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A MODIFIED MECHANICAL QUADRATURE FORMULA AND ITS EXTENSIONS

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Abstract. The typical mechanical quadrature formula is modified as

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n}\sum_{j=0}^{m} A_{(i,j)} f^{(j)}(x_i),$$

where $f(x) \in C^{(m)}[a,b]$ and $A_{(i,j)}$ are the quadrature weights. Based on the Taylor-series expansion technique, the methods for determining the quadrature weights $A_{(i,j)}$ with the known quadrature points x_i are given. The corresponding convergence and error estimate are made, then a sequence of Romberg-like quadrature formulae are analyzed. The modified mechanical quadrature formulae are further extended to solve the Riemann-Liouville fractional integral. Numerical results are carried out to show the effectiveness of the proposed methods by comparing some known methods. The proposed methods can be used to solve various linear and nonlinear integral equations with continuous and weakly singular kernels arising in practical physics, mechanics and engineering.

Keywords: modified mechanical quadrature formula; convergence and error estimate; Romberg-like quadrature formulae; Riemann-Liouville fractional integral

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1. Introduction

Since many definite integrals cannot be solved directly by using the Newton-Leibniz formula, numerical integration formulae are very important to evaluate approximately their values. For

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$f(x) \in C[a,b]$, the typical quadrature formula is given as

$$I[f] = \int_a^b f(x)dx \approx \sum_{i=0}^{n} A_i f(x_i), \tag{1.1}$$

with the weights A_i and the quadrature points $x_i \in [a,b]$ $(i = 0,1,\cdots,n)$. A method of giving A_i and x_i is that of evaluating the definite integral $I[f]$ such as the Newton-Cotes quadrature formulae, the Gaussian quadrature formulae and so on [1]. In the formula (1.1), one only considers the contributions of the values $f(x_i)$ to the integrals. However, in a practical application, it is not enough only to consider the values of $f(x)$ on the points x_i, since some experimental data may give the values of $f(x_i)$ and the derivatives $f^{(j)}(x_i)$ $(j = 1,2,\cdots,m)$. For example, in order to determine the displacement S of the moving object, one should measure the instantaneous velocity $V(t_i)$ and the acceleration $a(t_i)$ on the discretization points $t_i \in [a,b]$. According to (1.1), it is easy to evaluate the displacement S as

$$S = \int_a^b V(t)dt \approx \sum_{i=0}^{n} A_i V(t_i). \tag{1.2}$$

Clearly, in (1.2), one has neglected the values of $a(t_i)$. It motivates us strongly to consider the contributions of $a(t_i)$ to the displacement S. Then we may get more accurate results of the displacement S and decrease the quantity of the measuring points t_i.

Moreover, it is noted that a complicated function can be approximated by using the Hermite interpolation and the cubic spline interpolation, where the first-order derivative and the second-order derivative of the functions are used respectively [2]. In order to generally consider the jth-order derivatives of the function $f(x)$ for $j = 1,2,\cdots,m$, here we propose that the typical mechanical integration formula (1.1) is modified as

$$I[f] = \int_a^b f(x)dx \approx \sum_{i=0}^{n} \sum_{j=0}^{m} A_{(i,j)} f^{(j)}(x_i), \tag{1.3}$$

where $f(x) \in C^{(m)}[a,b]$ and $f^{(j)}(x_i)$ stand for the values of the jth-order derivative of $f(x)$ with respect to x on the points x_i. $A_{(i,j)}$ and x_i are the quadrature weights and the quadrature points respectively. When $m = 0$, the formula (1.2) is reduced to that in (1.1). On the other hand, it is noted that the fractional integrals and derivatives have attracted much attention due to their applications in engineering and physics [3,4]. It is very important to give numerical methods for evaluating the fractional integrals [5-8]. So the formula in (1.2) will be extended to compute the fractional integrals.

To achieve the objectives of formulating the formula in (1.2) together with its extensions and applications in evaluating the fractional integrals, the paper is organized as follows. In Section 2, a sequence of methods for determining $A_{(i,j)}$ and x_i will be given by using the Taylor-series expansion formula. The corresponding convergence and error estimate will also be addressed. Section 3 will give a sequence of Romberg-like quadrature formulae. The modified mechanical quadrature formula is extended to solve the Riemann-Liouville fractional integral in Section 4. Numerical results are carried out to show the effectiveness of the proposed methods in Section 5. Section 6 shows the main conclusions.

2. Determination of the quadrature weights

From the viewpoint of practical applications, the methods of determining the quadrature weights $A_{(i,j)}$ and the quadrature points x_i in the formula (1.2) should be given. Usually, similar to Newton-Cotes quadrature formulae, it is convenient to choose the equidistant quadrature points as

$$x_i = a + ih, \qquad i = 0, 1, \cdots, n, \qquad h = \frac{b-a}{n}. \tag{2.1}$$

When $f(x_i)$ and the first-order derivatives $f'(x_i)$ are known, the piecewise Hermite interpolation polynomial can be used to determine the coefficients $A_{(i,j)}$. Furthermore, if we know the values of $f(x_i)$, $f'(x_i)$ and $f''(x_i)$, the spline interpolation function of degree three is suitable to give the coefficients $A_{(i,j)}$. In what follows, generally if the values of $f^{(j)}(x_i)$ $(i = 0, 1, \cdots, n, j = 0, 1, \cdots, m)$ are known, we give a simple method of determining the coefficients $A_{(i,j)}$ based on the Taylor-series expansion technique.

Now by using the equidistant quadrature points, we obtain

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx = h \sum_{i=0}^{n-1} \int_0^1 f(x_i + h\xi)d\xi, \tag{2.2}$$

where the variable change of $x = x_i + h\xi$ is used. It is assumed that $f(x_i + \xi h)$ can be expanded as the following Taylor series

$$f(x_i+\xi h) = f(x_i) + \cdots + \frac{f^{(m)}(x_i)}{m!}(h\xi)^m + \frac{f^{(m+1)}(x_i+\theta_i h\xi)}{(m+1)!}(h\xi)^{m+1}, \tag{2.3}$$

where $0 < \theta_i < 1$. Then inserting (2.3) into (2.2), one has

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1}\sum_{j=0}^{m} \frac{h^{j+1}}{(j+1)!} f^{(j)}(x_i), \tag{2.4}$$

and the remainder

$$R_1 = \frac{h^{m+2}}{(m+1)!}\sum_{i=0}^{n-1}\int_0^1 f^{(m+1)}(x_i+\theta_i h\xi)\xi^{m+1}d\xi.$$

It is further supposed that $f^{(m+1)}(x)$ is continuous on $[a,b]$. Then based on the second integral mean value theorem, there exists $\xi_i \in (0,1)$ such that

$$\begin{aligned} R_1 &= \frac{h^{m+2}}{(m+1)!}\sum_{i=0}^{n-1} f^{(m+1)}(x_i+\theta_i h\xi_i)\int_0^1 \xi^{m+1}d\xi \\ &= \frac{h^{m+2}}{(m+2)!}\sum_{i=0}^{n-1} f^{(m+1)}(x_i+\theta_i h\xi_i). \end{aligned}$$

Under the consideration of

$$m \le \frac{1}{n}\sum_{i=0}^{n-1} f^{(m+1)}(x_i+\theta_i h\xi_i) \le M, \tag{2.5}$$

where M and m are the maximum and the minimum of $f^{(m+1)}(x)$ on $[a,b]$ respectively, we have

$$R_1 = \frac{nh^{m+2}}{(m+2)!}f^{(m+1)}(\eta_1) = \frac{(b-a)h^{m+1}}{(m+2)!}f^{(m+1)}(\eta_1), \tag{2.6}$$

with $\eta_1 \in (a,b)$.

Comparing (1.3) and (2.4), it is found that

$$A_{(i,j)} = \frac{h^{j+1}}{(j+1)!}, \qquad i = 0,1,\cdots,n-1;\, j = 0,1,2,\cdots,m,$$

and

$$A_{(n,j)} = 0, \qquad j = 0,1,2,\cdots,m. \tag{2.7}$$

From (2.7), it is seen that here we have neglected the values of $f^{(j)}(b)$ $(j = 0,1,2,\cdots,m)$. That is, the values of $f^{(j)}(b)(j = 0,1,2,\cdots,m)$ have no contributions to the numerical integration formula in (2.4).

On the other hand, application of the variable change $x = x_{i+1} - h\xi$ yields

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx = h \sum_{i=0}^{n-1} \int_0^1 f(x_{i+1} - h\xi)d\xi. \tag{2.8}$$

Under the assumption of

$$f(x_{i+1} - h\xi) = f(x_{i+1}) + \cdots + \frac{f^{(m)}(x_{i+1})}{m!}(-h\xi)^m + \frac{f^{(m+1)}(x_{i+1} - \bar{\theta}_i h\xi)}{(m+1)!}(-h\xi)^{m+1},$$

where $0 < \bar{\theta}_i < 1$, we have

$$\int_a^b f(x)dx \approx h \sum_{i=0}^{n-1} \sum_{j=0}^{m} \frac{(-h)^j}{(j+1)!} f^{(j)}(x_{i+1}), \tag{2.9}$$

with the remainder

$$R_2 = h\frac{(-h)^{m+1}}{(m+1)!} \sum_{i=0}^{n-1} \int_0^1 f^{(m+1)}(x_{i+1} - \bar{\theta}_i h\xi)\xi^{m+1}d\xi = \frac{(b-a)(-h)^{m+1}}{(m+2)!} f^{(m+1)}(\eta_2).$$

for $\eta_2 \in (a,b)$. Based on the formula (2.9), it follows

$$A_{(i,j)} = -\frac{(-h)^{j+1}}{(j+1)!}, \qquad i = 1, \cdots, n; j = 0, 1, 2, \cdots, m,$$

$$A_{(0,j)} = 0, \qquad j = 0, 1, 2, \cdots, m.$$

Generally, the sum of (2.4) and (2.9) with the parameter ω leads to

$$\begin{aligned}\int_a^b f(x)dx &\approx (1-\omega) \sum_{i=0}^{n-1} \sum_{j=0}^{m} \frac{h^{j+1}}{(j+1)!} f^{(j)}(x_i) - \omega \sum_{i=1}^{n} \sum_{j=0}^{m} \frac{(-h)^{j+1}}{(j+1)!} f^{(j)}(x_i) \\ &= \sum_{j=0}^{m} \frac{h^{j+1}}{(j+1)!} \Big\{ (1-\omega) f^{(j)}(a) + \omega(-1)^{j+2} f^{(j)}(b) \\ &+ \sum_{i=1}^{n-1} [(1-\omega) - \omega(-1)^{j+1}] f^{(j)}(x_i) \Big\},\end{aligned} \tag{2.10}$$

and for $i = 1, \cdots, n-1$, $j = 0, 1, 2, \cdots, m$, it gives

$$A_{(0,j)} = (1-\omega)\frac{h^{j+1}}{(j+1)!},$$

$$A_{(i,j)} = [(1-\omega) - \omega(-1)^{j+1}]\frac{h^{j+1}}{(j+1)!},$$

$$A_{(n,j)} = \omega(-1)^{j+2}\frac{(-h)^{j+1}}{(j+1)!},$$

together with the remainder

$$R_3 = (1-\omega)R_1 + \omega R_2.$$

When $\omega = 0$, the formula in (2.10) is reduced to that in (2.4) and when $\omega = 1$, the formula in (2.9) can be derived from that in (2.10). Furthermore, we have the following theorem.

Theorem 1 *It is assumed that*

$$\|f^{(m+1)}(x)\|_\infty = \max_{a\leq x\leq b} |f^{(m+1)}(x)| = M < +\infty.$$

Then when $m \to +\infty$ or $h \to 0$, the formulae (2.4), (2.9) and (2.10) are convergent to the definite integral $I[f]$ with

$$R_1 \to 0, \qquad R_2 \to 0, \qquad R_3 \to 0.$$

Proof From (2.5), we have

$$|R_1| \leq \frac{h^{m+2}}{(m+1)!} M \sum_{i=0}^{n-1} \int_0^1 \xi^{m+1} d\xi = \frac{nh^{m+2}}{(m+2)!} M = \frac{(b-a)h^{m+1}}{(m+2)!} M.$$

It is easy to see that when $m \to +\infty$ or $h \to 0$, one arrives at $R_1 \to 0$. Similarly, we can obtain $R_2 \to 0$ and $R_3 \to 0$. So the formulae (2.4), (2.9) and (2.10) are convergent for $m \to +\infty$ or $h \to 0$.

3. Romberg-like quadrature formulae

In the above section, several methods for determining the coefficients $A_{(i,j)}$ are given by using the Taylor-series expansion technique. In order to derive high-order approximation methods from low-order ones, it is seen that the extrapolation method of Richardson is effective. Here we further apply the ideas of Romberg quadrature formulae to the proposed methods and the derived ones are called as Romberg-like quadrature formulae.

Based on the formula (2.4), we define

$$I = \int_a^b f(x)dx \approx {}_1T_n^m = \sum_{i=0}^{n-1} \sum_{j=0}^{m} \frac{h^{j+1}}{(j+1)!} f^{(j)}(x_i), \quad n \geq 1. \tag{3.1}$$

From the remainder R_1, after some computations, one can get

$$\frac{I - {}_1T_n^m}{I - {}_1T_{2n}^m} \approx 2^{m+1},$$

and

$$I \approx RT_1 = \frac{2^{m+1}{}_1T_{2n}^m - {}_1T_n^m}{2^{m+1}-1}.$$

In what follows, we analyze the error estimate of the Romberg-type quadrature formula RT_1. One has the following theorem

Theorem 2 *Suppose that $f:[a,b]\to \mathbf{R}$ is $(m+2)$-times continuously differentiable. The Romberg-type quadrature formula RT_1 has the following error estimate*

$$|I - RT_1| \le \frac{(b-a)^2 h^{m+1}}{(m+2)!(2^{m+1}-1)} \parallel f^{(m+2)}(x) \parallel_\infty$$

with $h=(b-a)/n$.

proof We can calculate that

$$|I - RT_1| = \left|\frac{2^{m+1}(I - {}_1T_{2n}^m) - (I - {}_1T_n^m)}{2^{m+1}-1}\right|.$$

Moreover from the expression of R_1 and $h=(b-a)/n$, it is seen that

$$\begin{aligned}
I - {}_1T_{2n}^m &= \frac{h^{m+2}}{2^{m+2}(m+1)!}\sum_{i=0}^{2n-1}\int_0^1 f^{(m+1)}(x_i+\theta_i\xi h/2)\xi^{m+1}d\xi \\
&= \frac{h^{m+2}}{2^{m+2}(m+1)!}\sum_{i=0}^{2n-1} f^{(m+1)}(\bar{x}_i)\int_0^1 \xi^{m+1}d\xi \\
&= \frac{h^{m+2}}{2^{m+2}(m+2)!}\frac{\sum_{i=0}^{2n-1} f^{(m+1)}(\bar{x}_i)}{2n}2n \\
&= \frac{(b-a)h^{m+1}}{2^{m+1}(m+2)!}f^{(m+1)}(\eta_3)
\end{aligned}$$

where the integral mean value theorem and the intermediate value theorem have been used with $x_i = a+(hi)/2$, $x_i < \bar{x}_i < x_i + h/2$ and $\eta_3 \in [a,b]$. Similarly, we have

$$\begin{aligned}
I - {}_1T_n^m &= \frac{h^{m+2}}{(m+1)!}\sum_{i=0}^{n-1}\int_0^1 f^{(m+1)}(y_i+\vartheta_i h\xi)\xi^{m+1}d\xi \\
&= \frac{h^{m+2}}{(m+1)!}\sum_{i=0}^{n-1} f^{(m+1)}(\tilde{y}_i)\int_0^1 \xi^{m+1}d\xi \\
&= \frac{h^{m+2}}{(m+2)!}\frac{\sum_{i=0}^{n-1} f^{(m+1)}(\tilde{y}_i)}{n}n \\
&= \frac{(b-a)h^{m+1}}{(m+2)!}f^{(m+1)}(\eta_4),
\end{aligned}$$

where $y_i = a + ih$, $y_i < \tilde{y}_i < y_i + h$ and $\eta_4 \in [a,b]$. Now the error estimate can be rewritten as

$$
\begin{aligned}
|I - RT_1| &= \frac{1}{2^{m+1}-1}\frac{(b-a)h^{m+1}}{(m+2)!}\left|f^{(m+1)}(\eta_1) - f^{(m+1)}(\eta_2)\right| \\
&= \frac{1}{2^{m+1}-1}\frac{(b-a)h^{m+1}}{(m+2)!}\left|f^{(m+2)}(\bar{\eta})(\eta_1 - \eta_2)\right| \\
&\le \frac{1}{2^{m+1}-1}\frac{(b-a)^2h^{m+1}}{(m+2)!} \parallel f^{(m+2)}(x) \parallel_\infty
\end{aligned}
$$

where $\bar{\eta}$ is between η_3 and η_4. The proof is completed.

Moreover, from (2.9) and (2.10), we define

$$
\begin{aligned}
{}_2T_n^m &= h\sum_{i=0}^{n-1}\sum_{j=0}^{m}\frac{(-h)^j}{(j+1)!}f^{(j)}(x_{i+1}), \\
{}_3T_n^m &= \sum_{j=0}^{m}\frac{h^{j+1}}{(j+1)!}\Big\{(1-\omega)f^{(j)}(a) + \omega(-1)^{j+2}f^{(j)}(b) \\
&\quad + \sum_{i=1}^{n-1}[(1-\omega) - \omega(-1)^{j+1}]f^{(j)}(x_i)\Big\}.
\end{aligned}
$$

Then the Romberg-like quadrature formulae RT_2 and RT_3 can be given as

$$
RT_2 = \frac{2^{m+1}{}_2T_{2n}^m - {}_2T_n^m}{2^{m+1}-1}, \qquad RT_3 = \frac{2^{m+1}{}_3T_{2n}^m - {}_3T_n^m}{2^{m+1}-1}.
$$

For the errors of RT_2 and RT_3, we have the following theorem.

Theorem 3 *Let $f : [a,b] \to \mathbf{R}$ be $(m+2)$-times continuously differentiable. The Romberg-type quadrature formulae RT_2 and RT_3 have the following error estimate*

$$
\begin{aligned}
|I - RT_2| &\le \frac{(b-a)^2h^{m+1}}{(m+2)!(2^{m+1}-1)} \parallel f^{(m+2)}(x) \parallel_\infty, \\
|I - RT_3| &\le \frac{(b-a)^2h^{m+1}}{(m+2)!(2^{m+1}-1)} \parallel f^{(m+2)}(x) \parallel_\infty,
\end{aligned}
$$

with $h = (b-a)/n$.

Proof The error estimate of RT_2 can be obtained similar to that of RT_1 and the proof procedure has been omitted here. In what follows, we focus on the error estimate of RT_3. It is calculated that

$$
|I - RT_3| = \left|\frac{2^{m+1}(I - {}_3T_{2n}^m) - (I - {}_3T_n^m)}{2^{m+1}-1}\right|.
$$

Furthermore, under the consideration of R_3 and $h=(b-a)/n$, it follows

$$\begin{aligned}
I-{}_3T_{2n}^m &= \frac{(1-\omega)h^{m+2}}{2^{m+2}(m+1)!}\sum_{i=0}^{2n-1}\int_0^1 f^{(m+1)}(x_i+\theta_i\xi h/2)\xi^{m+1}d\xi \\
&- \frac{\omega(-h)^{m+2}}{2^{m+2}(m+1)!}\sum_{i=1}^{2n}\int_0^1 f^{(m+1)}(x_i-\bar{\theta}_i\xi h/2)\xi^{m+1}d\xi \\
&= \frac{(1-\omega)h^{m+2}}{2^{m+2}(m+1)!}\sum_{i=0}^{2n-1} f^{(m+1)}(\bar{x}_i)\int_0^1 \xi^{m+1}d\xi \\
&- \frac{\omega(-h)^{m+2}}{2^{m+2}(m+1)!}\sum_{i=1}^{2n} f^{(m+1)}(\tilde{x}_i)\int_0^1 \xi^{m+1}d\xi \\
&= \frac{2nh^{m+2}}{2^{m+2}(m+2)!}\left[\frac{(1-\omega)\sum_{i=0}^{2n-1} f^{(m+1)}(\bar{x}_i)}{2n}-\frac{\omega(-1)^{m+2}\sum_{i=1}^{2n} f^{(m+1)}(\tilde{x}_i)}{2n}\right] \\
&= \frac{(b-a)h^{m+1}}{2^{m+1}(m+2)!}\left[(1-\omega)f^{(m+1)}(\eta_5)-\omega(-1)^{m+2}f^{(m+1)}(\eta_6)\right]
\end{aligned}$$

where $\eta_{5,6}\in[a,b]$. On the other hand, we have

$$\begin{aligned}
I-{}_3T_{n}^m &= \frac{(1-\omega)h^{m+2}}{(m+1)!}\sum_{i=0}^{n-1}\int_0^1 f^{(m+1)}(y_i+\vartheta_i h\xi)\xi^{m+1}d\xi \\
&- \frac{\omega(-h)^{m+2}}{(m+1)!}\sum_{i=1}^{n}\int_0^1 f^{(m+1)}(y_i-\bar{\vartheta}_i h\xi)\xi^{m+1}d\xi \\
&= \frac{(1-\omega)h^{m+2}}{(m+1)!}\sum_{i=0}^{n-1} f^{(m+1)}(\bar{y}_i)\int_0^1 \xi^{m+1}d\xi \\
&- \frac{\omega(-h)^{m+2}}{(m+1)!}\sum_{i=1}^{n} f^{(m+1)}(\tilde{y}_i)\int_0^1 \xi^{m+1}d\xi \\
&= \frac{nh^{m+2}}{(m+2)!}\left[\frac{(1-\omega)\sum_{i=0}^{n-1} f^{(m+1)}(\tilde{y}_i)}{n}-\frac{\omega(-1)^{m+1}\sum_{i=1}^{n} f^{(m+1)}(\tilde{y}_i)}{n}\right] \\
&= \frac{(b-a)h^{m+1}}{(m+2)!}[(1-\omega)f^{(m+1)}(\bar{\eta}_5)-\omega(-1)^{m+2}f^{(m+1)}(\bar{\eta}_6)]
\end{aligned}$$

with $\bar{\eta}_{5,6} \in [a,b]$. So one can rewrite the error estimate of RT_3 as

$$
\begin{aligned}
|I - RT_3| &= \frac{1}{2^{m+1}-1}\frac{(b-a)h^{m+1}}{(m+2)!}\Big|(1-\omega)[f^{(m+1)}(\eta_5) - f^{(m+1)}(\bar{\eta}_5)] \\
&\quad +(-1)^{m+1}\omega[f^{(m+1)}(\eta_6) - f^{(m+1)}(\bar{\eta}_6)]\Big| \\
&= \frac{1}{2^{m+1}-1}\frac{(b-a)h^{m+1}}{(m+2)!}\Big|(1-\omega)f^{(m+2)}(\bar{\xi}_3)(\eta_5 - \bar{\eta}_5) \\
&\quad +(-1)^{m+1}\omega f^{(m+2)}(\bar{\xi}_4)(\eta_6 - \bar{\eta}_6)\Big| \\
&\le \frac{1}{2^{m+1}-1}\frac{(b-a)^2h^{m+1}}{(m+2)!} \parallel f^{(m+2)}(x) \parallel_\infty .
\end{aligned}
\tag{3.2}
$$

This completes the proof. Comparison between Theorems 1 and 2 shows that when $m \ge 1$, the Romberg-like quadrature formulae are suitable to accelerate the modified mechanical quadrature formula.

4. Quadrature formulae for the R-L fractional integral

Recently, fractional integrals and their applications have attracted much attention [3–6]. Numerical methods of evaluating fractional order integrals and solving fractional order differential equations are popular [7,8]. In the section, the modified mechanical integration formulae will be extended to compute the Riemann-Liouville fractional integral expressed as

$$
{}_aI^{\alpha}[f](t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(x)}{(t-x)^{1-\alpha}}dx, \quad 0 < \alpha < 1.
$$

Now the equidistant quadrature points are chosen as

$$
x_i = a + ih, \qquad i = 0, 1, \cdots, n, \qquad h = \frac{t-a}{n},
$$

and one has

$$
\begin{aligned}
\frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(x)}{(t-x)^{1-\alpha}}dx &= \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \frac{f(x)}{(t-x)^{1-\alpha}}dx \\
&= \frac{h^{\alpha}}{\Gamma(\alpha)}\sum_{i=0}^{n-1}\int_0^1 [(n-i)-\xi]^{\alpha-1} f(x_i + h\xi)d\xi.
\end{aligned}
\tag{4.1}
$$

Similar to the former section, application of the Taylor-series expansion (2.3) yields

$$
\frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(x)}{(t-x)^{1-\alpha}}dx \approx \sum_{i=0}^{n-1}\sum_{j=0}^{m}\frac{[(n-i)h]^{\alpha+j}}{\Gamma(\alpha)\Gamma(j+1)}B_{\beta_i}(j+1,\alpha)f^{(j)}(x_i), \tag{4.2}
$$

where $B_{\beta_i}(j+1,\alpha)$ is the incomplete Beta function defined as

$$B_x(p,q)=\int_0^x \tau^{p-1}(1-\tau)^{q-1}d\tau,$$

with

$$x=\beta_i=\frac{1}{n-i},\quad p=j+1,\quad q=\alpha.$$

Under the assumption of $f(x)\in C^{(m+1)}[a,t]$, the remainder can be computed as

$$\begin{aligned}
\tilde{R}_1 &= \sum_{i=0}^{n-1}\frac{h^{\alpha+m+1}}{\Gamma(\alpha)\Gamma(m+2)}\int_0^1[(n-i)-\xi]^{\alpha-1}\xi^{m+1}f^{(m+1)}(x_i+\theta_i h\xi)d\xi \\
&= \sum_{i=0}^{n-1}\frac{h^{\alpha+m+1}f^{(m+1)}(x_i+\theta_i h\xi_i)}{\Gamma(\alpha)\Gamma(m+2)}\int_0^1[(n-i)-\xi]^{\alpha-1}\xi^{m+1}d\xi \\
&\le \sum_{i=0}^{n-1}\frac{h^{\alpha+m+1}\hat{M}}{\Gamma(\alpha)\Gamma(m+2)}\int_0^1[(n-i)-\xi]^{\alpha-1}\xi^{m+1}d\xi \\
&= \sum_{i=0}^{n-1}\frac{[(n-i)h]^{\alpha+m+1}\hat{M}}{\Gamma(\alpha)\Gamma(m+2)}B_{\beta_i}(m+2,\alpha),
\end{aligned}\tag{4.3}$$

hereafter $\hat{M}=\max_{a\le x\le t}|f^{(m+1)}(x)|$.

On the other hand, applying the variable change $x=x_{i+1}-h\xi$, one has

$$\begin{aligned}
\frac{1}{\Gamma(\alpha)}\int_a^t\frac{f(x)}{(t-x)^{1-\alpha}}dx &= \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\frac{f(x)}{(t-x)^{1-\alpha}}dx \\
&= \frac{h^\alpha}{\Gamma(\alpha)}\sum_{i=1}^{n}\int_0^1[(n-i+1)-\xi]^{\alpha-1}f(x_i-h+h\xi)d\xi.
\end{aligned}\tag{4.4}$$

Based on the Taylor-series expansion in (2.3), it follows

$$\frac{1}{\Gamma(\alpha)}\int_a^t\frac{f(x)}{(t-x)^{1-\alpha}}dx\approx\sum_{i=1}^{n}\sum_{j=0}^{m}\frac{[(n-i)h]^{\alpha+j}}{\Gamma(\alpha)\Gamma(j+1)}\tilde{B}_{-\beta_i}(j+1,\alpha)f^{(j)}(x_{i+1}).\tag{4.5}$$

where

$$\tilde{B}_{-x}(p,q)=\int_{-x}^0\tau^{p-1}(1-\tau)^{q-1}d\tau,$$

with

$$x=\beta_i=\frac{1}{n-i},\quad p=j+1,\quad q=\alpha.$$

The remainder is

$$\begin{aligned}\tilde{R}_2 &= \sum_{i=1}^{n}\frac{[(n-i)h]^{\alpha+m+1}}{\Gamma(\alpha)\Gamma(m+2)}f^{(m+1)}(x_i+\bar{\theta}_i h\xi_i)\tilde{B}_{-\beta_i}(m+2,\alpha)\\ &\leq \sum_{i=1}^{n}\frac{[(n-i)h]^{\alpha+m+1}\hat{M}}{\Gamma(\alpha)\Gamma(m+2)}\tilde{B}_{-\beta_i}(m+2,\alpha)\end{aligned} \tag{4.6}$$

In addition, the convex combination of (4.2) and (4.4) with the weight ω is

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(x)}{(t-x)^{1-\alpha}}dx &\approx (1-\omega)\sum_{i=0}^{n-1}\sum_{j=0}^{m}\frac{[(n-i)h]^{\alpha+j}}{\Gamma(\alpha)\Gamma(j+1)}B_{\beta i}(j+1,\alpha)f^{(j)}(x_i)\\ &\quad+\omega\sum_{i=1}^{n}\sum_{j=0}^{m}\frac{[(n-i)h]^{\alpha+j}}{\Gamma(\alpha)\Gamma(j+1)}\tilde{B}_{-\beta_i}(j+1,\alpha)f^{(j)}(x_i)\\ &=\sum_{j=0}^{m}\frac{h^{\alpha+j}}{\Gamma(\alpha)\Gamma(j+1)}\{(1-\omega)n^{\alpha+j}B_{\beta_0}(j+1,\alpha)f^{(j)}(a)\\ &\quad+\sum_{i=1}^{n-1}[(1-\omega)B_{\beta_i}(j+1,\alpha)+\omega\tilde{B}_{-\beta_i}(j+1,\alpha)](n-i)^{\alpha+j}\\ &\quad\cdot f^{(j)}(x_i)+\omega(-1)^j B_{\beta_{n-1}}(1,\alpha+j)f^{(j)}(x_i)\},\end{aligned} \tag{4.7}$$

and the remainder

$$\tilde{R}_3=(1-\omega)\tilde{R}_1+\omega\tilde{R}_2. \tag{4.8}$$

In the end, we have the following theorem.

Theorem 4 *It is assumed that*

$$\|f^{(m+1)}(x)\|_\infty=\max_{a\leq x\leq b}\mid f^{(m+1)}(x)\mid=\hat{M}<+\infty.$$

Then when $m\to+\infty$ or $h\to 0$, the formulae (4.2), (4.4) and (4.6) are convergent with

$$\tilde{R}_1\to 0,\quad \tilde{R}_2\to 0,\quad \tilde{R}_3\to 0.$$

Proof From (4.3), we have

$$\tilde{R}_1\mid\leq\frac{nh^{\alpha+m+1}}{\Gamma(\alpha)\Gamma(m+2)}\hat{M}\sum_{i=0}^{n-1}B_{\beta_i}(m+2,\alpha)\leq\frac{(t-a)^2h^{\alpha+m-1}}{\Gamma(\alpha)\Gamma(m+3)}\hat{M}. \tag{4.9}$$

It is easy to see that when $m\to+\infty$ or $h\to 0$, one arrives at $\tilde{R}_1\to 0$. Similarly, we can obtain $\tilde{R}_2\to 0$ and $\tilde{R}_3\to 0$. So the formulae (4.2), (4.4) and (4.6) are convergent for $m\to+\infty$ or $h\to 0$.

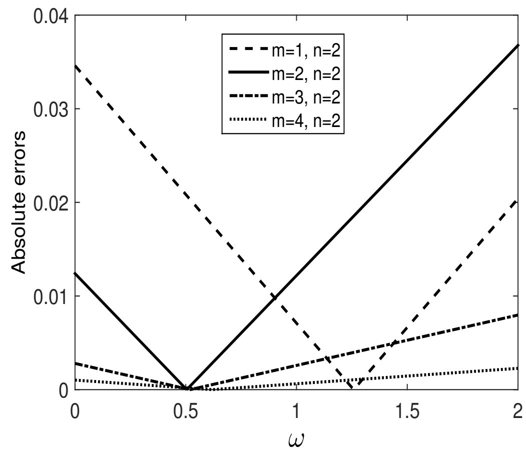

Fig. 1. The variations of absolute errors versus ω for $(m,n) = (1,2)$, $(2,2)$, $(3,2)$ and $(4,2)$ respectively.

5. Numerical examples

In order to illustrate the effectiveness of the proposed methods, several numerical examples are carried out in the section. For convenience, the parameters m and n are written as a pair of (m,n) in the following numerical computations.

Example 1 It is assumed that $f(x) = \dfrac{1}{x^2+1}$, and we calculate

$$I[f] = \int_0^1 f(x)dx,$$

with the exact result $I = \dfrac{\pi}{4}$. Based on the present method, we choose $(m,n) = (2,2)$, $(2,4)$, $(2,8)$, $(4,2)$, $(4,4)$ and $(4,8)$ to compute. The weight ω is chosen as $-1.0, -0.5, 0, 0.5$ and 1.0, respectively. The absolute errors between the approximate solution $I_{(m,n,\omega)}$ and the exact solution I are given in Table 1. It is seen from Table 1 that with the increasing of m or n for a fixed ω, the absolute errors are decreasing. In particular, it is found that the parameter of ω has large influences on the accuracy of the approximate solutions. The variations of the absolute errors versus ω are depicted in Figures 1 and 2 for some pairs of m and n. It is seen from Figures 1 and 2 that for each pair of m and n, there is a point of ω where the absolute error is tending to zero. That is to say, we can obtain a good approximation by choosing small values of m and n, then searching a suitable value of ω.

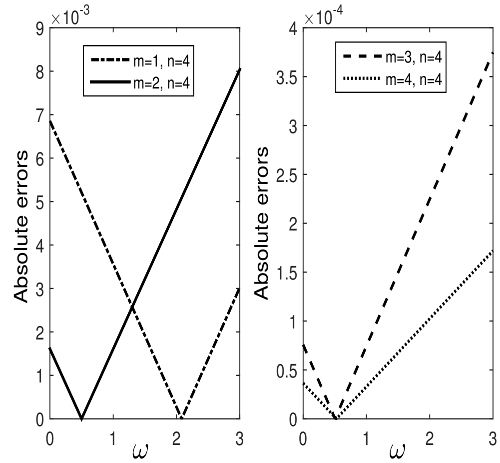

Fig. 2. The variations of absolute errors versus ω for $(m,n)=(1,4)$, $(2,4)$, $(3,4)$ and $(4,4)$ respectively.

Table 1 The absolute errors of the approximate and exact solutions for Example 1.

	$\lvert I-I_{(m,n,\omega)}\rvert$				
(m,n)	$\omega=-1$	$\omega=-0.5$	$\omega=0$	$\omega=0.5$	$\omega=1$
$(2,2)$	3.6981e-2	2.4690e-2	1.2398e-2	1.0650e-4	1.2185e-2
$(2,4)$	4.8263e-3	3.2178e-3	1.6093e-3	7.2557e-7	1.6078e-3
$(2,8)$	6.0863e-4	4.0576e-4	2.0288e-4	1.1352e-8	2.0286e-4
$(4,2)$	2.6678e-3	1.8438e-3	1.0198e-3	1.9588e-4	6.2808e-4
$(4,4)$	1.0606e-4	7.1309e-5	3.6562e-5	1.8151e-6	3.2932e-5
$(4,8)$	3.4208e-6	2.2900e-6	1.1592e-6	2.8381e-8	1.1024e-6

Moreover, it is significant to compare the present method with the composite trapezoid (CT) formula and the composite Simpson (CS) formula. In Table 2, we choose $m=2$ and $\omega=0.5$ to carry out Example 1 again by using the present method. The absolute errors in Table 2 show that the proposed method is effective.

Table 2 The absolute errors of approximate and exact solutions for Example 1.

	The absolute errors		
n	The present method: $(2,n,0.5)$	The CT formula	The CS formula
2	1.0650e-4	1.0398e-2	2.0648e-3
4	7.2557e-7	2.6040e-3	6.0065e-6
8	1.1352e-8	6.5104e-4	3.7783e-8

On the other hand, it is noted that the analysis and applications of Abel integral equation have been studied widely [9]. Abel integral equation is applied to model a fractional-order system and it is written as [3]

$$\frac{1}{\Gamma(\alpha)}\int_0^t \frac{\varphi(x)}{(t-x)^{1-\alpha}}dx = f(t), \quad t>0, \tag{5.1}$$

with $0<\alpha<1$. Its solution is expressed as the following well-known formula

$$\varphi(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f(x)}{(t-x)^{\alpha}}dx, \quad t>0. \tag{5.2}$$

Moreover, some numerical methods have been proposed to give the approximate solution of Abel integral equation (5.1) such as the quadrature methods [9], the Chebyshev polynomials method [10] and the Taylor expansion method [11]. Obviously, based on the present method, the approximate solution can be obtained effectively by evaluating the integral in (5.1) and the following example is given.

Example 2 Consider an Abel integral equation as follows [11]

$$\int_0^t \frac{\varphi(x)}{(t-x)^{1/2}}dx = e^t - 1.$$

The exact solution is $\varphi(t) = e^t erf(\sqrt{t})/\sqrt{\pi}$, where $erf(\sqrt{t})$ denotes the error function. Now based on the proposed method, the approximate solution can be computed as

$$\begin{aligned}\varphi_{(m,n,\omega)}(t) \approx \sum_{j=0}^{m} \frac{h^{\alpha+j}}{\Gamma(j+1)\pi}\{&(1-\omega)n^{\alpha+j}B_{\beta_0}(j+1,\alpha)\\ &+\sum_{i=1}^{n-1}[(1-\omega)B_{\beta_i}(j+1,\alpha)+\omega\tilde{B}_{-\beta_i}(j+1,\alpha)](n-i)^{\alpha+j}e^{x_i}\\ &+\omega(-1)^j B_{\beta_{n-1}}(1,\alpha+j)e^{x_i}\}.\end{aligned}$$

The absolute errors are given in Table 3 for $t = 0.2, 0.4, 0.6, 0.8, 1$ and $\omega = 0.5$ under $(m,n) = (2,4)$ and $(m,n) = (2,8)$, respectively. For convenience, the corresponding absolute errors by using the known method in [11] are computed and given in Table 3. It is found from Table 3 that the present method is effective and suitable to solve Abel integral equations.

Table 3 The absolute errors of approximate and exact solutions for Example 2.

	The absolute errors		
	The present method: $\omega = 0.5$		The known method
t	$(m,n) = (2,4)$	$(m,n) = (2,8)$	m=2 [11]
0.2	5.8811e-7	5.5128e-8	0.00290
0.4	7.5312e-6	7.2634e-7	0.00409
0.6	3.5292e-5	3.5034e-6	0.00528
0.8	1.0965e-4	1.1209e-5	0.00690
1.0	2.7211e-4	2.8658e-5	0.00940

6. Conclusions

The typical mechanical quadrature formula has been modified as a novel numerical integration formula by considering the derivatives of integrand. The Taylor-series expansion technique has been applied to obtain the coefficients of the modified mechanical quadrature formula. In order to accelerate the given quadrature formula, the Romberg-like quadrature formulae have been analyzed. The corresponding convergence and error estimate have been given. The proposed method is further extended to numerically solving Riemann-Liouville fractional integral. Numerical results show the effectiveness of the proposed formulae. In the future, the given methods will be used to numerically solve various linear and nonlinear integral equations arising from ordinary differential equations, physics, mechanical and engineering.

Conflict of Interests

The authors declare that there is no conflict of interests.

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REFERENCES

[1] P.K. Kythe, M.R. Schäferkotter, Handbook of computational methods for integration, Chapman & Hall/CRC Press, New York (2005).

[2] P.W. Zhang, T.J. Li, Numerical Analysis (In Chinese), Peking University Press, Beijing, (2007).

[3] I. Podbubny, Fractional Differetial Equations, Academic Press, San Diego, (1999).

[4] L. Debnath, Recent applications of fractional calculus to science and engineering, Int. J. Math. Math. Sci. 54 (2003) 3413-3442.

[5] C. H. Lubich, Discretized fractional calculus, SIAM J. Math. Anal. 17 (1986) 704-719.

[6] R. Khalil, M. A. Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math. 264 (2014) 65-70.

[7] K. Diethelm, N. J. Ford, A. D. Freed, Y. Luchko, Algorithms for the fractional calculus: A selection of numerical methods, Comput. Methods Appl. Mech. Engrg. 194 (2005) 743-773.

[8] J.R. Li, A fast time stepping method for evaluating fractional integrals, SIAM J. Sci. Comput. 31 (2.3) (2010) 4696-4714.

[9] R. Gorenflo, S. Vessella, Abel integral equation: Analysis and Applications, Springer-Verlag, Berlin (1991).

[10] R. Piessens, Computing integral transforms and solving integral equations using Chebyshev polynomial approximations, J. Comput. Appl. Math. 121 (2000) 113-124.

[11] L. Huang, Y. Huang, X.F. Li, Approximate solution of Abel integral equation, Comput. Math. Appl. 56 (2008) 1748-1777.