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THE ODD LINDLEY NADARAJAH-HAGHIGHI DISTRIBUTION

HAITHAM M. YOUSOF[1], MUSTAFA Ç. KORKMAZ[2], AND G. G. HAMEDANI[3,*]

[1]Department of Statistics, Mathematics and Insurance, Benha University, Benha, Egypt

[2]Department of Educational Measurement and Evaluation, Artvin Coruh University, Artvin, Turkey

[3]Department of Mathematics, Statistics and Computer Science, Marquette University, USA

Abstract: In this paper, a new three-parameter model which can be used in lifetime data analysis is introduced. Its failure rate function can be decreasing, increasing, constant and bathtub-shaped depending on its parameters. We derive explicit expressions for some of its statistical and mathematical quantities including the ordinary moments, generating function, incomplete moments, order statistics, moment of residual life and reversed residual life. Some useful characterizations are presented. Maximum likelihood method is used to estimate the model parameters. Simulation results to assess the performance of the maximum likelihood estimators are discussed in case of uncensored data. The censored maximum likelihood estimation is presented in the general case of the multi-censored data. We demonstrate empirically the importance and flexibility of the new model in modeling real data set.

Keywords: odd Lindley-G family; Nadarajah-Haghighi distribution; characterization; multicensored data; simulation study.

2010 AMS Subject Classification: 60E05, 60E10.

1. Introduction

Statistical distributions are very useful in describing and predicting real world phenomena. Numerous classical models have been extensively used over the past decades for modeling real data sets in several areas. Recent developments focus on definining the new families of

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*Corresponding author
E-mail address: gholamhoss.hamedani@marquette.edu

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distributions that extend well-known distributions and at the same time provide great flexibility in modeling real data.

Recently, a new generalization of the exponential distribution as an alternative to the gamma, Weibull and exponentiated-exponential distributions was proposed by Nadarajah and Haghighi (2011). The cumulative distribution function (cdf) is given by

$$G(x,\alpha,\lambda) = 1 - \exp[1-(1+\lambda\, x)^{\alpha}], x > 0, \tag{1}$$

and the corresponding probability density function (pdf) is

$$g(x,\alpha,\lambda) = \alpha\lambda\,(1+\lambda\, x)^{\alpha-1}\exp[1-(1+\lambda\, x)^{\alpha}], x > 0, \tag{2}$$

where the parameter $\alpha > 0$ control the shape of the distribution and $\lambda > 0$ is the scale parameter. Nadarajah and Haghighi (2011) pointed out that the density function (2) has the attractive feature of always having the zero mode. They also showed that larger values of α in (2) will lead to faster decay of the upper tail.

We shall refer to the new distribution using (1) and (2) as the Odd Lindley-Nadarajah-Haghighi (OLNH) model using the Odd Lindley-G (OL-G) family of distributions which introduced by Silva et al. (2017). The pdf and cdf of the OL-G family of distributions are given by

$$f(x;a,\xi) = \frac{a^2}{(1+a)}\frac{g(x;\xi)}{\bar{G}(x;\xi)^3}\exp\left[-a\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right] \tag{3}$$

and

$$F(x;a,\xi) = 1 - \frac{a+\bar{G}(x;\xi)}{(1+a)\bar{G}(x;\xi)}\exp\left[-a\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right], \tag{4}$$

respectively, where $G(x;\xi)$ and $\bar{G}(x;\xi)$are the baseline cdf and survival function which depends on a parameter vector ξ . where a is a positive shape parameter. To this end, we use (1), (2) and (3) to obtain the three-parameter OLNH pdf (for $x > 0$)

$$f(x;a,\alpha,\lambda) = \frac{a^2\alpha\lambda}{(1+a)}\frac{(1+\lambda\, x)^{\alpha-1}}{\exp\{2[1-(1+\lambda\, x)^{\alpha}]\}}\exp\left[-a\frac{1-\exp[1-(1+\lambda\, x)^{\alpha}]}{\exp[1-(1+\lambda\, x)^{\alpha}]}\right]. \tag{5}$$

The corresponding cdf is given by

$$F(x;a,\alpha,\lambda) = 1 - \frac{a+\exp[1-(1+\lambda\, x)^{\alpha}]}{(1+a)\exp[1-(1+\lambda\, x)^{\alpha}]}\exp\left[-a\frac{1-\exp[1-(1+\lambda\, x)^{\alpha}]}{\exp[1-(1+\lambda\, x)^{\alpha}]}\right], x \geq 0. \tag{6}$$

Note that the Type **I** odd Lindley exponential model (T**I**OLE) arises when $\alpha = 1$ and the T**II**OLE model arises when $a = \alpha = 1$.

The OLNH density function can be expressed as an infinite mixture of exponentiated-G (exp-G) density functions

$$f(x) = \sum_{m,k=0}^{\infty} \upsilon_{m,k} \pi_{m+k+1}(x), \tag{7}$$

where

$$\upsilon_{m,k} = \frac{(-1)^k a^{2+k} \Gamma(m+k+3)}{(a+1)(m+k+1) m!\, k!\, \Gamma(k+3)}$$

and

$$\pi_\gamma(x) = \gamma \underbrace{\alpha\lambda\,(1+\lambda\,x)^{\alpha-1} \exp[1-(1+\lambda\,x)^{\alpha}]}_{g(x,\alpha,\lambda)} \underbrace{\{1-\exp[1-(1+\lambda\,x)^{\alpha}]\}^{\gamma-1}}_{G(x,\alpha,\lambda)^{\gamma-1}}$$

represents the exp-NH density with power parameter $\gamma > 0$. The exp-NH distribution was introduced and studied by Lemonte (2013).

The properties of exp-G distributions have been studied by many authors in recent years, see Mudholkar and Srivastava (1993) and Mudholkar et al. (1995) for exponentiated Weibull, Gupta et al. (1998) for exponentiated Pareto, Gupta and Kundu (1999) for exponentiated exponential, Nadarajah (2005) for the exponentiated-type distributions, Nadarajah and Kotz (2006) for exponentiated Gumbel, Shirke and Kakade (2006) for exponentiated log-normal and Nadarajah and Gupta (2007) for exponentiated gamma distributions, among others. The cdf of OLNH model can be given by integrating (7) as

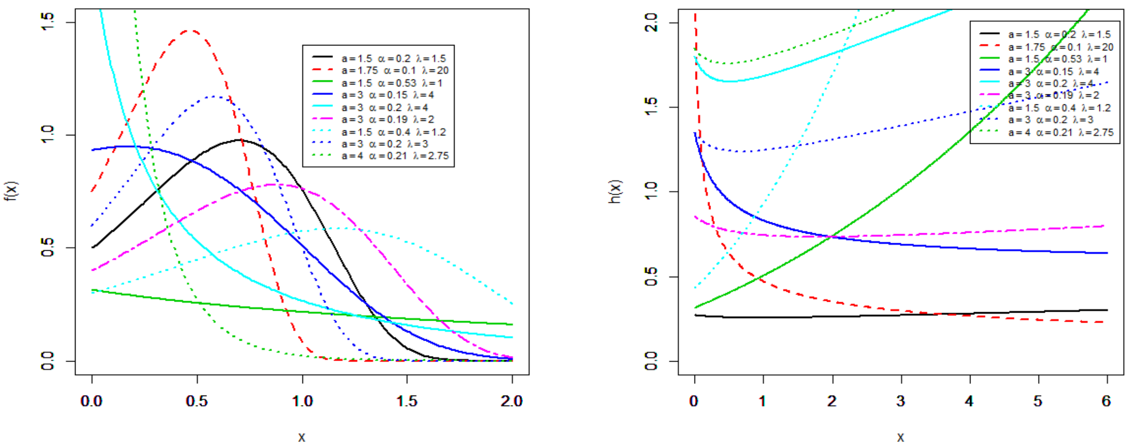
$$F(x) = \sum_{m,k=0}^{\infty} \upsilon_{m,k} \mathbf{\Pi}_{m+k+1}(x), \tag{8}$$

where

$$\Pi_\gamma(x) = \underbrace{\{1-\exp[1-(1+\lambda\,x)^{\alpha}]\}^{\gamma}}_{G(x,\alpha,\lambda)^{\gamma}}$$

is the cdf of the exp-NH model with power parameter γ. Equation (8) reveals that the OLNH cdf is a linear combination of exp-G cdf's. So, some mathematical properties of this family can be determined from those of the exp-G distribution. Equations (7) and (8) are the main results of this section.

Equation (7) is easily obtained by the general result given by Silva et al. (2017) in their relation (19). Moreover, Equation (8) is easily obtained by the general result given by Silva et al. (2017) in their relation (20).



(a) (b)

Fig. 1: Plots of the OLNH pdf (a) and OLNH hrf (b) for some parameter values.

Figure 1. (a) shows that the OLNH distribution has various pdf shapes. Further, Fig. 1. (b) shows that the OLNH model produces flexible hazard rate shapes such as constant, increasing, decreasing and bathtub. These plots indicate that the OLNH model is very useful in fitting different data sets with various shapes.

2. Properties

2.1 Moments and generating function

The rth moment of X, say $\mu_r^{'}$, follows from (7) as

$$\mu_r^{'} = E(X^r) = \sum_{m,k=0}^{\infty} \sum_{j=0}^{\gamma-1} \sum_{i=0}^{r} \upsilon_{m,k}\, \beta_{j,i}^{(m+k+1)} \Gamma\left(\frac{i}{\alpha} + 1, 1 + j\right), \tag{9}$$

where $\beta_{j,i}^{(\gamma)} = \frac{\gamma}{\lambda^r} \frac{(-1)^{r+j-i} e^{1+j}}{(1+j)^{\frac{i}{\alpha}+1}} \binom{\gamma-1}{j} \binom{r}{i}$ and $r > 0$ integer. The nth central moment of X, say M_n, is given by

$$M_n = E\left(X - \mu_1^{'}\right)^n = \sum_{r=0}^{n} \sum_{m,k=0}^{\infty} \upsilon_{m,k} \binom{n}{r} (-1)^r \left(\mu_1^{'}\right)^r \mu_{n-r}^{'}$$

The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The mgf $M_X(t) = E(e^{t\,X})$ of X can be derived using (7) as

$$M_X(t) = \sum_{m,k,r=0}^{\infty} \sum_{j=0}^{\gamma-1} \sum_{i=0}^{r} \upsilon_{m,k} \frac{t^r}{r!} \beta_{j,i}^{(m+k+1)} \Gamma\left(\frac{i}{\alpha} + 1, 1 + j\right).$$

For more details about the properties Exp NH see Lemonte (2013, p.155).

2.2 Incomplete moments

The main applications of the first incomplete moment refer to the mean deviations and the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The rth incomplete moment, say $I_r(t)$, of X can be expressed using (7) as

$$I_r(t) = \int_{-\infty}^{t} x^r f(x) dx,$$

then

$$I_r(t) = \sum_{m,k=0}^{\infty} \sum_{j=0}^{\gamma-1} \sum_{i=0}^{r} \upsilon_{m,k} \beta_{j,i}^{(m+k+1)} \Gamma\left(\frac{i}{\alpha} + 1, (1+j)(1+\lambda\, t)^{\alpha}\right). \qquad (10)$$

2.3 moment of residual life and reversed residual life

The rth moment of the residual life, say $z_r(t) = E[(X-t)^r | X > t]$, $r = 1,2,\ldots$, uniquely determines $F(x)$. The rth moment of the residual life of X is given by $z_r(t) = \frac{1}{1-F(t)} \int_t^{\infty} (x - t)^r dF(x)$. Therefore

$$z_r(t) = \frac{1}{1-F(t)} \sum_{m,k=0}^{\infty} \sum_{j=0}^{\gamma-1} \sum_{i=0}^{r} \upsilon_{m,k}^{*} \beta_{j,i}^{(m+k+1)} \Gamma\left(\frac{i}{\alpha} + 1, (1+j)(1+\lambda\, t)^{\alpha}\right),$$

where $\upsilon_{m,k}^{*} = \upsilon_{m,k} \sum_{h=0}^{n} \binom{n}{h} (-t)^{n-h}$. Another interesting function is the *mean residual life* (MRL) function or the life expectation at age t defined by $z_1(t) = E[(X-t)|X > t]$, which represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained by setting $r = 1$ in the last equation.

The rth moment of the reversed residual life, say $Z_r(t) = E[(t-X)^r | X \leq t]$, for $t > 0$ and $r = 1,2,\ldots$, uniquely determines $F(x)$. We have $Z_r(t) = \frac{1}{F(t)} \int_0^{t} (t-x)^r dF(x)$. Then, the rth moment of the reversed residual life of X becomes

$$Z_r(t) = \frac{1}{F(t)} \sum_{m,k=0}^{\infty} \sum_{j=0}^{\gamma-1} \sum_{i=0}^{r} \upsilon_{m,k}^{**} \beta_{j,i}^{(m+k+1)} \Gamma\left(\frac{i}{\alpha} + 1, (1+j)(1+\lambda\, t)^{\alpha}\right),$$

where $\upsilon_{k+1}^{**} = \upsilon_{m,k} \sum_{h=0}^{n} (-1)^h \binom{n}{h} t^{n-h}$. The *mean inactivity time* (MIT), also called the mean reversed residual life function, is given by $Z_1(t) = E[(t-X)|X \leq t]$, and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$. The MIT of the ONH-G family can be obtained easily by setting $r = 1$ in the above equation. For more details see Navarro et al. (1998).

2.4 Order statistics and quantile spread order

Suppose $X_1, \dots, X_n$ is a random sample from an OLNH model. Let $X_{i:n}$ denote the ith order statistic. The pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \frac{f(x)}{B(i,n-i+1)}\, F(x)^{i-1}[1-F(x)]^{n-i}, \tag{11}$$

we can write the density function of $X_{i:n}$ in (11) as

$$f_{i:n}(x) = \sum_{m,p=0}^{\infty} \sum_{j=0}^{k+n-i} \upsilon_{j,m,p}\, \pi_{j+m+p}(x), \tag{12}$$

Where

$$\upsilon_{j,m,p} = \frac{a^{j+m+2}\binom{j+m+p}{j+m}\sum_{k=0}^{n-1}(-1)^{k+m}\binom{k+n-i}{j}\binom{i-1}{k}}{B(i,n-i+1)m!\,(1+a)^{j+1}(j+m+p+1)}.$$

Equation (12) is the main result of this section, it reveals that the pdf of the OLNH order statistics is a linear combination of exp-G density functions. So, several mathematical quantities of the OLNH order statistic such as ordinary, incomplete and factorial moments, mean deviations and several others can be determined from those quantities of the exp-G distribution. The pth moment of $X_{i:n}$ is given by

$$E\left(X_{i:n}^{p}\right) = \sum_{m,p=0}^{\infty} \sum_{j=0}^{k+n-i} \sum_{w=0}^{p} \sum_{l=0}^{r} \upsilon_{j,m,p}\, \beta_{w,l}^{(j+m+p)} \Gamma\left(\frac{l}{\alpha}+1,1+w\right). \tag{13}$$

3 Characterizations

This section deals with various characterizations of OLNH distribution. These characterizations are presented in two directions: (i) based on a truncated moment and (ii) in terms of the hazard function. It should be noted that characterization (i) can be employed also when the cdf does not have a closed form. We present our characterizations (i) and (ii) in two subsections.

3.1 Characterizations based on a truncated moment

Our first characterization employs a version of a theorem due to Glanzel (1987), see Theorem 1 of Appendix A .The result, however, holds also when the interval H is not closed since the condition of Theorem 1 is on the interior of H. We like to mention that this kind of characterization based on a truncated moment is stable in the sense of weak convergence (see, Glanzel 1990).

Proposition 3.1. Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q(x)=\exp\left\{-a\frac{1-e^{1-(1+\lambda x)^{\alpha}}}{e^{1-(1+\lambda x)^{\alpha}}}\right\}$ for $x>0$. The random variable X belongs to the family (5) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\exp\left\{-a\frac{1-e^{1-(1+\lambda x)^{\alpha}}}{e^{1-(1+\lambda x)^{\alpha}}}\right\},\quad x>0.$$

Proof. Let X be a random variable with pdf (5), then

$$\big(1-F(x)\big)E[q(X)|X\geq x]=\frac{1}{2(1+a)}\left(\frac{a+\exp[1-(1+\lambda x)^{\alpha}]}{\exp[1-(1+\lambda x)^{\alpha}]}\right)\times$$
$$\exp\left\{-2a\frac{1-\exp[1-(1+\lambda\, x)^{\alpha}]}{\exp[1-(1+\lambda\, x)^{\alpha}]}\right\},$$

and hence

$$\eta(x)=E[q(X)|X\geq x]=\frac{1}{2}\exp\left\{-a\frac{1-\exp[1-(1+\lambda\, x)^{\alpha}]}{\exp[1-(1+\lambda\, x)^{\alpha}]}\right\},\quad x>0,$$

and finally

$$\eta(x)-q(x)=-\frac{1}{2}\exp\left\{-a\frac{1-\exp[1-(1+\lambda\, x)^{\alpha}]}{\exp[1-(1+\lambda\, x)^{\alpha}]}\right\}<0 for\ x>0.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)}{\eta(x)-q(x)}=a\alpha\lambda(1+\lambda x)^{\alpha-1},\ x>0,$$

and hence

$$s(x)=a(1+\lambda x)^{\alpha},\ x>0.$$

Now, according to Theorem 1, X has density (5).

Corollary 3.1. Let $X:\Omega\to(0,\infty)$ be a continuous random variable. Then, X has pdf (5) if and only if there exist functions q and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)}{\eta(x)-g(x)}=a\alpha\lambda(1+\lambda x)^{\alpha-1},\ x>0.$$

The general solution of the differential equation in Corollary 3.1 is

$$\eta(x)=\exp\left\{a\frac{1-\exp[1-(1+\lambda\, x)^{\alpha}]}{\exp[1-(1+\lambda\, x)^{\alpha}]}\right\}$$
$$\times\left[-\int a\alpha\lambda(1+\lambda x)^{\alpha-1}\exp\left\{-a\frac{1-\exp[1-(1+\lambda\, x)^{\alpha}]}{\exp[1-(1+\lambda\, x)^{\alpha}]}\right\}q(x)dx+D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 3.1 with $D=0$.

3.2 Characterization in terms of the hazard function

It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h'_F(x)}{h_F(x)} - h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterization establishes a non-trivial characterization of OLNH in terms of the hazard function which is not of the above trivial form.

Proposition 3.2. Let $X: \Omega \to (0, \infty)$ be a continuous random variable. Then, X has pdf (5) if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h'_F(x) - \frac{\lambda(\alpha-1)}{1+\lambda x} h_F(x) = -\frac{a^2\alpha^2\lambda^2(1+\lambda x)^{2(\alpha-1)}\{a+2\exp[1+(1+\lambda x)^{\alpha}]\}}{e^{1+(1+\lambda x)^{\alpha}}\left[a+\exp[1-(1+\lambda x)^{\alpha}]\right]^2}, x > 0,$$

with the initial condition $h_F(0) = \frac{a^2\alpha\lambda}{1+a}$.

Proof. If X has pdf (5), then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left\{(1+\lambda x)^{-(\alpha-1)} h_F(x)\right\} = a^2\alpha\lambda \frac{d}{dx}\left[\begin{pmatrix} a\exp[1-(1+\lambda x)^{\alpha}] \\ +\exp\{2[1-(1+\lambda x)^{\alpha}]\}\end{pmatrix}^{-1}\right]$$

or

$$h_F(x) = \frac{a^2\alpha\lambda(1+\lambda x)^{\alpha-1}}{a\exp[1-(1+\lambda x)^{\alpha}]+\exp\{2[1-(1+\lambda x)^{\alpha}]\}}, x > 0,$$

which is the hazard function of (5).

for characterizations of other well-known continuous distributions based on the hazard function, the reader should refer to Hamedani (2004) and Hamedani and Ahsanullah (2005).

4. Estimation

Subection 4.1 gives procedures for maximum likelihood estimation of the OLNH distribution. Subection 4.2 assesses the performance of the maximum likelihood estimators (MLEs) in terms of biases, mean squared errors, coverage probabilities and coverage lengths by means of a simulation study. Subection 4.3 gives procedures for maximum likelihood estimation in the presence of censored data.

4.1 maximum likelihood estimation

We consider the estimation of the unknown parameters of the OLNH model from complete samples only by maximum likelihood method. The MLEs of the parameters of the OLNH

(a, α, β) model is now discussed. Let $x_1, \dots, x_n$ be a random sample from this distribution with parameter vector $\mathbf{\Psi} = (a, \alpha, \beta,)^{\mathrm{T}}$. The log-likelihood function for $\mathbf{\Psi}$, say $\ell(\mathbf{\Psi})$, is given by

$$\ell(\mathbf{\Psi}) = 2n\log(a) + n\log(\alpha) + n\log(\lambda) - n\log(1 + a)$$
$$+(\alpha - 1)\sum_{i=0}^{n} \log(1 + \lambda\, x_i) - 2\sum_{i=0}^{n} [1 - (1 + \lambda\, x_i)^{\alpha}]$$
$$-a \textstyle\sum_{i=0}^{n} (\{\exp[(1 + \lambda\, x_i)^{\alpha} - 1]\} - 1). \quad (14)$$

The last equation can be maximized either by using the different programs like R (optim function), SAS (PROC NLMIXED) or by solving the nonlinear likelihood equations obtained by differentiating (14). The score vector elements, $\mathbf{U}(\mathbf{\Psi}) = (\frac{\partial}{\partial a}, \frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \lambda})^{\mathrm{T}}$, are given in Appendix B. We can obtain the estimates of the unknown parameters by setting the score vector to zero, $\mathbf{U}(\widehat{\mathbf{\Psi}}) = \mathbf{0}$. Solving these equations simultaneously gives the MLEs $\hat{a}, \hat{\alpha}$ and $\hat{\lambda}$. For the OLNH distribution, all the second order derivatives exist. The interval estimation of the model parameters requires the 3×3 observed information matrix $J(\mathbf{\Psi}) = \{J_{ij}\}$ for $i, j = a, \alpha, \lambda$.

The multivariate normal $N_3(0, J(\widehat{\mathbf{\Theta}})^{-1})$ distribution, under standard regularity conditions, can be used to provide approximate confidence intervals for the unknown parameters, where $J(\widehat{\mathbf{\Theta}})$ is the total observed information matrix evaluated at $\widehat{\mathbf{\Psi}}$. Then, approximate $100(1 - \delta)\%$ confidence intervals for a, α and λ can be determined by: $\hat{a} \pm z_{\delta/2}\sqrt{\hat{J}_{aa}}$, $\hat{\alpha} \pm z_{\delta/2}\sqrt{\hat{J}_{\alpha\alpha}}$ and $\hat{\beta} \pm z_{\delta/2}\sqrt{\hat{J}_{\lambda\lambda}}$, where $z_{\delta/2}$ is the upper δ^{th} percentile of the standard normal distribution.

4.2 Simulation studies

This Subsection assesses the performance of the maximum likelihood estimators (MLEs) in terms of biases, mean squared errors, coverage probabilities and coverage lengths by means of a simulation study ". But We didn't give coverage probability and coverage lengths. We only gave empirical means, sd, biases, and mean squared errors.

In this subsection, we perform two simulation studies using the OLNH distribution to see the performance of MLE’s of this distributions. All results were obtained using optim routine in the R programme. We generate 1,000 samples of sizes 20, 50, 100, 200, 300 and 500 from OLNH distribution with α =1.5, λ =0.5 and a=2. Secondly, we also generate 1000 samples of size n=20,21,…,100 from OLNH with α =0.25, λ =5 and a=10. The random number procedure is obtained by using the inversion method of its cdf. We also compute the biases and mean squared errors (MSE) of the MLEs with

$$\widehat{Bias}_h = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{h}_i - h\right),$$

and

$$\widehat{MSE}_h = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{h}_i - h)^2,$$

respectively, h = α, λ and a. The results of the simulation are reported in Table 1 and Figure 2. From this Table and Figure 2, we observe that when the sample size increases, the empirical means come close to true values whereas sd, biases and MSEs decrease in all cases, as expected.

Table 1. Empirical means, sd, bias and mean squared errors

n	Parameter	Mean	SD	Bias	MSE
20	α	1.4431	0.4408	0.0554	0.1299
	λ	0.5873	0.1962	0.0873	0.0456
	a	1.8015	0.6781	0.6434	0.3919
50	α	1.5276	0.2667	0.0276	0.0709
	λ	0.5392	0.0948	0.0392	0.0095
	a	1.9122	0.4660	0.4503	0.2141
100	α	1.4891	0.2483	-0.0201	0.0616
	λ	0.5286	0.0809	0.0286	0.0072
	a	2.0807	0.3530	-0.1427	0.1506
200	α	1.5124	0.1988	0.0124	0.0386
	λ	0.5262	0.0614	0.0122	0.0037
	a	2.0742	0.2444	0.2278	0.0637
300	α	1.4912	0.1378	0.0088	0.0189
	λ	0.5170	0.0397	0.0094	0.0016
	a	1.9960	0.2249	0.2064	0.0502
500	α	1.4988	0.1019	-0.0011	0.0103
	λ	0.5057	0.0374	0.0058	0.0014
	a	1.9970	0.1676	-0.2060	0.0280

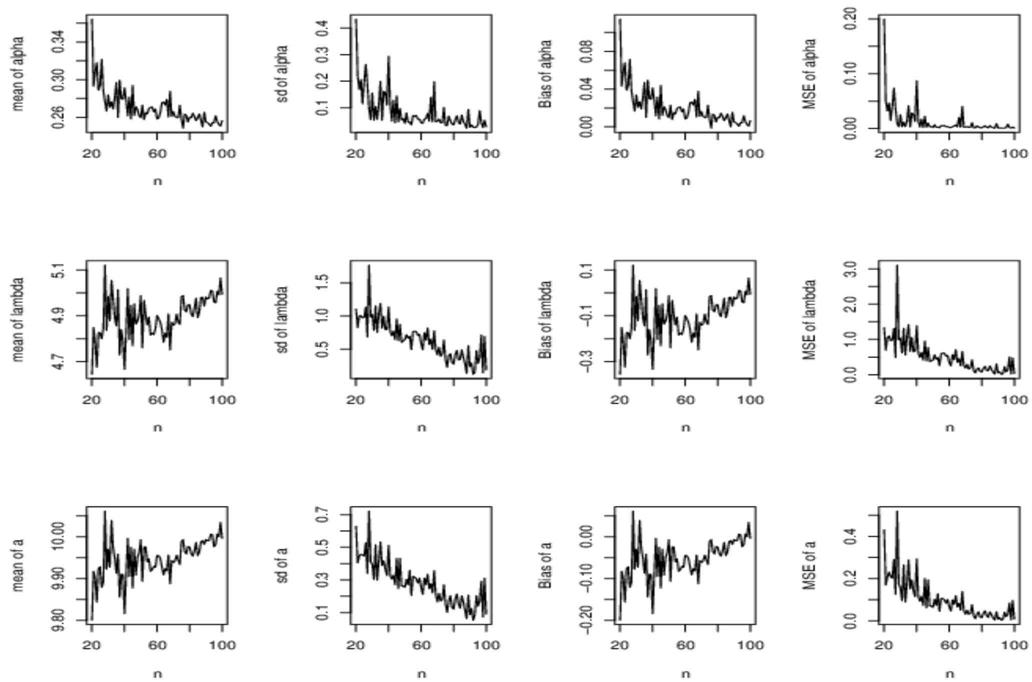


Figure 2: Plots of the empirical mean, sd, biases and MSE of alpha, lambda and a versus n.

4.3 Multi-censored maximum likelihood estimation

Often with lifetime data, we encounter censored observations. There are different forms of censoring: type **I** censoring, type **II** censoring, etc. Here, we consider the general case of multi-censored data: there are m subjects of which m_0 are known to have failed at the times $x_1, \ldots, x_{m_0}$. m_1 are known to have failed in the interval $[s_{j-1}, s_j]$, $j = 1, \ldots, m_1$. m_2 survived to a time r_j, $j = 1, \ldots, m_2$ but not observed any longer. Note that $m = m_0 + m_1 + m_2$ and that type **I** censoring and type **II** censoring are contained as particular cases of multi-censoring. The log-likelihood function for $\mathbf{\Psi}$ is

$$\ell_m(\mathbf{\Psi}) = 2m_0 \log(a) + m_0 \log(\alpha) + m_0 \log(\lambda) - m_0 \log(1+a)$$

$$+(\alpha-1)\sum_{i=0}^{m_0} \log(1+\lambda\, x_i) - 2\sum_{i=0}^{m_0} [1-(1+\lambda\, x_i)^{\alpha}] + \sum_{i=0}^{m_2} \log\left\{\frac{a+\theta_{r_i}}{(1+a)\theta_{r_i}} \exp\left[-a\frac{1-\theta_{r_i}}{\theta_{r_i}}\right]\right\}$$

$$-a\sum_{i=0}^{m_0} (\{\exp[(1+\lambda\, x_i)^{\alpha} - 1]\} - 1)$$

$$+\sum_{i=0}^{m_1}\log\left(\begin{array}{c}\left\{1-\frac{a+\theta_{s_i}}{(1+a)\theta_{s_i}}\exp\left[-a\frac{1-\theta_{s_i}}{\theta_{s_i}}\right]\right\}\\ -\left\{1-\frac{a+\theta_{s_{i-1}}}{(1+a)\theta_{s_{i-1}}}\exp\left[-a\frac{1-\theta_{s_{i-1}}}{\theta_{s_{i-1}}}\right]\right\}\end{array}\right).$$

The normal equations are given in Appendix C.

5. Data analysis

In this section, we present an application based on the real data set to show the flexibility of the OLNH distribution. We compare OLNH with Gamma-NH (GNH) (Ortega et al., 2015), Marshall-Olkin-NH (MONH) (Lemonte et al., 2016), exponentiated-NH (ENH)(Lemonte, 2013) and beta-NH (BNH)(Dias et al., 2017), generalized Lindley (GL) (Zakerzadeh and Dolati, 2009), extended Lindley (EL) (Bakouch et al., 2012) and power Lindley (PL) (Ghitany et al,. 2013) distributions. The model selection is applied using the estimated log-likelihood $(\hat{\ell})$, Kolmogorov-Smirnov (K-S) statistics, Akaike information criterion (AIC), Consistent Akaike information criteria (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC). AIC, CAIC, BIC and HQIC.

$$AIC=-2\hat{\ell}+2p\ ,\ CAIC=-2\hat{\ell}+2pn/(n-p-1), BIC=-2\hat{\ell}+p\log n$$

and

$$HQIC=-2\hat{\ell}+2p\log(\log n),$$

where p is the number of the estimated model parameters and n is sample size. When searching the best fit among others to data, the distribution with the smallest AIC, CAIC, BIC, HQIC and K-S values and the biggest log-likelihood and p values of the K-S statistics is chosen. All calculations are obtained by maxLik routine in R programme. The data consist of 72 exceedances for the years 1958–1984, rounded to one decimal place (Choulakian and Stephens, 2001): 1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0. This data also has been analyzed by Lemonte (2013) for the ENH distribution. In the applications, the information about the hazard shape can help in selecting a particular model. For this aim, a device called the total time on test (TTT) plot (Aarset, 1987) is useful. The TTT plot is obtained by plotting $T\left(\frac{i}{n}\right)$ against $\frac{i}{n}$, where

$$T\left(\frac{i}{n}\right) = \frac{\sum_{i=1}^{i} y_{j:n} + (n-r) y_{j:n}}{\sum_{i=1}^{i} y_{j:n}}$$

and $i = 1, \dots, n$ and $y_{j:n}$ are the order statistics of the sampleIt is convex shape for decreasing hazards and concave shape for increasing hazards. The TTT plot for the exceedances of flood peaks data in Figure 2 agree with a bathtub-shaped failure rate function. We also, analyzed the ordinary NH distribution with this data. For ordinary NH distribution, we obtained K-S statistics and its p-values as 0.1244 and 0.2153 respectively. The other results of this application are listed in Table 2. These results show that the OLNH distribution has the lowest AIC, CAIC, BIC, HQIC and K-S values and has the biggest estimated log-likelihood and p-value of the K-S statistics among all the fitted models. So it could be chosen as the best model under these criteria.and p-value of the K-S statistics among all the fitted models. So it could be chosen as the best model under these criteria.

Table 2. The application results of the exceedances of flood peaks data (and the corresponding standard deviations (sd) in parentheses)

Model	alpha (sd)	lambda (sd)	a (sd)	b (sd)	loglike	AIC	CAIC	BIC	HQIC	K-S (p-value)
OLNH	0.2519 (0.0520)	1.8065 (3.3549)	0.7293 (0.6059)	-	**-250.5885**	**507.1770**	**507.5300**	**514.0070**	**509.8961**	**0.1009 (0.4565)**
GNH	1.9299 (1.7591)	0.0242 (0.0312)	0.7286 (0.1385)	-	-250.9172	507.8344	508.1874	514.6644	510.5535	0.1065 (0.3880)
MONH	23.7701 (5.5053)	0.0011 (0.0003)	0.2660 (0.0895)	-	-251.0874	508.1747	508.5277	515.0047	510.8938	0.1074 (0.3771)
ENH	1.7126 (1.2607)	0.0309 (0.0330)	0.7289 (0.1404)	-	-250.9246	507.8492	508.2021	514.6792	510.5682	0.1067 (0.3859)
BNH	0.6396 (0.8227)	0.0003 (0.0004)	0.8381 (0.1215)	316.0285 (4.2194)	-251.3564	510.7129	511.3099	519.8195	514.3382	0.1044 (0.4127)
GL	0.7874 (0.1532)	0.0875 (0.0204)	0.0342 (0.0537)	-	-251.1690	508.3379	508.6909	515.1679	511.0570	0.6173 (0.0000)
EL	0.3684 (0.5725)	0.1129 (0.0551)	0.8607 (0.0961)	-	-251.3664	508.7327	509.0857	515.5627	511.4518	0.1037 (0.4214)
PL	0.6999 (0.0570)	-	0.3385 (0.0558)	-	-252.2218	508.4436	508.6175	512.9969	510.2563	0.1050 (0.4050)

Finally, we plot the estimated pdf, cdf and hrf of the OLNH for the exceedances of flood peaks data with Figure 4. Clearly, the OLNH distribution provides a closer fit to the empirical pdf and cdf. Also, from this figures, we have a bathtub-shaped failure rate function for the exceedances of flood peaks data, which are in accordance with TTT plot.

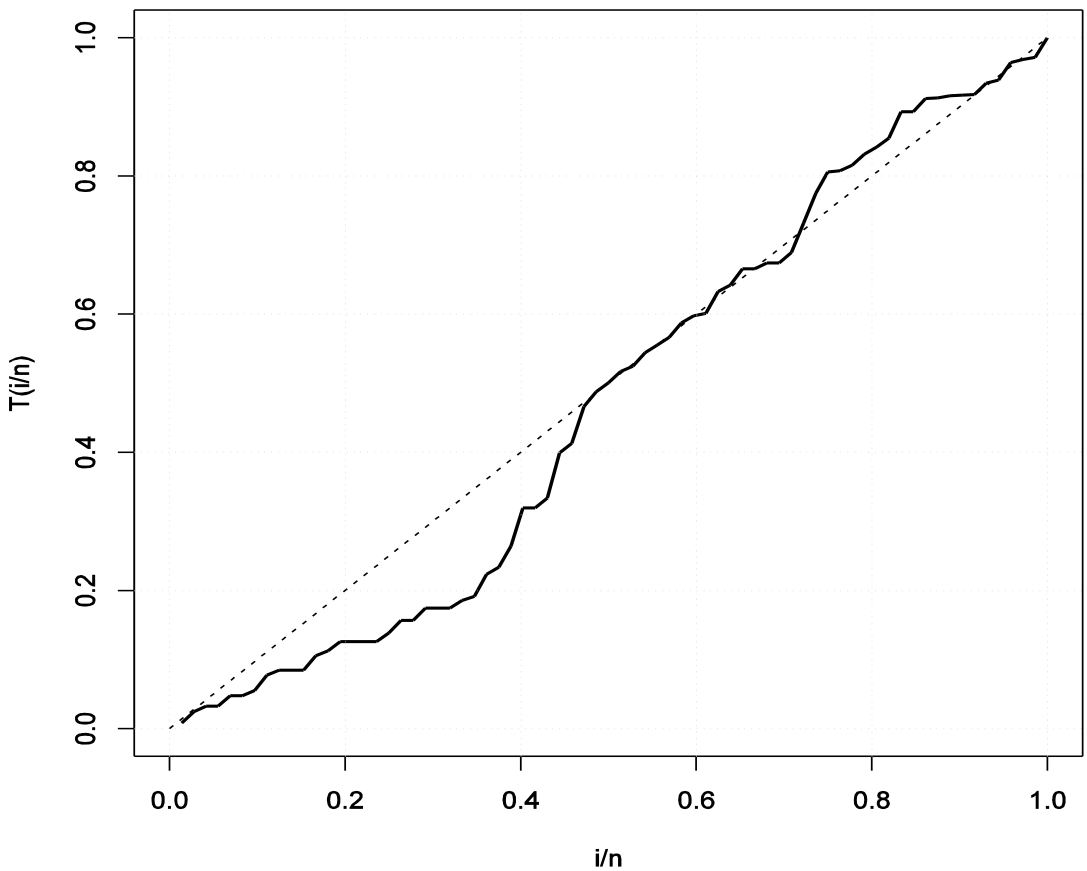


Fig. 3: TTT plot of the exceedances of flood peaks data

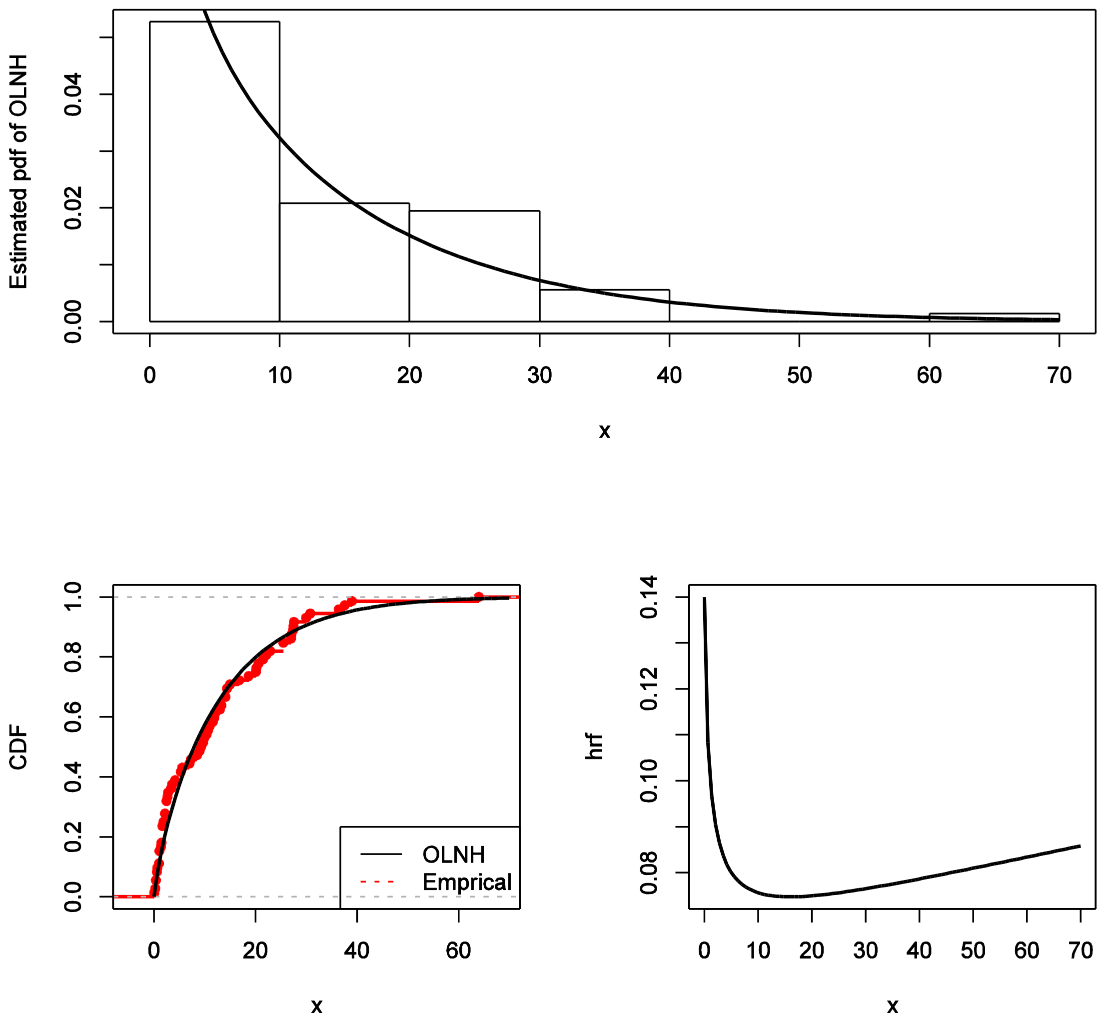


Fig. 4: Estimated pdf (top), cdf (left) and hrf (right) of the OLNH for the exceedances of flood peaks data

6. Conclusions

In this article, we introduce a new three-parameter model called Odd Lindley-Nadarajah-Haghighi (OLNH) model which extends the Nadarajah-Haghighi (N-H) model. The OLNH density function can be expressed as a straightforward linear mixture of exponentiated Nadarajah-Haghighi density. We derive explicit expressions for some of its statistical and mathematical quantities including the ordinary moments, generating function, incomplete moments, order statistics, moment of residual life and reversed residual life. Some useful characterizations are presented. Maximum likelihood method is used to estimate the model parameters. Simulation results to assess the performance of the maximum likelihood estimators are discussed in case of uncensored data. The censored maximum likelihood estimation is presented in the general case of the multi-censored data. We demonstrate empirically the importance and flexibility of the new model in modeling real data set. We hope that the proposed distribution will attract wider applications in areas such as economics (income inequality), survival and lifetime data, engineering, hydrology, meteorology and others. As a future work we will consider the bivariate and the multivariate extensions of the OLNH distribution.

Conflict of Interests

The authors declare that there is no conflict of interests.

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Appendix A

Theorem 1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ $(d = -\infty, e = \infty$ mightaswellbeallowed$)$. Let $X: \Omega \to H$ be a continuous random variable with the distribution function F and let q be a real function defined on H such that

$$\mathbf{E}[q(X)|X \geq x] = \eta(x), \quad x \in H,$$

is defined with some real function η. Assume that $q \in C^1(H)$, $\xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta = q$ has no real solution in the interior of H. Then F is uniquely determined by the functions q and η, particularly

$$F(x) = \int_a^x C \left|\frac{\eta'(u)}{\eta(u) - q(u)}\right| \exp(-s(u)) du,$$

where the function s is a solution of the differential equation $s' = \frac{\eta'}{\eta - q}$ and C is the normalization constant, such that $\int_H dF = 1$.

Appendix B

$$\frac{\partial \ell(\mathbf{\Psi})}{\partial a} = \frac{2n}{a} - \frac{n}{1+a} - \sum_{i=0}^{n} (\{\exp[(1+\lambda\, x_i)^\alpha - 1]\} - 1),$$

$$\frac{\partial \ell(\mathbf{\Psi})}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=0}^{n} \log(1+\lambda\, x_i) + \sum_{i=0}^{n} \frac{2 - a\exp[(1+\lambda\, x_i)^\alpha - 1]}{(1+\lambda\, x_i)^{-\alpha}[\log(1+\lambda\, x_i)]^{-1}}$$

and

$$\frac{\partial \ell(\mathbf{\Psi})}{\partial \lambda} = \frac{n}{\lambda} + (\alpha - 1)\sum_{i=0}^{n} \frac{x_i}{1+\lambda\, x_i} + \alpha \sum_{i=0}^{n} \frac{x_i\{2 - a\exp[(1+\lambda\, x_i)^\alpha - 1]\}}{(1+\lambda\, x_i)^{1-\alpha}}.$$

Appendix C

$$\frac{\partial \ell_m(\mathbf{\Psi})}{\partial a} = \sum_{i=0}^{m_2} A^{(a)}(r_i) \left[\frac{a+\theta_{r_i}}{(1+a)\theta_{r_i}}\right]^{-1} \exp\left[a\frac{1-\theta_{r_i}}{\theta_{r_i}}\right] + \frac{2m_0}{a} - \frac{m_0}{1+a} - \sum_{i=0}^{m_0} (\{\exp[(1+\lambda\, x_i)^\alpha - 1]\} - 1)$$

$$+ \sum_{i=0}^{m_1} \left\{ \frac{\left[A^{(a)}(s_i) - A^{(a)}(s_{i-1})\right]}{\left[\begin{array}{c} \left\{1 - \frac{a+\theta_{s_i}}{(1+a)\theta_{s_i}} \exp\left[-a\frac{1-\theta_{s_i}}{\theta_{s_i}}\right]\right\} \\ -\left\{1 - \frac{a+\theta_{s_{i-1}}}{(1+a)\theta_{s_{i-1}}} \exp\left[-a\frac{1-\theta_{s_{i-1}}}{\theta_{s_{i-1}}}\right]\right\} \end{array}\right]} \right\},$$

$$\frac{\partial \ell_m(\boldsymbol{\Psi})}{\partial \alpha} = \sum_{i=0}^{m_2} B^{(\alpha)}(r_i)\left[\frac{a+\theta_{r_i}}{(1+a)\theta_{r_i}}\right]^{-1} \exp\left[a\frac{1-\theta_{r_i}}{\theta_{r_i}}\right] + \frac{m_0}{\alpha} + \sum_{i=0}^{m_0} \log(1+\lambda\, x_i)$$

$$+ \sum_{i=0}^{m_0} \frac{2 - a\exp[(1+\lambda\, x_i)^{\alpha} - 1]}{(1+\lambda\, x_i)^{-\alpha}[\log(1+\lambda\, x_i)]^{-1}}$$

$$+ \sum_{i=0}^{m_1} \left\{\frac{\left[B^{(\alpha)}(s_i) - B^{(\alpha)}(s_{i-1})\right]}{\left(\begin{array}{l}\left[\left\{1 - \frac{a+\theta_{s_i}}{(1+a)\theta_{s_i}}\exp\left[-a\frac{1-\theta_{s_i}}{\theta_{s_i}}\right]\right\}\right. \\ \left.-\left\{1 - \frac{a+\theta_{s_{i-1}}}{(1+a)\theta_{s_{i-1}}}\exp\left[-a\frac{1-\theta_{s_{i-1}}}{\theta_{s_{i-1}}}\right]\right\}\right]\end{array}\right)}\right\},$$

$$\frac{\partial \ell_m(\boldsymbol{\Psi})}{\partial \lambda} = \sum_{i=0}^{m_2} C^{(\lambda)}(r_i)\left[\frac{a+\theta_{r_i}}{(1+a)\theta_{r_i}}\right]^{-1} \exp\left[a\frac{1-\theta_{r_i}}{\theta_{r_i}}\right] + \frac{m_0}{\lambda} + (\alpha-1)\sum_{i=0}^{m_0} \frac{x_i}{1+\lambda\, x_i}$$

$$+ \alpha\sum_{i=0}^{m_0} \frac{x_i\{2 - a\exp[(1+\lambda\, x_i)^{\alpha} - 1]\}}{(1+\lambda\, x_i)^{1-\alpha}}$$

$$+ \sum_{i=0}^{m_1} \left\{\frac{\left[C^{(\lambda)}(s_i) - C^{(\lambda)}(s_{i-1})\right]}{\left(\begin{array}{l}\left[\left\{1 - \frac{a+\theta_{s_i}}{(1+a)\theta_{s_i}}\exp\left[-a\frac{1-\theta_{s_i}}{\theta_{s_i}}\right]\right\}\right. \\ \left.-\left\{1 - \frac{a+\theta_{s_{i-1}}}{(1+a)\theta_{s_{i-1}}}\exp\left[-a\frac{1-\theta_{s_{i-1}}}{\theta_{s_{i-1}}}\right]\right\}\right]\end{array}\right)}\right\}$$

where

$$A^{(a)}(s_i) = \frac{\left(\frac{\{a+\theta_{s_i}\}\{\exp[(1+\lambda\, s_i)^{\alpha}-1]-1\}}{(1+a)\theta_{s_i}} - \frac{a\{\exp[1-(1+\lambda s_i)^{\alpha}]-1\}}{(1+a)^2\exp\{2[1-(1+\lambda\, s_i)^{\alpha}]\}}\right)}{\exp[a\{\exp[(1+\lambda s_i)^{\alpha}-1]-1\}]},$$

$$B^{(\alpha)}(s_i) = \frac{\left(\begin{array}{l}\frac{(1+\lambda s_i)^{\alpha}\log(1+\lambda s_i)\exp[1-(1+\lambda s_i)^{\alpha}]}{[(1+a)\exp(a\{\exp[(1+\lambda\, s_i)^{\alpha}-1]-1\}-(1+\lambda s_i)^{\alpha}+1)]^{-1}} \\ + \frac{(1+a)\exp(a\{\exp[(1+\lambda s_i)^{\alpha}-1]-1\}-(1+\lambda s_i)^{\alpha}+1)\log(1+\lambda s_i)}{\{a+\exp[1-(1+\lambda s_i)^{\alpha}]\}^{-1}\{a\exp[(1+\lambda s_i)^{\alpha}-1]\ \ -1\}^{-1}(1+\lambda s_i)^{-\alpha}}\end{array}\right)}{[(1+a)\exp(a\{\exp[(1+\lambda s_i)^{\alpha}-1]-1\}-(1+\lambda s_i)^{\alpha}+1)]^2},$$

$$C^{(\lambda)}(s_i) = -\frac{\left(\begin{array}{l}\frac{\alpha s_i\left((1+a)\{a+\theta_{s_i}\}-(1+a)\theta_{s_i}\right)}{\{(1+a)\theta_{s_i}\}^2(1+\lambda s_i)^{1-\alpha}\exp[(1+\lambda\, s_i)^{\alpha}-1]} \\ -\frac{a\alpha s_i(1+\lambda s_i)^{\alpha-1}\{a+\theta_{s_i}\}}{(1+a)\theta_{s_i}[\exp[(1+\lambda s_i)^{\alpha}-1]-1]^{-1}}\end{array}\right)}{\exp(-a\{\exp[(1+\lambda\, s_i)^{\alpha}-1]-1\})},$$

$$A^{(a)}(s_{i-1}) = \frac{\left(\frac{\{a+\exp[1-(1+\lambda s_{i-1})^{\alpha}]\}\{\exp[(1+\lambda s_{i-1})^{\alpha}-1]-1\}}{(1+a)\theta_{s_{i-1}}} - \frac{a\{\exp[1-(1+\lambda s_{i-1})^{\alpha}]-1\}}{(1+a)^2\exp\{2[1-(1+\lambda s_{i-1})^{\alpha}]\}}\right)}{\exp[a\{\exp[(1+\lambda s_{i-1})^{\alpha}-1]-1\}]},$$

$$B^{(\alpha)}(s_{i-1}) = \frac{\left(\begin{array}{c} \dfrac{(1+\lambda\, s_{i-1})^{\alpha}\log(1+\lambda\, s_{i-1})\theta_{s_{i-1}}}{[(1+a)\exp(a\{\exp[(1+\lambda\, s_{i-1})^{\alpha}-1]-1\}-(1+\lambda\, s_{i-1})^{\alpha}+1)]^{-1}} \\ +\dfrac{(1+a)\exp(a\{\exp[(1+\lambda\, s_{i-1})^{\alpha}-1]-1\}-(1+\lambda\, s_{i-1})^{\alpha}+1)\log(1+\lambda\, s_{i-1})}{\left\{a+\theta_{s_{i-1}}\right\}^{-1}\{a\exp[(1+\lambda\, s_{i-1})^{\alpha}-1]-1\}^{-1}(1+\lambda\, s_{i-1})^{-\alpha}} \end{array} \right)}{[(1+a)\exp(a\{\exp[(1+\lambda\, s_{i-1})^{\alpha}-1]-1\}-(1+\lambda\, s_{i-1})^{\alpha}+1)]^{2}},$$

$$C^{(\lambda)}(s_{i-1}) = -\frac{\left(\begin{array}{c} \dfrac{\alpha s_{i-1}\left((1+a)\left\{a+\theta_{s_{i-1}}\right\}-(1+a)\theta_{s_{i-1}}\right)}{\left\{(1+a)\theta_{s_{i-1}}\right\}^{2}(1+\lambda s_{i})^{1-\alpha}\exp[(1+\lambda\, s_{i-1})^{\alpha}-1]} \\ -\dfrac{a\alpha s_{i-1}(1+\lambda s_{i-1})^{\alpha-1}\left\{a+\theta_{s_{i-1}}\right\}}{(1+a)\theta_{s_{i-1}}\{\exp[(1+\lambda\, s_{i-1})^{\alpha}-1]-1\}^{-1}} \end{array} \right)}{\exp(-a\{\exp[(1+\lambda\, s_{i-1})^{\alpha}-1]-1\})},$$

$$A^{(a)}(r_i) = -\frac{\left(\dfrac{\{a+\exp[1-(1+\lambda r_i)^{\alpha}]\}\{\exp[(1+\lambda r_i)^{\alpha}-1]-1\}}{(1+a)\exp[1-(1+\lambda r_i)^{\alpha}]} - \dfrac{a\{\exp[1-(1+\lambda r_i)^{\alpha}]-1\}}{(1+a)^{2}\exp\{2[1-(1+\lambda r_i)^{\alpha}]\}}\right)}{\exp[a\{\exp[(1+\lambda r_i)^{\alpha}-1]-1\}]},$$

$$B^{(\alpha)}(r_i) = -\frac{\left(\begin{array}{c} \dfrac{(1+\lambda\, r_i)^{\alpha}\log(1+\lambda\, s_i)\exp[1-(1+\lambda r_i)^{\alpha}]}{\left[(1+a)\exp\left(a\left\{\exp\left[(1+\lambda\, r_i)^{\alpha}-1\right]-1\right\}-(1+\lambda r_i)^{\alpha}+1\right)\right]^{-1}} \\ +\dfrac{(1+a)\exp\left(a\left\{\exp\left[(1+\lambda r_i)^{\alpha}-1\right]-1\right\}-(1+\lambda r_i)^{\alpha}+1\right)\log(1+\lambda r_i)}{\{a+\exp[1-(1+\lambda r_i)^{\alpha}]\}^{-1}\{a\exp[(1+\lambda r_i)^{\alpha}-1]\ -1\}^{-1}(1+\lambda r_i)^{-\alpha}} \end{array} \right)}{[(1+a)\exp(a\{\exp[(1+\lambda r_i)^{\alpha}-1]-1\}-(1+\lambda r_i)^{\alpha}+1)]^{2}},$$

$$C^{(\lambda)}(r_i) = \frac{\left(\begin{array}{c} \dfrac{\alpha r_i\left((1+a)\{a+\exp[1-(1+\lambda r_i)^{\alpha}]\}-(1+a)\exp[1-(1+\lambda r_i)^{\alpha}]\right)}{\{(1+a)\exp[1-(1+\lambda r_i)^{\alpha}]\}^{2}(1+\lambda r_i)^{1-\alpha}\exp[(1+\lambda r_i)^{\alpha}-1]} \\ -\dfrac{a\alpha r_i(1+\lambda r_i)^{\alpha-1}\{a+\exp[1-(1+\lambda r_i)^{\alpha}]\}}{(1+a)\exp[1-(1+\lambda r_i)^{\alpha}]\{\exp[(1+\lambda r_i)^{\alpha}-1]-1\}^{-1}} \end{array} \right)}{\exp(-a\{\exp[(1+\lambda r_i)^{\alpha}-1]-1\})}$$

and

$$\theta_{\bullet} = \exp[1-(1+\lambda\ \bullet)^{\alpha}],$$