-2}\right]*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{r,r-1}^{\ell}A_m^{\beta}(\ell)b_j^{\beta_{m-\ell}}\right)$$

$$+\frac{\lambda h}{3}\sum_{p=1}^{(r-1)/2}\left\{\mathcal{K}_{r,2p}^{m}y_{2p}+4\mathcal{K}_{r,2p-1}^{m}y_{2p-1}+\mathcal{K}_{r,2p-2}^{m}y_{2p-2}\right\}\quad \ldots(16)$$

- For r is even :

$$\left[\sum_{i=0}^{n-1} \mathcal{P}_i(x)\, {}_a^C D_x^{\alpha_{n-i}} y(x) + \mathcal{P}_n(x) y(x)\right]_{x=x_r} = f(x_r) + \lambda \sum_{\ell=0}^{m-1} \left\{ h \sum_{p=1}^{r/2} \sum_{k=0}^{2} w_k^s * \right.$$

$$\left. \mathcal{K}_\ell(x_r, t_{2p-k}) \left[{}_a^C D_t^{\beta_{m-\ell}} y(t) \right]_{t=t_{2p-k}} \right\} + \lambda h \sum_{p=1}^{r/2} \sum_{k=0}^{2} w_k^s\, \mathcal{K}_m(x_r, t_{2p-k}) y(t_{2p-k}) \dots (17)$$

Using the weights of Simpson's rule with proposition (1) furthermore $\left[{}_a^C D_t^{\beta_{m-\ell}} y(t) \right]_{t=t_0}$ is vanishes by proposition (2). Then equation (17) becomes

$$\sum_{i=0}^{n-1} \mathcal{P}_{ir} \frac{h^{-\alpha_{n-i}}}{\Gamma(2-\alpha_{n-i})} \left\{ [y_r - y_{r-1}] + \sum_{j=1}^{r-1} [y_{r-j} - y_{r-j-1}] b_j^{\alpha_{n-i}} \right\} + \mathcal{P}_{nr} y_r$$

$$= f_r + \lambda h \sum_{\ell=0}^{m-1} \left\{ \frac{1}{3} \sum_{p=1}^{r/2} \mathcal{K}_{r,2p}^{\ell}\ \frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})} \sum_{j=0}^{2p-1} \left[y_{2p-j} - y_{2p-j-1} \right] b_j^{\beta_{m-\ell}} \right.$$

$$+ \frac{4}{3} \sum_{p=1}^{r/2} \mathcal{K}_{r,2p-1}^{\ell}\ \frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})} \sum_{j=0}^{2p-2} \left[y_{2p-j-1} - y_{2p-j-2} \right] b_j^{\beta_{m-\ell}}$$

$$\left. + \frac{1}{3} \sum_{p=2}^{r/2} \mathcal{K}_{r,2p-2}^{\ell} * \frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})} + \sum_{j=0}^{2p-3} \left[y_{2p-j-2} - y_{2p-j-3} \right] b_j^{\beta_{m-\ell}} \right\}$$

$$+ \frac{\lambda h}{3} \sum_{p=1}^{r/2} \left\{ \mathcal{K}_{r,2p}^{m} y_{2p} + 4 \mathcal{K}_{r,2p-1}^{m} y_{2p-1} + \mathcal{K}_{r,2p-2}^{m} y_{2p-2} \right\} \qquad \dots (18)$$

where $\mathcal{K}_{rp}^{\ell} = \mathcal{K}_\ell(x_r, t_p)$ for all ℓ, r and p .since r even then $r = 2\bar{r}$; $\bar{r} \in \mathbb{Z}^+$, then from equation (18) calculate it to find $y_{2\bar{r}}$ so :

$$\left[\sum_{i=0}^{n-1} \mathcal{P}_{i(2\bar{r})} \frac{h^{-\alpha_{n-i}}}{\Gamma(2-\alpha_{n-i})} + \mathcal{P}_{n(2\bar{r})} - \frac{\lambda h}{3} \mathcal{K}_{2\bar{r},2\bar{r}}^{m} \right] y_{2\bar{r}} =$$

$$\left[\sum_{i=0}^{n-1} \mathcal{P}_{i(2\bar{r})} \frac{h^{-\alpha_{n-i}}}{\Gamma(2-\alpha_{n-i})} \right] y_{2\bar{r}-1} + f_{2\bar{r}}$$

$$- \sum_{j=1}^{2\bar{r}-1} \left[y_{2\bar{r}-j} - y_{2\bar{r}-j-1} \right] * \left(\sum_{i=0}^{n-1} \mathcal{P}_{i(2\bar{r})} \frac{h^{-\alpha_{n-i}}}{\Gamma(2-\alpha_{n-i})} b_j^{\alpha_{n-i}} \right)$$

$$+ \frac{\lambda h}{3} \sum_{p=1}^{\bar{r}-1} \mathcal{K}_{2\bar{r},2p}^{m} y_{2p} + \frac{4\lambda h}{3} \sum_{p=1}^{\bar{r}} \mathcal{K}_{2\bar{r},2p-1}^{m} y_{2p-1} + \frac{\lambda h}{3} \sum_{p=1}^{\bar{r}} \mathcal{K}_{2\bar{r},2p-2}^{m} y_{2p-2}$$

$$+\frac{\lambda h}{3}\sum_{p=1}^{\bar{r}}\left\{\sum_{j=0}^{2p-1}\left[y_{2p-j}-y_{2p-j-1}\right]*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{2\bar{r},2p}^{\ell}\ \frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})}b_j^{\beta_{m-\ell}}\right)\right\}$$

$$+\frac{4\lambda h}{3}\sum_{p=1}^{\bar{r}}\left\{\sum_{j=0}^{2p-2}\left[y_{2p-j-1}-y_{2p-j-2}\right]*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{2\bar{r},2p-2}^{\ell}\ \frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})}b_j^{\beta_{m-\ell}}\right)\right\}$$

$$+\frac{\lambda h}{3}\sum_{p=2}^{\bar{r}}\left\{\sum_{j=0}^{2p-3}\left[y_{2p-j-2}-y_{2p-j-3}\right]*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{2\bar{r},2p-2}^{\ell}\ \frac{h^{-\beta_{m-\ell}}}{\Gamma(2-\beta_{m-\ell})}b_j^{\beta_{m-\ell}}\right)\right\} \quad \ldots(19)$$

After some simple manipulation and rechange every $\bar{r}$ to $r/2$ we obtain the approximation y_r for all $r=1,2,\ldots,N$ .for ther more using the assuming equations (10 and 11) we obtain:

$$\left[\mathcal{H}_n^{\alpha}(r)-\frac{\lambda h}{3}\sum_{\ell=0}^{m}\mathcal{K}_{\ell,r}A_m^{\beta}(\ell)\right]y_r=f_r+\left[\mathcal{H}_n^{\alpha}(r)-\mathcal{P}_{nr}-\frac{\lambda h}{3}\sum_{\ell=0}^{m-1}\mathcal{K}_{r,r}^{\ell}A_m^{\beta}(\ell)\right]y_{r-1}$$

$$+\sum_{j=1}^{r-1}\left[y_{r-j}-y_{r-j-1}\right]*\left(\frac{\lambda h}{3}\sum_{\ell=0}^{m-1}\mathcal{K}_{r,r}^{\ell}A_m^{\beta}(\ell)\,b_j^{\beta_{m-\ell}}-\sum_{i=0}^{n-1}\mathcal{P}_{ir}A_n^{\alpha}(i)b_j^{\alpha_{n-i}}\right)$$

$$+\frac{\lambda h}{3}\left\{\sum_{p=1}^{\frac{r}{2}-1}\mathcal{K}_{r,2p}^{m}y_{2p}+4\sum_{p=1}^{\frac{r}{2}}\mathcal{K}_{r,2p-1}^{m}y_{2p-1}+\sum_{p=1}^{\frac{r}{2}}\mathcal{K}_{r,2p-2}^{m}y_{2p-2}\right\}$$

$$+\frac{\lambda h}{3}\sum_{p=1}^{\frac{r}{2}-1}\left\{\sum_{j=0}^{2p-1}\left[y_{2p-j}-y_{2p-j-1}\right]*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{r,2p}^{\ell}A_m^{\beta}(\ell)\,b_j^{\beta_{m-\ell}}\right)\right.$$

$$+\frac{4\lambda h}{3}\sum_{p=1}^{\frac{r}{2}}\sum_{j=0}^{2p-2}\left[y_{2p-j-1}-y_{2p-j-2}\right]*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{r,2p-1}^{\ell}A_m^{\beta}(\ell)\,b_j^{\beta_{m-\ell}}\right)$$

$$+\frac{\lambda h}{3}\sum_{p=2}^{r/2}\sum_{j=0}^{2p-3}\left[y_{2p-j-2}-y_{2p-j-3}\right]*\left(\sum_{\ell=0}^{m-1}\mathcal{K}_{r,2p-2}^{\ell}A_m^{\beta}(\ell)\,b_j^{\beta_{m-\ell}}\right) \quad \ldots(20)$$

The Algorithm [*AVIFS*]:

The finite difference approximation solution for multi-linear VIFDE using Simpson's rule is computed as follows:

Step 1: a. Take $h=\frac{b-a}{N}$; $N\in\mathbb{N}$.

b. Set $y_0=y(x_0)$, [from initial condition (2) is given].

Step 2: a. To compute $A_k^{\sigma}(s)$, for each $s=0,1,2,\ldots,k-1,\ k\in\mathbb{Z}^+$ and all

$\sigma's$ (fractional order lins in (0,1) and $\sigma_k>\sigma_{k-1}>\sigma_1>\sigma_0=0$), here σ is (α or β) and k is (n or m) respectively, we use step 1 and equation (10).

b. Using equation (11) and step $(2, a)$ for $\alpha_k (k = 0,1,2, \dots, n)$, to evaluate $H_n^{\alpha}(r)$, $r \in \mathbb{Z}^+$

Step 3: for each $r = 1,2,3, \dots, N$:

a. Put $T[r-1] = y(x_{r-1}) = y_{r-1}$

b1. If r is odd, Evaluate:

$$R_m[r] = \lambda h \sum_{p=1}^{(r-1)/2} \sum_{k=0}^{2} w(k) \mathcal{K}_m(x_r, x_{2p-k})\, T[2p-k]$$

and $$Q_m^k[r,p,j] = \frac{\lambda h}{2} \sum_{\ell=0}^{k} \mathcal{K}_\ell\left(x_r, x_p\right) A_m^{\beta}(\ell) b_j^{\beta_{m-\ell}}$$ with

$$L_m[r] = \sum_{j=1}^{r-1} (T[r-j] - T[r-j-1])\, (Q_m^{m-1}[r,r,j] + Q_m^{m-1}[r,r-1,j-1])$$

c1. Calculate $B_{n,m}^{\alpha,\beta}[r] = B_n^{\alpha}[r] + B_m^{\beta}[r]$, where

$$B_n^{\alpha}[r] = -\sum_{j=1}^{r-1} (T[r-j] - T[r-j-1]) * \left(\sum_{i=0}^{n-1} p_i(x_r) A_n^{\alpha}(i) b_j^{\alpha_{n-i}} \right)$$

$$\begin{aligned} B_m^{\beta}[r] = 2 &\sum_{p=1}^{(r-1)/2} \sum_{k=0}^{1} \sum_{j=0}^{2p-1-k} w(k)\, (T[2p-j-k] - T[2p-j-k-1]) \\ &* Q_m^{m-1}[r, 2p-k, j] \\ &+ 2 \sum_{p=2}^{(r-1)/2} \sum_{j=0}^{2p-3} (T[2p-j-2] - T[2p-j-3])\, Q_m^{m-1}[r, 2p-2, j]. \end{aligned}$$

d1. Compute $y_r = y(x_r)$ for each $r = 1,2,3, \dots, N$ and odd, from:

$$\begin{aligned} (H_n^{\alpha}(r) - Q_m^m[r,r,0]) y_r &= \left(H_n^{\alpha}(r) - \mathcal{P}_n(x_r) + \frac{\lambda h}{2} \mathcal{K}_m(x_r, x_{r-1}) - Q_m^{m-1}[r,r,0] \right) T[r-1] \\ &+ B_{n,m}^{\alpha,\beta}[r] + R_m[r] + L_m[r] + f(x_r) \end{aligned}$$

with putting $R_m[1] = 0$; $L_m[1] = 0$ and $B_{n,m}^{\alpha,\beta}[1] = 0$

b2. If r is even, Evaluate:

$$Q_m^k[r,p,j] = \frac{\lambda h}{3} \sum_{\ell=0}^{k} \mathcal{K}_\ell\left(x_r, x_p\right) A_m^{\beta}(\ell) b_j^{\beta_{m-\ell}}$$

c2. Calculate $B_{n,m}^{\alpha,\beta}[r] = B_n^{\alpha}[r] + B_m^{\beta}[r]$, where

$$B_n^{\alpha}[r] = -\sum_{j=1}^{r-1}(T[r-j] - T[r-j-1]) * \left(\sum_{i=0}^{n-1} p_i(x_r) A_n^{\alpha}(i) b_j^{\alpha_{n-i}}\right)$$

and $$B_m^{\beta}[r] = \sum_{j=1}^{r-1}(T[r-j] - T[r-j-1]) * Q_m^{m-1}[r,r,j] +$$

$$\sum_{k=0}^{1}\sum_{p=1}^{r/2+k-1}\sum_{j=0}^{2p-k-1} w(k)\ (T[2p-j-k] - T[2p-j-k-1]) Q_m^{m-1}[r,2p-k,j]$$

$$+\sum_{p=1}^{r/2-1}\sum_{j=0}^{2p-1}(T[2p-j] - T[2p-j-1]) * Q_m^{m-1}[r,2p,j]$$

$$L_m[r] = \frac{\lambda h}{3}\left\{\sum_{p=1}^{r/2-1}\mathcal{K}_m(x_r, x_{2p})T[2p] + 4\sum_{p=1}^{r/2}\mathcal{K}_m(x_r, x_{2p-1})T[2p-1] + \sum_{p=1}^{r/2}\mathcal{K}_m(x_r, x_{2p-2})T[2p-2]\right\}$$

d1. Compute $y_r = y(x_r)$ for each $r = 1,2,\dots,N$ and even, from

$$(H_n^{\alpha}(r) - Q_m^m[r,r,0])y_r = (H_n^{\alpha}(r) - \mathcal{P}_n(x_r) - Q_m^{m-1}[r,r,0])T[r-1] + B_{n,m}^{\alpha,\beta}[r] + L_m[r] + f(x_r)$$

Where $b_\theta^{\sigma} = (1+\theta)^{1-\sigma} - \theta^{1-\sigma}$, $0 < \sigma \le 1$ and $\theta = 0,1,2,\dots$

5. Numerical Examples:

Here, some numerical results were presented for linear VIFDE of variable coefficients using Quadrature methods: Trapezoidal and Simpson's rules. Their results are obtained applying algorithms ***AVIFT*** and ***AVIFS*** respectively.

Example (1):

We consider a multi-order linear VIFDE with variable coefficients:

$$ {}_0^C D_x^{0.4} y(x) + x\,{}_0^C D_x^{0.2} y(x) - 2y(x) = f(x) + \int_0^x [(x-2t^2)\,{}_0^C D_t^{0.2} y(t) + (tx^2-1)y(t)]dt$$

where $$f(x) = \frac{2}{3}x^5 - \frac{1}{2}x^4 - x^2 + 5x - 2 - \frac{100}{\Gamma(0.8)}\left(\frac{1}{76}x^2 - \frac{1}{72}x + \frac{1}{40}\right)x^{1.8} - \frac{10}{3\Gamma(0.6)}x^{0.6}$$

together with initial condition: $y(0) = 1$; $0 \le x \le 1$

while the exact solution is $y(x) = 1 - 2x$

which is a linear VIFDE's with variable coefficients for various fractional order between 0 and 1 : Take $N = 10$, $h = 0.1$ and $x_r = a + rh$, $r = 0,1,2, \dots, N$. From equations (10) and (11) , where $n = 2$ and $m = 1$ with $\alpha_2 = 0.4$, $\alpha_1 = 0.2$, $\alpha_0 = 0$; $\beta_1 = 0.2$, $\beta_0 = 0$, by running the programs VIFT and VIFS the following obtained:

$$A_2^{\alpha}(0) = 2.8112403816 \quad , \quad A_2^{\alpha}(1) = 1.7016542932 \quad , \quad A_2^{\alpha}(2) = 1$$

$$A_1^{\beta}(0) = 1.7016542932 \quad , \quad A_1^{\beta}(1) = 1$$

with

$$\mathcal{H}_2^{\alpha}(r) = \mathcal{P}_2(x_r) + \sum_{s=0}^{1} \mathcal{P}_s(x_r) A_2^{\alpha}(s) \quad ; \quad r = \overline{1\colon N}$$

Table (1) contain all values of $\mathcal{H}_2^{\alpha}$ for each $x_r = 0.1(0.1)1$ for $r = 0,1,2, \dots, 10$.

Table (1)

x_r	0.1	0.2	0.3	0.4	0.5
$\mathcal{H}_2^{\alpha}(r)$	0.9814058109	1.1515712402	1.3217366696	1.4919020989	1.6620675282
x_r	0.6	0.7	0.8	0.9	1.0
$\mathcal{H}_2^{\alpha}(r)$	1.8322329575	2.0023983868	2.1725638161	2.3427292455	2.5128946748

Table (2) presents a comparison between the exact solution and numerical solution of the two types of quadrature methods: Trapezoidal and Simpson's rules for $y(x)$ depending on least square error and running time.

Table (2)

Quadrature Methods			
x_r	**Exact**	**VIFT**	**VIFS**
0	1	1	1
0.1	0.8	0.800338635492	0.800338635492
0.2	0.6	0.601201996421	0.600504169622
0.3	0.4	0.402620488654	0.401192790813
0.4	0.2	0.204527187232	0.201450510459
0.5	0	0.006790955582	0.002296902597
0.6	-0.2	-0.190744353036	-0.19748422478
0.7	-0.4	-0.388222891688	-0.396652774768
0.8	-0.6	-0.585748774149	-0.596542492442
0.9	-0.8	-0.783370182785	-0.795780128674
1	-1	-0.98107523516	-0.995740402573
$\boldsymbol{L.S.E}$		$0.113720\ e-002$	$0.746102\ e-004$
$\boldsymbol{R.Time/Sec}$		5.985896	0.587096

Table (3) lists the least square errors and running times for quadrature methods with different values of steps size h .

Table (3)

h / QM	0.1		0.02		0.01		0.002	
	L.S.E	R.Time /Sec	L.S.E	R.Time /Sec	L.S.E	R.Time /Sec	L.S.E	R.Time /Sec
VIFT	$0.113720e-002$	0.528507	$0.104468e-004$	4.759228	$0.154297e-005$	17.67052	$0.196154e-007$	460.0662
VIFS	$0.746102e-004$	0.613206	$0.222049e-006$	4.820356	$0.287381e-007$	17.87806	$0.366407e-009$	464.9959

Example (2):

Consider a multi-order linear VIFDE's for fractional order lies in (0,1) on closed interval $[0,1]$:

$$ {}^{C}D_x^{0.6}y(x)+(x-1)y(x)=f(x) + \int_0^x \left[(x^2+t)\,{}^{C}D_t^{0.7}y(t)+(1-xt)\,{}^{C}D_t^{0.4}y(t)-\sin(x-t)\,y(t)\right]dt $$

where $$ f(x)=2\cos x+x^3-2+\frac{2}{\Gamma(2.4)}\,x^{1.4}-\frac{1}{5\Gamma(4.3)}\,[33x+23]x^{3.3} - \frac{2}{\Gamma(4.6)}\,[18-13x^2]x^{2.6} $$

Together with initial condition: $y(0)=0$; while the exact solution is : $y(x)=x^2$.

Take $N=10$ and $h=0.1$. use equations (10 and 11) where $n=1$ and $m=2$ and by applying algorithms ***AVIFT*** and ***AVIFS*** for $\alpha_1=0.6$, $\alpha_0=0$ and $\beta_2=0.7$, $\beta_1=0.4$, $\beta_0=0$ we obtain :

$$ A_1^{\alpha}(0)=4.486908659 \quad , \quad A_1^{\alpha}(1)=1 $$

$$ A_2^{\beta}(0)=5.5844412045 \quad , \quad A_2^{\beta}(1)=2.8112403816 \quad , \quad A_2^{\beta}(2)=1 $$

with tabulate all values of $\mathcal{H}_1^{\alpha}(r=\overline{1:N})$ in the table (4) as follows :

Table (4)

x_r	0.1	0.2	0.3	0.4	0.5
$\mathcal{H}_1^{\alpha}(r)$	3.586908659	3.6869086589	3.786908658	3.8869086589	3.9869086589
x_r	0.6	0.7	0.8	0.9	1.0
$\mathcal{H}_1^{\alpha}(r)$	4.0869086589	4.1869086589	4.2869086589	4.3869086589	4.4869086589

Table (5) shows a comparison between the exact solution and numerical solution using Trapezoidal and Simpson's Methods for $y(x)$ depending on least square error and running time.

Table (5)

Quadrature Methods			
x_r	Exact	VIFT	VIFS
0	0	0	0
0.1	0.01	0.0156473327586	0.0156473327586
0.2	0.04	0.0508925732614	0.0506043858389
0.3	0.09	0.106122509597	0.105576670662
0.4	0.16	0.18139928134	0.180315272293
0.5	0.25	0.276688204177	0.275072955583
0.6	0.36	0.391906666315	0.389458522819
0.7	0.49	0.526936452023	0.523612625074
0.8	0.64	0.681623803088	0.677083764842
0.9	0.81	0.855772988398	0.849885667857
1	1	1.04913456954	1.04148654673
$L.S.E$		$0.102049\ e-001$	$0.811316e-002$
$R.Time/Sec$		0.419637	0.431129

The result in table (6) shows the least square errors and running times for quadrature methods with different values of steps size h .

Table (6)

h / QM	0.1		0.02		0.01		0.002	
	L.S.E	R.Time /Sec	L.S.E	R.Time /Sec	L.S.E	R.Time /Sec	L.S.E	R.Time /Sec
VIFT	$0.102049\ e-001$	0.411406	$0.481241\ e-003$	2.661740	$0.132383\ e-003$	9.438810	$0.644849\ e-005$	225.6094
VIFS	$0.811316\ e-002$	0.431129	$0.437695\ e-003$	2.775692	$0.124180\ e-003$	9.546942	$0.628550\ e-005$	230.0403

6. Discussion:

In this chapter, two numerical algorithms have been applied to solve the linear VIFDEs with variable coefficients containing: trapezoidal and Simpson's rules with aid of forward finite difference scheme for Caputo derivative for each algorithm a computer program was written and several examples are included for illustration.

For the comparison of computing a accuracy and the speed, the least square error and running time are also given in tabular forms, we conclude the following points:

1. With equal step sizes, the ***VIFS*** algorithm gives better accuracy than ***VIFT*** algorithm.
2. The accuracy of the results depends on the method used as well as the step length h ,i.e. as we reduce h , the accuracy is increased (see tables (2) and (3)).
3. We can classify these two algorithms as multi- step methods.

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