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PLANT-PEST-NATURAL ENEMY DYNAMICS WITH DISEASE IN PEST AND GESTATION DELAY FOR NATURAL ENEMY

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Abstract. In this present study, a food chain system with the disease in pest species and gestation delay for the natural enemy is proposed. Here the boundedness and positivity of the system are studied. Stability analysis for all feasible equilibrium points is carried out. The Hopf bifurcation at interior equilibrium point is presented and analyzed. The sensitivity analysis is performed to find the respective sensitive indices of the variables of the proposed system. Further, simulations have been carried out to support our analytic results.

Keywords: a food chain model; disease in pest; gestation delay; boundedness; positivity; Hopf bifurcation; sensitivity study.

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1. Introduction

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Since pest species are harmful to plants and their control has become a challenge for us. Pest population is responsible for severe environmental and realistic problems. [1, 3]. Also, many authors have discussed the models based on chemical pesticides, which are less harmful to humanity and environment [2, 4, 5, 6, 7]. For productive use of biological or natural methods to manage pest populations, without any adverse effects, it is essential to understand the biology of beneficial species or natural enemy and pests [8]. Our most important aim is to control negative impacts of agriculture pests, for both humanity and agriculture, which harms the environment and generating different types of pollution. Researchers must have to produce, the natural systems to control pests by taking into account the communications between solid Allee effect in pests with natural methods: alternative food support for the natural enemy, introduction of infected pests to control healthy pests[9, 10] The interactions between pests and natural enemies in the same biological environment is an ample exciting area of research as per Lotka and Volterra. Natural enemies are more vulnerable to the infected pest since infectious pest population is weak and less active. Therefore natural enemy efficiently harvests pests. Due to the interaction between infected pests and natural enemy, the natural enemies must be infected. Hence natural enemy populations may live on other food resources for their growth and survival. Also, the species do not grow instantaneously; some time is taken by the species to give a new generation, called gestation lag period [11]. Functional responses play an important role to develop a predator-prey system in population dynamics. Various factors like hiding technique of pests from the natural enemy, shooting ability of the predator to harvest insect, etc., have a large influence on functional responses. Functional responses are of different types: for example, Holling type I-III, etc. Also, people are more conscious and choose, the modern methods to manage agricultural pests, for example, less harmful chemical pesticides and natural techniques[12, 13, 14, 15, 16], whereas biological techniques are simple and safer to control pests than pesticide practices. Also time lag factors are of great significance to produce population models and used by numerous authors[17, 18, 19, 20, 22, 23, 24, 25, 26, 27]. According to many authors, models with continuous lag factors are practical[22] than instant delays[27].

In the light of above literature survey, here, dynamics of a food chain model with infection in pest species and gestation delay for the natural enemy is proposed and analyzed. The model is represented as follows: Section 1, consists of an introduction. The proposed mathematical system is presented, in section 2. In section 3, the boundedness and positivity of the model have been discussed. Equilibrium points and their stability analysis is investigated for possible steady states, in section 4. The sensitivity analysis of the system at interior equilibrium point for system parameters is presented, in section 5. In section 6, numerical simulations have been carried out to support our analytic results. Finally, the results have been concluded in the conclusion section.

2. The Proposed Mathematical System

The assumptions of the proposed model are:

(i) In a particular habitat, there are four types of populations, namely, plant $X(t)$, healthy pest $P_h(t)$, infected pest $P_i(t)$ and natural enemy$N(t)$.

(ii) Plants grow logistically with α as the intrinsic growth rate and k being carrying capacity. Thus, when the system is free from pest population, plants grow with rate $\alpha X\left(1-\frac{X}{k}\right)$.

(iii)Plants are harvested by healthy pests with Holling type-I, response function.

(iv) Pests can hide from the natural enemy, hence the natural enemy harvesting pests with Holling type-II response function.

(v) Let β be the predation rate of the plant by healthy pest; β_1 is the conversion rate for healthy pest; γ is the contact rate of infected pest with healthy pest; δ is the harvesting rate of healthy pests by the natural enemy. Let a be the half saturation constant, and δ_1 be the predation rate of the infected pest by the natural enemy. Let δ_2 be the conversion rate for the natural enemy; μ_1, μ_2 and μ_3 are the natural death rates for healthy pest, infected pest, and natural enemy respectively. Let η be the alternative food resource for the growth of natural enemies. Natural enemies die with rate η_1 due to consumption of infected pest.

(vi) Finally, τ is the gestation delay for the natural enemy.

Keeping in view the assumptions and interactions, our proposed system is of the form:

$$\frac{dX}{dt} = \alpha X\left(1-\frac{X}{k}\right) - \beta X P_h, \tag{1}$$

$$\frac{dP_h}{dt} = \beta_1 X P_h - \gamma P_h P_i - \frac{\delta P_h N(t-\tau)}{a+P_h} - \mu_1 P_h, \tag{2}$$

$$\frac{dP_i}{dt} = \gamma P_h P_i - \delta_1 P_i N - \mu_2 P_i, \tag{3}$$

$$\frac{dN}{dt} = \frac{\delta_2 \delta P_h N(t-\tau)}{a+P_h} + \eta N - \eta_1 P_i N - \mu_3 N, \tag{4}$$

with initial conditions: $X(0) > 0, P_h(0) > 0, P_i(0)$ and $N(0) > 0$.

3. Positivity and boundedness

Here, the positivity and boundedness of solution of the system $(1)-(4)$, is discussed with the help of following lemmas:

Lemma 3.1. *The solution of the mathematical system* $(1)-(4)$*, with non-negative initial populations for all* $t \geq 0$.

Proof. Let the solution of the proposed system $(1)-(4)$ with non-negative initial populations be $(X(t), P_h(t), P_i(t), N(t))$. For $t \in [0,\tau]$, the equation (1) may be written as:

$$\frac{dX}{dt} \geq -\frac{\alpha X^2}{k} - \beta X P_h,$$

it follows that

$$X(t) \geq \frac{\exp\left\{-\int_0^t (\beta P_h)\, du\right\}}{X(0) + \int_0^t \frac{\alpha}{k} \exp\left\{-\int_0^t (\beta P_h)\, du\right\} dv} > 0.$$

For $t \in [0,\tau]$, the equation (2) of system can be written as

$$\frac{dP_h}{dt} \geq -\gamma P_h P_i - \frac{\delta P_h N}{a+P_h} - \mu_1 P_h,$$

which evidences that

$$P_h(t) \geq \frac{\exp\left\{-\int_0^t (\mu_1)\, du\right\}}{P_h(0) + \int_0^t \frac{-\gamma P_i(a+P_h)-\delta N}{P_h(a+P_h)} \exp\left\{-\int_0^t (\mu_1)\, du\right\} dv} > 0.$$

The equation (3) of model, for $t \in [0,\tau]$ may be represented as

$$\frac{dP_i}{dt} \geq -\delta_1 P_i N - \mu_2 P_i,$$

which implies that

$$P_i(t) \geq P_i(0)\exp\left\{-\int_0^t (\delta_1 N + \mu_2)du\right\} > 0.$$

From equation (4), for $t \in [0,\tau]$, we have

$$\frac{dN}{dt} \geq -\eta_1 P_i N - \mu_3 N,$$

which results that

$$N(t) \geq N(0)\exp\left\{-\int_0^t (\eta_1 P_i + \mu_3)du\right\} > 0.$$

Clearly, $X(t) > 0,\ P_h(t) > 0,\ P_i(t) > 0$ and $N(t) > 0$ for all $t \geq 0$, by induction.

Lemma 3.2. *The solution of proposed model* $(1)-(4)$ *is uniformly bounded in* Ω*, where*

$$\Omega = \left\{(X,P_h,P_i,N) : 0 \leq X(t)+P_h(t)+P_i(t)+N(t) \leq \frac{(\alpha+\mu)^2 k}{4\alpha\mu}\right\},$$

$\mu = min\{\mu_1,\mu_2,-(\mu_3-\eta)\},\ \beta_1 << \beta,\ \delta_2 << \delta.$

Proof. Let $V(t) = X(t)+P_h(t)+P_i(t)+N(t)$. Now, differentiating $V(t)$ w.r.t. t, we have

$$\frac{dV(t)}{dt} = \alpha X\left(1-\frac{X}{k}\right) - \mu_1 P_h - \delta_1 P_i N - \mu_2 P_i + \eta N - \eta_1 P_i N - \mu_3 N.$$

As alternate food resource η, for natural enemy is limited, so assuming η is small, we have

$$\frac{dV(t)}{dt} + \mu V \leq \frac{(\alpha+\mu^2)k}{4\alpha}.$$

Therefore,

$$0 \leq V(t) \leq V(0)e^{-ut} + \frac{(\alpha+\mu)^2 k}{4\alpha\mu}.$$

As $t \to \infty$, we have

$$0 \leq V(t) \leq \frac{(\alpha+\mu)^2 k}{4\alpha\mu}.$$

Hence, $V(t)$ is bounded, i.e., the proposed system is bounded.

4. Equilibrium points and their stability analysis

The system of equations $(1)-(4)$ have six feasible equilibrium points:

(i) The equilibrium point $E_0(0,0,0,0)$ always exists.

(ii) The equilibrium point $E_1(k,0,0,0)$ exists.

(iii) The equilibrium point $E_2(X_2, P_{h2}, 0, 0)$ exists only when $k\beta_1 > \mu_1$, where $X_2 = \frac{\mu_1}{\beta_1}, P_{h2} = \frac{\alpha(1-\frac{\mu_1}{k\beta_1})}{\beta}$.

(iv) The natural enemy free equilibrium $E_3(X_3, P_{h3}, P_{i3}, 0)$ exists only when $\gamma > max\{\frac{\beta\mu_2}{\alpha}, \frac{k\beta\beta_1\mu_2}{(k\beta_1-\mu_1)\alpha}\}$, where $X_3 = k - \frac{k\beta\mu_2}{\alpha\gamma}$, $P_{h3} = \frac{\mu_2}{\gamma}$, $P_i = \frac{-\alpha\gamma\mu_1+k\beta_1(\alpha\gamma-\beta\mu_2)}{\alpha\gamma^2}$.

(v) The equilibrium $E_4(X_4, P_{h4}, 0, N_4)$ exists only when $\frac{\alpha}{\beta} > max\{\frac{a(-\eta+\mu_3)}{\eta+\delta\delta_2-\mu_3}, \frac{ak\beta_1(-\eta+\mu_3)}{(k\beta_1-\mu_1)(\eta+\delta\delta_2-\mu_3)}\}$ and $\eta < \mu_3 < \eta + \delta\delta_2$; where $X_4 = k + \frac{ak\beta}{\alpha} - \frac{ak\beta\delta\delta_2}{\alpha\eta+\alpha\delta\delta_2-\alpha\mu_3)}$, $P_{h4} = \frac{a(-\eta+\mu_3)}{\eta+\delta\delta_2-\mu_3}$, $N_4 = \frac{a\delta_2(k\beta_1(\alpha\delta\delta_2(\alpha+a\beta)(\eta-\mu_3))-\alpha\mu_1(\eta+\delta\delta_2-\mu_3))}{\alpha(\eta+\delta\delta_2-\mu_3)^2}$.

(vi) Interior equilibrium point $E^*(X^*, P_h^*, P_i^*, N^*)$ exists, where X^*, P_h^*, P_i^*, N^* are given by

$$\begin{pmatrix} \alpha\left(1-\dfrac{X^*}{k}\right) - \beta P_h^* = 0, \\ \beta_1 X^* - \gamma P_i^* - \dfrac{\delta N^*}{a+P_h^*} - \mu_1 = 0, \\ \gamma P_h^* - \delta_1 N^* - \mu_2 = 0, \\ \dfrac{\delta\delta_2 P_h^*}{a+P_h^*} + \eta - \eta_1 P_i^* - \mu_3 = 0. \end{pmatrix} \tag{5}$$

Now, the local behavior of non-negative equilibria of the system $(1)-(4)$ is as follows: using the lemmas [11, 28], for the transcendental polynomials. For the transcendental polynomial equation of first degree of the form

$$\lambda + r + qe^{-\lambda\tau} = 0, \tag{6}$$

we will verify the following conditions:

$(A1)$ $q + r > 0$;

$(A2)$ $r^2 - q^2 > 0$;

$(A3)$ $r^2 - q^2 < 0$.

Now, we will state the following lemmas similar to [11, 28]:

Lemma 4.1. *For equation* (6)*;*

(i) If $(A1)-(A2)$ holds, then all the roots of equation (6) have negative real parts for all $\tau \geq 0$.

(ii) If $(A1)$and$(A3)$ hold and $\tau = \tau_j^+$, then equation (6) has a pair of purely imaginary roots $\pm iw_+$. When $\tau = \tau_j^+$ then all roots of (6) except $\pm iw_+$ have negative real parts.

Now, for second degree polynomial equation

$$\lambda^2 + p\lambda + r + (s\lambda + q)e^{-\lambda\tau} = 0, \tag{7}$$

we will check the following relations:

$(B1)$ $p + s > 0$;

$(B2)$ $q + r > 0$;

$(B3)$ either $s^2 - p^2 + 2r < 0$ and $r^2 - q^2 > 0$ or $(s^2 - p^2 + 2r)^2 < 4(r^2 - q^2)$;

$(B4)$ either $r^2 - q^2 < 0$ or $s^2 - p^2 + 2r > 0$ and $(s^2 - p^2 + 2r)^2 = 4(r^2 - q^2)$;

$(B5)$ either $r^2 - q^2 > 0$, $s^2 - p^2 + 2r > 0$ and $(s^2 - p^2 + 2r)^2 > 4(r^2 - q^2)$.

Lemma 4.2. [11, 28] *For equation (7);*

(i) *If $(B1)-(B3)$ holds, then all the roots of (7) have negative real parts for all $\tau \geq 0$.*

(ii) *If $(B1)$, $(B2)$ and $(B4)$ hold and $\tau = \tau_j^+$, then equation (7) has a pair of purely imaginary roots $\pm iw_+$. When $\tau = \tau_j^+$ then all roots of (7) except $\pm iw_+$ have negative real parts.*

(iii) *If $(B1)$, $(B2)$ and $(B5)$ hold and $\tau = \tau_j^+$($\tau = \tau_j^-$ respectively) then equation (7) has a pair of purely imaginary roots $\pm iw_+$ ($\pm iw_-$, respectively). Furthermore $\tau = \tau_j^+$ (τ_j^-, respectively),then all roots of (7) except $\pm iw_+$ ($\pm iw_-$, respectively) have negative real parts.*

Theorem 4.3. *The local behavior of different equilibrium points of the system $(1)-(4)$ is as follows:*

(i) *The equilibrium point $E_0(0,0,0,0)$ is always exist and unstable.*

(ii) *The equilibrium point $E_1(k,0,0,0)$ is stable only when $\mu_1 > k\beta_1$.*

Proof. (i) The characteristic equation for $E_0(0,0,0,0)$ is

$$(-\lambda + \alpha)(-\lambda - \mu_1)(-\lambda - \mu_2)(-\lambda + \eta - \mu_3) = 0. \tag{8}$$

Here, the characteristic roots are $\lambda = \alpha$, $\lambda = -\mu_1$, $\lambda = -\mu_2$ and $\lambda = -\mu_3$. The equilibrium $E_0(0,0,0,0)$ is always unstable, since one of the characteristic roots, i.e., $\lambda = \alpha$, of (8) is positive.

(ii)The characteristic equation for $E_1(k,0,0,0)$ is

$$(-\lambda-\alpha)(-\lambda-\mu_1+k\beta_1)(-\lambda-\mu_2)(-\lambda-\mu_3)=0. \tag{9}$$

The characteristic roots are $\lambda = -\alpha$, $\lambda = k\beta_1-\mu_1$, $\lambda = -\mu_2$ and $\lambda = -\mu_3$. Hence, the equilibrium point $E_1(k,0,0,0)$ is stable only when $\mu_1 > k\beta_1$.

Theorem 4.4. *For the system* $(1)-(4)$*, if* $\mu_3 > \eta+\delta\delta_2\frac{P_h}{a+P_h}$*,* $\eta < min\{\mu_3+\delta\delta_2\frac{P_h}{a+P_h}, \mu_3-\delta\delta_2\frac{P_h}{a+P_h}\}$*,* $\gamma P_h < \mu_2$ *and* $k\beta_1 > \mu_1$ *hold, then the equilibrium* $E_2(X_2,P_{h2},0,0)$ *is locally asymptotically stable for all* τ*, unstable otherwise.*

Proof. The characteristic equation at E_2 may be written as:

$$(-\lambda-\mu_2+\gamma P_h)F_1(\lambda)F(\lambda)=0, \tag{10}$$

where

$$F_1(\lambda)=\lambda^2+\frac{\alpha\mu_1}{k\beta_1}\lambda+\alpha\mu_1\left(1-\frac{\mu_1}{k\beta_1}\right), \tag{11}$$

and

$$F(\lambda)=-\lambda-\mu_3+\eta+\delta\delta_2\frac{P_h}{a+P_h}e^{-\lambda\tau}. \tag{12}$$

The one eigen value of equation (10) is $\lambda = -(\mu_2-\gamma P_h)$ and other two eigen values are obtained from $F_1(\lambda)=0$, implies $\lambda^2+\frac{\alpha\mu_1}{k\beta_1}\lambda+\alpha\mu_1\left(1-\frac{\mu_1}{k\beta_1}\right)=0$; clearly by Routh-Hurwitz's criteria and using the existence condition, $k\beta_1 > \mu_1$ of the equilibrium point $E_2(X_2,P_{2h},0,0)$, the roots of $F_1(\lambda)=0$ are negative. Also we have $F(\lambda)=-\lambda-\mu_3+\eta+\delta\delta_2\frac{P_h}{a+P_h}e^{-\lambda\tau}=0$. On comparing with equation (6), here $r=\mu_3-\eta$, $q=-\delta\delta_2\frac{P_h}{a+P_h}$. It is observed that (**A1**) is hold only when $\mu_3 > \eta+\delta\delta_2\frac{P_h}{a+P_h}$ and subsequently it will satisfy (**A2**) i.e. $\eta < min\{\mu_3+\delta\delta_2\frac{P_h}{a+P_h}, \mu_3-\delta\delta_2\frac{P_h}{a+P_h}\}$. Hence by using Lemma 4.1., the equilibrium point, $E_2(X_2,P_{h2},0,0)$, is stable only when $\mu_3 > \eta+\delta\delta_2\frac{P_h}{a+P_h}$, $\eta < min\{\mu_3+\delta\delta_2\frac{P_h}{a+P_h}, \mu_3-\delta\delta_2\frac{P_h}{a+P_h}\}$, $\gamma P_h < \mu_2$ and $k\beta_1 > \mu_1$ hold and unstable, otherwise .

Theorem 4.5. *Let* (S_1) *holds, for the system* $(1)-(4)$*, then the equilibrium* E_3 *is locally asymptotically stable for all* τ*.*

Proof. The characteristic equation at the equilibrium point E_3 may be written as:

$$G(\lambda)G_1(\lambda) = 0, \tag{13}$$

where $G(\lambda) = \lambda^3 - \lambda^2(a_1 + b_1 + c_2) + \lambda(a_1b_1 + a_1c_2 + b_1c_2 - b_2c_1 - a_2a_3) + (a_1b_2c_1 + a_2a_3c_2 - a_1b_1c_2) = 0$; $G_1(\lambda) = \lambda - d_2 - d_1e^{-\lambda\tau}$ and $a_1 = -\frac{2\alpha X}{k} + \alpha - \beta P_h,\ a_2 = -\beta X,\ a_3 = \beta_1 P_h,\ b_1 = \beta_1 X - \gamma P_i - \mu_1,\ b_2 = -\gamma P_h,\ b_3 = -\frac{\delta P_h}{a+P_h},\ c_1 = \gamma P_i, c_2 = \gamma P_h - \mu_2, c_3 = -\delta_1 P_i,\ d_1 = \delta\delta_2\frac{P_h}{(a+P_h)}$ and $d_2 = \eta - \eta_1 P_i - \mu_3$. Now, from relations $G_1(\lambda) = 0$ and $G(\lambda) = 0$, we proceed as follows: When $\tau = 0$, from transcendental polynomial $G_1(\lambda) = \lambda - d_2 - d_1e^{-\lambda\tau}$=0, we obtain, $\lambda = d_2 + d_1$, if negative, i.e., one eigen value of the equation (13) is negative. For remaining three eigen values, the equation $G(\lambda) = 0$ can be written as

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \tag{14}$$

where, $A_1 = -a_1 - b_1 - c_2$, $A_2 = a_1b_1 + a_1c_2 + b_1c_2 - b_2c_1 - a_2a_3$, $A_3 = a_1b_2c_1 + a_2a_3c_2 - a_1b_1c_2$. By Routh-Hurwitz criteria, all the roots of equation (14) have negative real parts and the equilibrium E_3 is locally asymptotically stable for all τ, if (S_1): $A_1, A_2, A_3 > 0$ and $A_1A_2 - A_3 > 0$ holds.

Theorem 4.6. *Let (S_2) holds, for the system $(1)-(4)$, then the equilibrium point E_4 is locally asymptotically stable for all τ.*

Proof. The characteristic equation of the variational matrix at E_4 may be represented as:

$$(\gamma P_h - \delta_1 N - \mu_2 - \lambda)(G_2(\lambda)) = 0, \tag{15}$$

where $G_2(\lambda) = (\lambda^3 + A\lambda^2 + B\lambda + C) + (F\lambda^2 + E\lambda + D)e^{-\lambda\tau}$ and $A = -a_1 - b_1 - d_2$, $B = a_1b_1 + a_1d_2 + b_1d_2 - a_2a_3, C = a_2a_3d_2 - a_1b_1d_2, D = a_2a_3d_1 + c_2b_3 - a_1b_1d_1, E = a_1d_1 + b_1d_1,$ $F = -d_1$ and $a_1 = -\frac{2\alpha X}{k} + \alpha - \beta P_h, a_2 = -\beta X, a_3 = \beta_1 P_h, b_1 = \beta_1 X - \frac{\delta Na}{(a+P_h)^2} - \mu_1, b_2 = -\gamma P_h,$ $b_3 = -\frac{\delta P_h}{a+P_h}, c_1 = \gamma P_h - \delta_1 N - \mu_2, c_2 = \frac{\delta\delta_2 Na}{(a+P_h)^2}, c_3 = -\eta_1 N, d_1 = \frac{\delta\delta_2 P_h}{a+P_h}, d_2 = \eta - \mu_3.$
When $\tau = 0$, then from equation (15) we have, i.e.,

$$G_2(\lambda) = 0 \Rightarrow \lambda^3 + A_{11}\lambda^2 + A_{22}\lambda + A_{33} = 0, \tag{16}$$

$A_{11} = -a_1 - b_1 - d_2 - d_1$, $A_{22} = a_1b_1 + a_1d_2 + b_1d_2 - a_2a_3 + a_1d_1 + b_1d_1$ and $A_{33} = a_2a_3d_2 - a_1b_1d_2 + a_2a_3d_1 + c_2b_3 - a_1b_1d_1$. Moreover from equation (15), if $(\gamma P_h - \delta_1 N - \mu_2 - \lambda) < 0$ implies that $\gamma P_h < \mu_2 + \delta_1 N$, i.e., one of the eigen value of equation (15) is negative. Also, by Routh-Hurwitz criterion, all the roots of equation (16) have negative real parts, if (S_2): $A_{11}, A_{22}, A_{33} > 0$ and $A_{11}A_{22} - A_{33} > 0$ holds. Therefore, the steady state E_4 is locally asymptotically stable for all τ.

Theorem 4.7. *Let* (S_3) *holds, for the system* $(1)-(4)$*, then interior equilibrium* E^* *is locally asymptotically stable for all* $\tau \in (0, \tau_0^+)$*. If* $\tau \geq \tau_0^+$*, then the interior equilibrium* E^* *is unstable and undergoes Hopf bifurcation.*

Proof. The characteristic equation of the variational matrix at E^* may represented as:

$$(\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D) + (E\lambda^3 + F\lambda^2 + G\lambda + H)e^{-\lambda\tau} = 0, \tag{17}$$

where $A = -a_1 - b_1 - d_3$, $B = a_1b_1 + a_1d_3 - c_2d_1 - b_2c_1 + b_1d_3 - a_2a_3$, $C = a_1c_2d_1 + a_1b_2c_1 - a_1b_1d_3 - b_2c_2c_3 + b_1c_2d_1 + b_2d_3c_1 + a_2a_3d_3$, $D = a_1b_2c_2c_3 - a_1b_1c_2d_1 - a_1b_2d_3c_1 + a_2c_2d_1a_3$, $E = -d_2$, $F = a_1d_2 - c_3b_3 + b_1d_2$, $G = a_1c_3b_3 - a_1b_1d_2 - b_3c_1d_1 + b_2c_1d_2 + a_2a_3d_2$, $H = a_1b_3c_1d_1 - a_1b_2c_1d_2$, and $a_1 = \frac{-\alpha X^*}{k}$, $a_2 = -\beta X^*$, $a_3 = \beta_1 P_h^*$, $b_1 = \beta_1 X^* - \gamma P_i^* - \delta N^* \frac{a}{(a+P_h^*)^2} - \mu_1$, $b_2 = -\gamma P_h^*$, $b_3 = \frac{-\delta P_h^*}{a+P_h^*}$, $c_1 = \gamma P_i^*$, $c_2 = -\delta_1 P_i^*$, $c_3 = \delta\delta_2 N^* \frac{a}{(a+P_h^*)^2}$, $d_1 = -\eta_1 N^*$, $d_2 = \delta\delta_2 \frac{P_h^*}{a+P_h^*}$, $d_3 = \eta - \eta_1 P_i^* - \mu_3$.

Case I: When $\tau = 0$, the equation (17) reduces to

$$\lambda^4 + (A+E)\lambda^3 + (B+F)\lambda^2 + (C+G)\lambda + (D+H) = 0, \tag{18}$$

i.e.,

$$\lambda^4 + A_{111}\lambda^3 + A_{222}\lambda^2 + A_{333}\lambda + A_{444} = 0, \tag{19}$$

where $A_{111} = A+E$, $A_{222} = B+F$, $A_{333} = C+G$, $A_{444} = D+H$. By Routh-Hurwitz Criterion, all the roots of equation (18) have negative real parts, if (S_3): A_{111}, A_{222}, A_{333}, $A_{444} > 0$ and $A_{111}A_{222}A_{333} - A_{333}^2 - A_{111}^2 A_{444} > 0$ holds. Thus the steady state E^* is locally asymptotically stable for all τ.

Case II: If $\tau > 0$, due to complexity and lack of tools and techniques, the transcendental equation (17) can not be solved analytically, it can be discussed numerically only in the numerical section. This completes the proof.

5. Sensitivity Analysis

In this section, the sensitivity analysis of the system $(1)-(4)$ at the interior equilibrium point is carried out. The respective sensitive parameters of the state variables of the system at interior equilibrium point are given in the Table 1, using the values of parameters: $\alpha = 0.2$; $k = 5$; $\beta = 0.05$; $\beta_1 = 0.1$; $\gamma = 0.3$; $\delta = 0.01$; $a = 0.9$; $\mu_1 = 0.0002$; $\delta_1 = 0.04$; $\mu_2 = 0.01$; $\delta_2 = 0.3$; $\eta = 0.01$; $\eta_1 = 0.02$; $\mu_3 = 0.001$. It is clear that α, k, γ, δ, μ_1, δ_2, η have a positive impact on X^*. Whereas the impact of remaining parameters on X^* is negative. The parameters β_1 and γ are more sensitive to X^*. Also α, k, β_1, a, δ_1, μ_2, η_1, μ_3 have a positive impact on P_h^*. The impact of other remaining parameters on P_h^* is negative; α and β are more sensitive to P_h^*. Again, the impact of α, k, β_1, δ, δ_1, μ_2, δ_2, η on P_i^* is positive and the impact of rest of parameters on P_i^* is negative. Clearly, η and η_1 are more sensitive to P_i^*. Now, the impact of α, k, β_1, γ, a, η_1, μ_3 on N^* is positive and the impact of remaining parameters on N^* is negative. Obviously, α is the more sensitive parameter to N^*.

6. Numerical Simulations

Numerical simulations of the system $(1)-(4)$ are performed to support our analytic findings with the help of MATLAB software. The interior equilibrium $E^*(4.92, 0.066, 1.62, 0.49)$ is stable for parameter values: $\alpha = 0.2$; $k = 5$; $\beta = 0.05$; $\beta_1 = 0.1$; $\gamma = 0.3$; $\delta = 0.01$; $a = 0.9$; $\mu_1 = 0.0002$; $\delta_1 = 0.04$; $\mu_2 = 0.01$; $\delta_2 = 0.3$; $\eta = 0.01$; $\eta_1 = 0.02$; $\mu_3 = 0.001$ and result is shown in Figure 1. Moreover, the boundary equilibrium $E_1(3, 0, 0, 0)$ is stable for the parameters: $\alpha = 0.2$; $k = 3$; $\beta = 0.005$; $\beta_1 = 0.1$; $\gamma = 0.03$; $\delta = 0.01$; $a = 0.5$; $\mu_1 = 0.6$; $\delta_1 = 0.04$; $\mu_2 = 0.4$; $\delta_2 = 0.3$; $\eta = 0.01$; $\eta_1 = 0.02$; $\mu_3 = 0.1$, see Figure 2. The steady state $E_2(2, 8.25, 0, 0)$ is stable for parametric values: $\alpha = 2.2$; $k = 8$; $\beta = 0.2$; $\beta_1 = 0.001$; $\gamma = 0.03$; $\delta = 0.01$; $a = 0.5$; $\mu_1 = 0.002$; $\delta_1 = 0.04$; $\mu_2 = 0.6$; $\delta_2 = 0.3$; $\eta = 0.01$; $\eta_1 = 0.02$; $\mu_3 = 0.1$ and

TABLE 1. The sensitive indices $\gamma_{y_u}^{x_v} = \frac{\partial x_v}{\partial y_u} \times \frac{y_u}{x_v}$ of the model $(1)-(4)$ to the parameters y_u for the parameter values: $\alpha = 0.2$; $k = 5$; $\beta = 0.05$; $\beta_1 = 0.1$; $\gamma = 0.3$; $\delta = 0.01$; $a = 0.9$; $\mu_1 = 0.0002$; $\delta_1 = 0.04$; $\mu_2 = 0.01$; $\delta_2 = 0.3$; $\eta = 0.01$; $\eta_1 = 0.02$; $\mu_3 = 0.001$.

Parameter (y_u)	$\gamma_{y_u}^{X^*}$	$\gamma_{y_u}^{P_h^*}$	$\gamma_{y_u}^{P_i^*}$	$\gamma_{y_u}^{N^*}$
α	0.104128	0.917076	0.0506418	0.931013
k	0.0829237	0.730324	0.0403291	0.741423
β	-0.104128	-0.917076	-0.0506418	-0.931013
β_1	-0.917076	0.730324	0.0403291	0.741423
γ	0.91211	-0.726369	-0.0401107	0.27779
δ	0.350263	-0.278936	0.176442	-0.283175
a	-0.10082	0.0802893	-0.0507872	0.0815095
μ_1	0.00827464	-0.0065896	-0.000363883	-0.00668974
δ_1	-0.217674	0.173347	0.00957238	-0.824018
μ_2	-0.00330803	0.00263439	0.000145473	-0.0125227
δ_2	0.132589	-0.105589	0.186014	-0.107194
η	0.620598	-0.49422	0.870659	-0.501731
η_1	-0.691128	0.550387	-0.969607	0.558751
μ_3	-0.0620598	0.049422	-0.0870659	0.0501731

result is shown in Figure 3. The natural enemy free equilibrium $E_3(1.38, 0.33, 0.30, 0)$ is stable for parametric values: $\alpha = 0.54$; $k = 2$; $\beta = 0.5$; $\beta_1 = 0.02$; $\gamma = 0.09$; $\delta = 0.3$; $a = 0.8$; $\mu_1 = 0.001$; $\delta_1 = 0.9$; $\mu_2 = 0.03$; $\delta_2 = 0.3$; $\eta = 0.001$; $\eta_1 = 0.5$; $\mu_3 = 0.04$, see Figure 4. It is clear from Figure 5 that the equilibrium point $E_4(0.2, 5.76, 0, 1.49 * 10^{-15})$ is stable for parametric values: $\alpha = 1.2$; $k = 5$; $\beta = 0.2$; $\beta_1 = 0.01$; $\gamma = 0.03$; $\delta = 0.01$; $a = 0.5$; $\mu_1 = 0.002$; $\delta_1 = 0.04$; $\mu_2 = 0.6$; $\delta_2 = 0.3$; $\eta = 0.01$; $\eta_1 = 0.02$; $\mu_3 = 0.1$. It is obvious from Figure 6 that the interior equilibrium point $E^*(9.65, 2.26, 8.58, 3.41)$ is stable for the parametric values: $\alpha = 3.2$; $k = 10$; $\beta = 0.05$; $\beta_1 = 0.1$; $\gamma = 0.1$; $\delta = 0.1$; $a = 1$; $\mu_1 = 0.002$; $\delta_1 = 0.04$; $\mu_2 = 0.09$; $\delta_2 = 0.01$; $\eta = 0.01$; $\eta_1 = 0.003$; $\mu_3 = 0.0001$; $\tau = 1000 < \tau_0^+ = 1500$. Moreover, Figure 7 suggests that

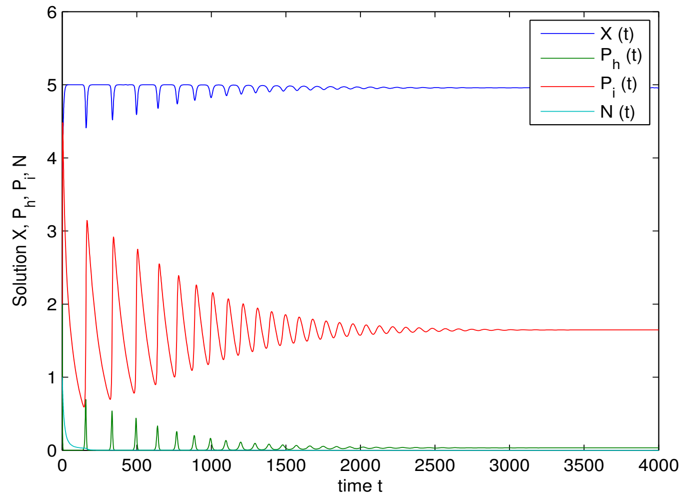

FIGURE 1. The interior equilibrium $E^*(4.92, 0.066, 1.62, 0.49)$ is stable for parameter values: $\alpha = 0.2$; $k = 5$; $\beta = 0.05$; $\beta_1 = 0.1$; $\gamma = 0.3$; $\delta = 0.01$; $a = 0.9$; $\mu_1 = 0.0002$; $\delta_1 = 0.04$; $\mu_2 = 0.01$; $\delta_2 = 0.3$; $\eta = 0.01$; $\eta_1 = 0.02$; $\mu_3 = 0.001$.

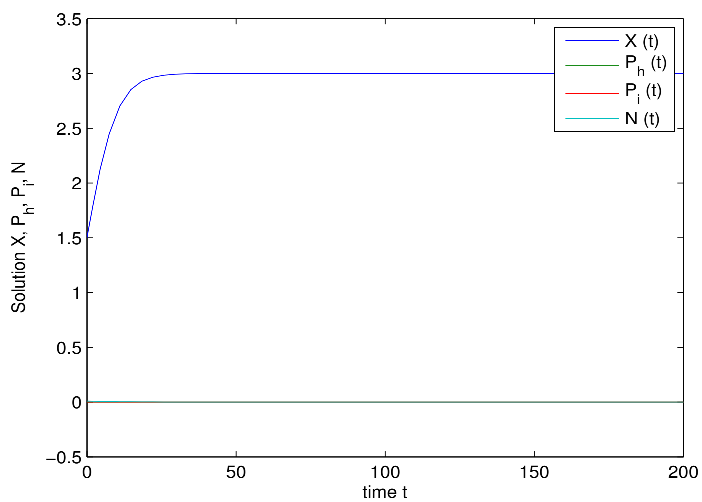

FIGURE 2. The boundary equilibrium $E_1(3, 0, 0, 0)$ is stable for the parametric values: $\alpha = 0.2$; $k = 3$; $\beta = 0.005$; $\beta_1 = 0.1$; $\gamma = 0.03$; $\delta = 0.01$; $a = 0.5$; $\mu_1 = 0.6$; $\delta_1 = 0.04$; $\mu_2 = 0.4$; $\delta_2 = 0.3$; $\eta = 0.01$; $\eta_1 = 0.02$; $\mu_3 = 0.1$.

the interior equilibrium $E^*(9.65, 2.26, 8.58, 3.41)$ is unstable and Hopf Bifurcation appears for the parametric values: $\alpha = 3.2$; $k = 10$; $\beta = 0.05$; $\beta_1 = 0.1$; $\gamma = 0.1$; $\delta = 0.1$; $a = 1$; $\mu_1 = 0.002$; $\delta_1 = 0.04$; $\mu_2 = 0.09$; $\delta_2 = 0.01$; $\eta = 0.01$; $\eta_1 = 0.003$; $\mu_3 = 0.0001$;$\tau = 1900 > \tau_0^+ = 1500$.

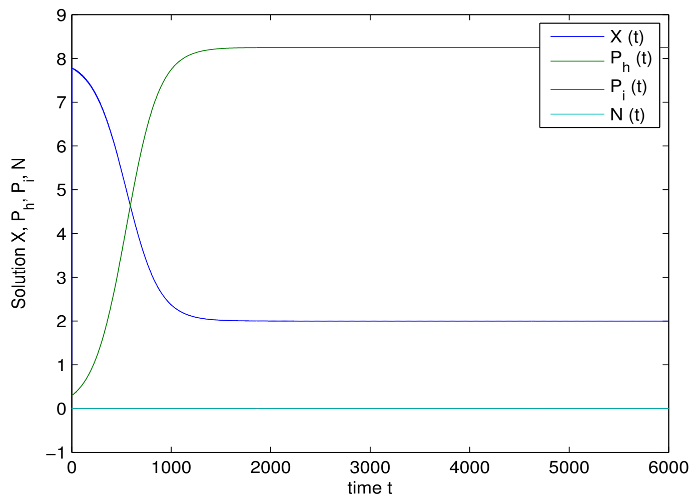

FIGURE 3. The equilibrium point $E_2(2, 8.25, 0, 0)$ is stable for parametric values: $\alpha = 2.2$; $k = 8$; $\beta = 0.2$; $\beta_1 = 0.001$; $\gamma = 0.03$; $\delta = 0.01$; $a = 0.5$; $\mu_1 = 0.002$; $\delta_1 = 0.04$; $\mu_2 = 0.6$; $\delta_2 = 0.3$; $\eta = 0.01$; $\eta_1 = 0.02$; $\mu_3 = 0.1$.

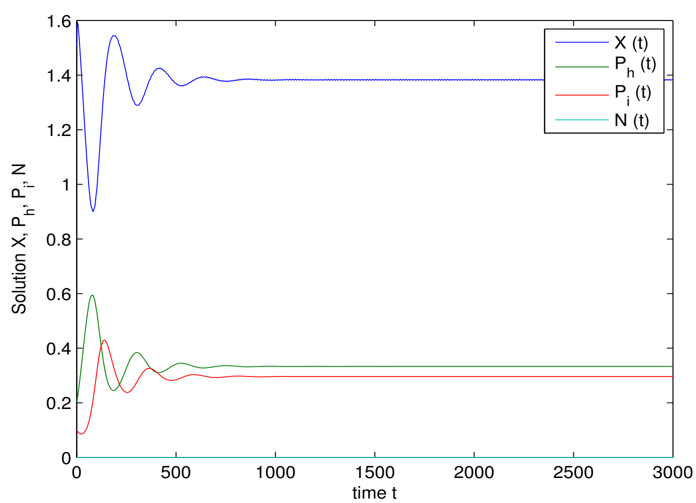

FIGURE 4. The natural enemy free equilibrium point $E_3(1.38, 0.33, 0.30, 0)$ is stable for parametric values: $\alpha = 0.54$; $k = 2$; $\beta = 0.5$; $\beta_1 = 0.02$; $\gamma = 0.09$; $\delta = 0.3$; $a = 0.8$; $\mu_1 = 0.001$; $\delta_1 = 0.9$; $\mu_2 = 0.03$; $\delta_2 = 0.3$; $\eta = 0.001$; $\eta_1 = 0.5$; $\mu_3 = 0.04$.

7. Conclusion

Here, a plant-pest-natural enemy system with disease in pest and gestation delay for natural enemy is proposed. There are six feasible steady states and asymptotic stability of the system for

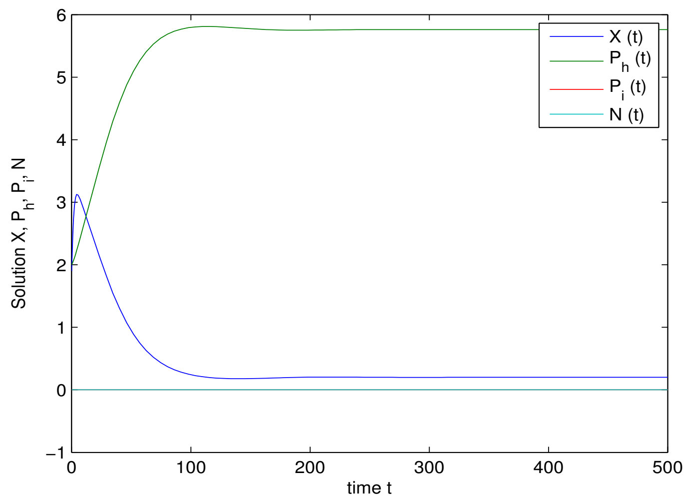

FIGURE 5. The equilibrium point $E_4(0.2, 5.76, 0, 1.49 * 10^{-15})$ is stable for parametric values: $\alpha = 1.2$; $k = 5$; $\beta = 0.2$; $\beta_1 = 0.01$; $\gamma = 0.03$; $\delta = 0.01$; $a = 0.5$; $\mu_1 = 0.002$; $\delta_1 = 0.04$; $\mu_2 = 0.6$; $\delta_2 = 0.3$; $\eta = 0.01$; $\eta_1 = 0.02$; $\mu_3 = 0.1$.

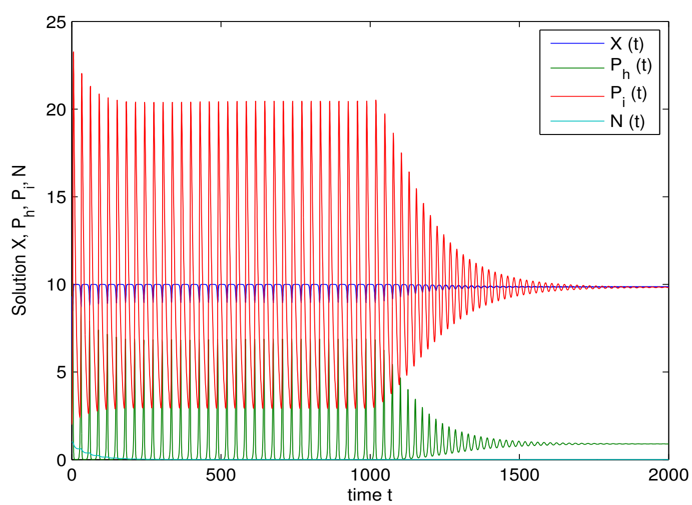

FIGURE 6. The interior equilibrium point $E^*(9.65, 2.26, 8.58, 3.41)$ is stable for the parametric values: $\alpha = 3.2$; $k = 10$; $\beta = 0.05$; $\beta_1 = 0.1$; $\gamma = 0.1$; $\delta = 0.1$; $a = 1$; $\mu_1 = 0.002$; $\delta_1 = 0.04$; $\mu_2 = 0.09$; $\delta_2 = 0.01$; $\eta = 0.01$; $\eta_1 = 0.003$; $\mu_3 = 0.0001$;$\tau = 1000 < \tau_0^+ = 1500$; discussion is numerical only.

all equilibria are studied and analyzed. It is established that boundary and interior equilibrium points are asymptotically stable under certain conditions. The existence of Hopf bifurcation at interior equilibrium point is explored and determined the critical limits for gestation delay

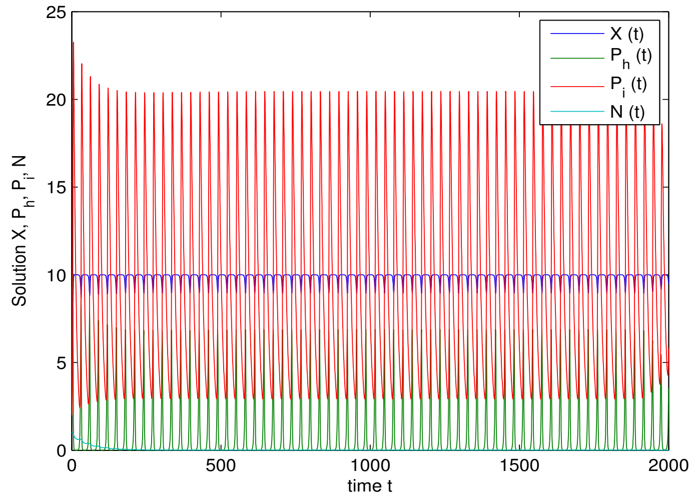

FIGURE 7. The interior equilibrium point $E^*(9.65, 2.26, 8.58, 3.41)$ is unstable and Hopf bifurcation appears for parametric values: $\alpha = 3.2$; $k = 10$; $\beta = 0.05$; $\beta_1 = 0.1$; $\gamma = 0.1$; $\delta = 0.1$; $a = 1$; $\mu_1 = 0.002$; $\delta_1 = 0.04$; $\mu_2 = 0.09$; $\delta_2 = 0.01$; $\eta = 0.01$; $\eta_1 = 0.003$; $\mu_3 = 0.0001$; $\tau = 1900 > \tau_0^+ = 1500$; discussed numerically only.

τ. Finally, the normalized forward sensitivity indices are calculated for the state variables at interior equilibrium point with respect to various system parameters. Numerical simulations of the proposed system are presented by taking a particular set of parameter values to verify our analytic results.

Conflict of Interests

The authors declare that there is no conflict of interests.

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