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MAPPINGS ON FUZZY T_0 TOPOLOGICAL SPACES IN QUASI-COINCIDENCE SENSE

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Abstract. In this paper, we introduce two notions of T_0 property in fuzzy topological spaces by using quasi-coincidence sense and we show relations among ours and other such notions. Then, we establish that all these notions satisfy good extension property. Also hereditary, productive and projective properties are satisfied by these notions. We observe that all these concepts are preserved under one-one, onto, fuzzy open and fuzzy continuous mappings. Finally, we discuss initial and final fuzzy topological spaces on our concepts.

Keywords: fuzzy topological space; quasi-coincidence; fuzzy T_0 topological space; good extension; initial fuzzy topology; final fuzzy topology.

2010 AMS Subject Classification: 54A40.

1. Introduction

Chang [5] defined fuzzy topological spaces (fts, in short) in 1968 by using the concept of fuzzy sets introduced by Zadeh [26] in 1965. Since then extensive work on fuzzy topological spaces has been carried out by many researchers like Gouguen [7], Wong [24], Lowen [13],

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Warren [23], Hutton [10] and others. Separation axioms are important parts in fuzzy topological spaces. Many works [6, 1, 8, 3, 19, 20, 21] on separation axioms have been done by researchers. Among those axioms, fuzzy T_0 type is one and it has been already introduced in the literature. There are many articles on fuzzy T_0 topological space which are created by many authors like P. Wuyts and R. Lowen [25], D. M. Ali [1], Srivastava et al. [22], M. S. Hossain and D. M. Ali [9] and many others.

The purpose of this paper is to further contribute to the development of fuzzy topological spaces specially on fuzzy T_0 topological spaces. In the present paper, fuzzy T_0 topological space is defined by using quasi-coincidence sense and relations among the given and other such notions are shown here. It is showed that the good extension property is satisfied on our notions. It is also showed that the hereditary, order preserving, productive, and projective properties hold on the new concepts. In the last section of this paper, initial and final fuzzy topological spaces are discussed on author's concept.

2. Preliminaries

In this section, we recall some concepts occurring in the papers [1, 26] which will be needed in the sequel. In the present paper, X and Y always denote non empty sets and $I = [0,1], I_1 = [0,1)$. The class of all fuzzy sets on a non empty set X is denoted by I^X and fuzzy sets on X are denoted as u, v, w etc. Crisp subsets of X are denoted by capital letters U, V, W etc. throughout this paper.

Definition 2.1. [26] A function u from X into the unit interval I is called a fuzzy set in X. For every $x \in X, u(x) \in I$ is called the grade of membership of x in u.

Definition 2.2. [16] A fuzzy set u in X is called a fuzzy singleton if and only if $u(x) = r, 0 < r \leq 1$, for a certain $x \in X$ and $u(y) = 0$ for all points y of X except x. The fuzzy singleton is denoted by x_r and x is its support. The class of all fuzzy singletons in X will be denoted by $S(X)$. If $u \in I^X$ and $x_r \in S(X)$, then we say that $x_r \in u$ if and only if $r \leq u(x)$.

Definition 2.3. [11] A fuzzy singleton x_r is said to be quasi-coincidence with u, denoted by $x_r qu$ if and only if $u(x)+r>1$. If x_r is not quasi-coincidence with u, we write $x_r \bar{q} u$ and defined as $u(x)+r \leq 1$.

Definition 2.4. [5] Let f be a mapping from a set X into a set Y and u be a fuzzy subset of X. Then f and u induce a fuzzy subset v of Y defined by

$v(y) = \sup\{u(x)\}$ if $x \in f^{-1}[\{y\}] \neq \phi, x \in X$

$= 0$ otherwise.

Definition 2.5. [5] Let f be a mapping from a set X into a set Y and v be a fuzzy subset of Y. Then the inverse of v written as $f^{-1}(v)$ is a fuzzy subset of X defined by $f^{-1}(v)(x) = v(f(x))$, for $x \in X$.

Definition 2.6. [5] Let $I = [0,1], X$ be a non empty set and I^X be the collection of all mappings from X into I, *i.e.* the class of all fuzzy sets in X. A fuzzy topology on X is defined as a family t of members of I^X, satisfying the following conditions.

(i) $1, 0 \in t$,

(ii) If $u \in t$ for each $i \in \Lambda$, then $\bigcup_{i\in\Lambda} u_i \in t$, where Λ is an index set.

(iii) If $u, v \in t$ then $u \cap v \in t$.

The pair (X,t) is called a fuzzy topological space (in short fts) and members of t are called $t-open$ fuzzy sets.

Definition 2.7. [17] The function $f : (X,t) \to (Y,s)$ is called fuzzy continuous if and only if for every $v \in s, f^{-1}(v) \in t$, the function f is called fuzzy homeomorphic if and only if f is bijective and both f and f^{-1} are fuzzy continuous.

Definition 2.8. [15] The function $f : (X,t) \to (Y,s)$ is called fuzzy open if and only if for every open fuzzy set u in $(X,t), f(u)$ is open fuzzy set in (Y,s).

Definition 2.9. [12] Let $\{X_i, i \in \Lambda\}$, be any class of sets and let X denotes the Cartesian product of these sets, $i.e. X = \prod_{i\in\Lambda} X_i$. Note that X consists of all points $p = < a_i, i \in \Lambda >$, where $a_i \in X_i$. Recall that, for each $j_0 \in \Lambda$, we define the projection π_{j_0} from the product set X to the coordinate space X_{j_0}, i.e. $\pi_{j_0} : X \longrightarrow X_{j_0}$ by $\pi_{j_0}(< a_i, i \in \Lambda >) = a_{j_0}$.

Definition 2.10. [24] Let $\{X_i, i \in \Lambda\}$ be a family of non empty sets. Let $X = \prod_{i\in\Lambda} X_i$ be the usual product of X_i's and let π_i be the projection from X into X_i. Further assume that each

X_i is a fuzzy topological space with fuzzy topology t_i. Now, the fuzzy topology generated by $\{\pi_i^{-1}(b_i) : b_i \in t_i, i \in \Lambda\}$ as a sub basis, is called the product fuzzy topology on X. Clearly if w is a basis element in the product, then there exist $i_1, i_2, i_3, ..., i_n \in \Lambda$ such that $w(x) = \min\{b_i(x_i) : i = 1, 2, 3, ..., n\}$, where $x = (x_i)_{i \in \Lambda} \in X$.

Definition 2.11. [18] Let f be a real valued function on a topological space. If $\{x : f(x) > \alpha\}$ is open for every real α, then f is called lower semi continuous function.

Definition 2.12. [13] Let X be a non empty set and T be a topology on X. Let $t = \omega(T)$ be the set of all lower semi continuous functions from (X, T) to I (with usual topology). Thus $\omega(T) = \{u \in I^X : u^{-1}(\alpha, 1] \in T\}$ for each $\alpha \in I_1$. It can be shown that $\omega(T)$ is a fuzzy topology on X.

Definition 2.13. [14] The initial fuzzy topology on a set X for the family of fuzzy topological spaces $\{(X_i, t_i)_{i \in \Lambda}\}$ and the family of functions $\{f_i : X \to (X_i, t_i)\}_{i \in \Lambda}$ is the smallest fuzzy topology on X making each f_i fuzzy continuous. It is easily seen that it is generated by the family $\{f_i^{-1}(u_i) : u_i \in t_i\}_{i \in \Lambda}$.

Definition 2.14. [14] The final fuzzy topology on a set X for the family of fuzzy topological spaces $\{(X_i, t_i)_{i \in \Lambda}\}$ and the family of functions $\{f_i : (X_i, t_i) \to X\}_{i \in \Lambda}$ is the finest fuzzy topology on X making each f_i fuzzy continuous.

Theorem 2.1. [2] A bijective mapping from an fts (X, t) to an fts (Y, s) preserves the value of a fuzzy singleton (fuzzy point). Note: Preimage of any fuzzy singleton (fuzzy point) under bijective mapping preserves its value.

3. Main results

In this section, we discuss about our notions and findings. Some well-known properties are discussed here by using our concepts.

Definition 3.1. A fuzzy topological space (X, t) is called

(a) $FT_0(i)$ if and only if for any pair $x_r, y_s \in S(X)$ with $x \neq y$, there exists $u \in t$ such that $x_r q u, y_s \bar{q} u$, or there exists $v \in t$ such that $y_s q v, x_r \bar{q} v$.

(b) $FT_0(ii)$ if and only if for any pair $x_r, y_s \in S(X)$ with $x \neq y$, there exists $u \in t$ such that $x_r q u, y_s \cap u = 0$ or there exists $v \in t$ such that $y_s q v, x_r \cap v = 0$.

(c) ([22]) $FT_0(iii)$ if and only if for any pair $x, y \in X$ with $x \neq y$, there exists $u \in t$ such that $u(x) = 1$, $u(y) = 0$ or there exists $v \in t$ such that $v(y) = 1$, $v(x) = 0$.

Example 3.1. Let $X = \{x, y\}$, $u \in I^X$, where $u(x) = 1$, $u(y) = 0$. Consider the fuzzy topology t on X generated by $\{0, u, 1\}$. Let x_r, y_s be fuzzy points in X with $x \neq y$.Then $u(x) + r > 1$ and $u(y) + s \leq 1$ for $r, s \in (0, 1]$. Therefore $x_r qu$, $y_s \bar{q} u$. This shows that (X, t) is $FT_0(i)$. Also, as $u(y) = 0$, $y_s \cap u = 0$. Thus, (X, t) is $FT_0(ii)$.

Theorem 3.1. For a fuzzy topological space (X, t) the following implications are true:
$FT_0(ii) \Rightarrow FT_0(i)$, $FT_0(iii) \Rightarrow FT_0(i)$, $FT_0(iii) \Rightarrow FT_0(ii)$
But, in general the converse is not true.

Proof: $FT_0(ii) \Rightarrow FT_0(i)$: Let (X, t) be a fuzzy topological space and (X, t) is $FT_0(ii)$. Also let x_r, y_s be fuzzy singletons in X with $x \neq y$. Since (X, t) is $T_0(ii)$ fuzzy topological space, there exists $u \in t$ such that $x_r qu, y_s \cap u = 0$ or there exists $v \in t$ such that $y_s qv, x_r \cap v = 0$.
To prove (X, t) is $FT_0(i)$, it is only needed to prove that $y_s \bar{q} u$.
Now, $y_s \cap u = 0 \Rightarrow u(y) = 0 \Rightarrow u(y) + s \leq 1 \Rightarrow y_s \bar{q} u$
It follows that there exists $u \in t$ such that $x_r qu, y_s \bar{q} u$. Hence, it is clear that (X, t) is $FT_0(i)$. To show $FT_0(i) \not\Rightarrow FT_0(ii)$, we give a counter example.

Counter example 3.1. Let $X = \{x, y\}$ and $u \in I^X$ be given by $u(x) = 1$, $u(y) = 0.1$. Consider the fuzzy topology t on X generated by $\{0, u, 1\}$.
For $0 < r \leq 1, 0 < s < 0.9$,
$u(x) + r > 1 \Rightarrow x_r qu$ and, $u(y) + s \leq 1 \Rightarrow y_s \bar{q} u$
Hence, (X, t) is $FT_0(i)$. But $u(y) \neq 0 \Rightarrow y_s \cap u \neq 0$ Hence, (X, t) is not $FT_0(ii)$.

Proof: $FT_0(iii) \Rightarrow FT_0(i)$: Let (X, t) be a fuzzy topological space and (X, t) is $FT_0(iii)$. Also let x_r, y_s be fuzzy singletons in X with $x \neq y$. Since (X, t) is $FT_0(iii)$ fuzzy topological space, there exists $u \in t$ such that $u(x) = 1$, $u(y) = 0$ or there exists $v \in t$ such that $v(y) = 1$, $v(x) = 0$.
To prove (X, t) is $FT_0(i)$, it is needed to prove that $x_r qu$, $y_s \bar{q} u$.
Now, $u(x) = 1 \Rightarrow u(x) + r > 1$, for any $r \in (0, 1] \Rightarrow x_r qu$
and $u(y) = 0 \Rightarrow u(y) + s \leq 1$, for any $s \in (0, 1] \Rightarrow y_s \bar{q} u$.
It follows that there exists $u \in t$ such that $x_r qu$, $y_s \bar{q} u$. Hence, (X, t) is $FT_0(i)$. To show $FT_0(i) \not\Rightarrow$

$FT_0(iii)$, we give a counter example.

Counter example 3.2. Let $X = \{x, y\}$ and $u \in I^X$ be given by
$u(x) = 1-\varepsilon, u(y) = 0$, where $\varepsilon = \frac{r}{2}\ for\ r \in (0,1]$. Consider the fuzzy topology t on X generated by $\{0,u,1\}$.
Then, $u(x) = 1-\frac{r}{2} \Rightarrow u(x)+\frac{r}{2} = 1 \Rightarrow u(x)+r > 1 \Rightarrow x_r qu$ and, $u(y)+s \leq 1 \Rightarrow y_s \bar{q} u$. Hence, (X,t) is $FT_0(i)$. But $u(x) \neq 1$. Thus, (X,t) is not $FT_0(iii)$.

Proof: $FT_0(iii) \Rightarrow FT_0(ii)$: Let (X,t) be a fuzzy topological space and (X,t) is $FT_0(iii)$. Also let x_r, y_s be fuzzy singletons in X with $x \neq y$. Since (X,t) is $FT_0(iii)$ fuzzy topological space, there exists $u \in t$ such that $u(x) = 1, u(y) = 0$ or there exists $v \in t$ such that $v(y) = 1, v(x) = 0$. To prove (X,t) is $FT_0(ii)$, it is needed to prove that $x_r qu, y_s \cap u = 0$.
Now, $u(x) = 1 \Rightarrow u(x)+r > 1,$ for any $r \in (0,1] \Rightarrow x_r qu$
and $u(y) = 0 \Rightarrow y_s \cap u = 0$. It follows that there exists $u \in t$ such that $x_r qu, y_s \cap u = 0$. Hence, (X,t) is $FT_0(ii)$. To show $FT_0(ii) \not\Rightarrow FT_0(iii)$, we give a counter example.

Counter example 3.3. Let $X = \{x, y\}$ and $u \in I^X$ be given by
$u(x) = 1-\varepsilon, u(y) = 0$, where $\varepsilon = \frac{r}{2}\ for\ r \in (0,1]$. Consider the fuzzy topology t on X generated by $\{0,u,1\}$.
Then, $u(x) = 1-\frac{r}{2} \Rightarrow u(x)+\frac{r}{2} = 1 \Rightarrow u(x)+r > 1 \Rightarrow x_r qu$ and, $u(y) = 0 \Rightarrow y_s \cap u = 0$. Hence, (X,t) is $FT_0(ii)$. But, $u(x) \neq 1$. Hence, (X,t) is not $FT_0(iii)$. This completes the proof.

Now, we shall show that our notions satisfy the good extension property.

Theorem 3.2. Let (X,T) be a topological space. Consider the following statements:
(1) (X,T) be a T_0 Topological Space
(2) $(X,\omega(T))$ be an $FT_0(i)$ space.
(3) $(X,\omega(T))$ be an $FT_0(ii)$ space.
The following implications are true:
$(1) \Leftrightarrow (2), (1) \Leftrightarrow (3)$.

Proof of $(1) \Leftrightarrow (2)$: Let (X,T) be a topological space and (X,T) is T_0. We have to prove that $(X,\omega(T))$ is $FT_0(i)$. Let x_r, y_s be fuzzy points in X with $x \neq y$. Since (X,T) is T_0 topological space, we have, there exists $U \in T$ such that $x \in U, y \notin U$. From the definition of lower semi continuous we have $1_U \in \omega(T)$ and $1_U(x) = 1, 1_U(y) = 0$. Then

$1_U(x) + r > 1 \Rightarrow x_r q 1_U$ and $1_U(y) + s \leq 1 \Rightarrow y_s \bar{q} 1_U$

It follows that there exists $1_U \in \omega(T)$ such that $x_r q 1_U, y_s \bar{q} 1_U$. Hence $(X,\omega(T))$ is $FT_0(i)$. Thus $(1) \Rightarrow (2)$ holds.

Conversely, let $(X,\omega(T))$ be a fuzzy topological space and $(X,\omega(T))$ is $FT_0(i)$. We have to prove that (X,T) is T_0. Let x,y be points in X with $x \neq y$. Since $(X,\omega(T))$ is $FT_0(i)$ topological space, we have, for any fuzzy singletons x_r, y_r in X, there exists $u \in t$ such that $x_r qu, y_r \bar{q} u$ or there exists $v \in t$ such that $y_r qv, x_r \bar{q} v$.

Now, $x_r qu \Rightarrow u(x) + r > 1 \Rightarrow u(x) > 1 - r = \alpha \Rightarrow x \in u^{-1}(\alpha, 1]$

and $y_r \bar{q} u \Rightarrow u(y) + r \leq 1 \Rightarrow u(y) \leq 1 - r = \alpha \Rightarrow u(y) \leq \alpha \Rightarrow y \notin u^{-1}(\alpha, 1]$

Also, $u^{-1}(\alpha, 1] \in T$. It follows that $\exists\, u^{-1}(\alpha, 1] \in T$ such that $x \in u^{-1}(\alpha, 1]$, $y \notin u^{-1}(\alpha, 1]$. Thus $(2) \Rightarrow (1)$ holds. Similarly, we can prove that $(1) \Leftrightarrow (3)$.

Now, we shall show that our notions satisfy the hereditary property.

Theorem 3.3. Let (X,t) be a fuzzy topological space, $A \subseteq X$, $t_A = \{u/A : u \in t\}$, then

(a) (X,t) is $FT_0(i) \Rightarrow (A,t_A)$ is $FT_0(i)$

(b) (X,t) is $FT_0(ii) \Rightarrow (A,t_A)$ is $FT_0(ii)$.

Proof of (a) : Let (X,t) be a fuzzy topological space and (X,t) is $FT_0(i)$. We have to prove that (A,t_A) is $FT_0(i)$. Let x_r, y_s be fuzzy singletons in A with $x \neq y$. Since $A \subseteq X$, these fuzzy singletons are also fuzzy singletons in X. Also since (X,t) is $FT_0(i)$ fuzzy topological space, we have, there exists $u \in t$ such that $x_r qu, y_s \bar{q} u$ or there exists $v \in t$ such that $y_s qv, x_r \bar{q} v$. For $A \subseteq X$, we have $u/A \in t$.

Now, $x_r qu \Rightarrow u(x) + r > 1, x \in X \Rightarrow u/A(x) + r > 1, x \in A \subseteq X \Rightarrow x_r qu/A$

and $y_s \bar{q} u \Rightarrow u(y) + s \leq 1, y \in X \Rightarrow u/A(y) + s \leq 1, y \in A \subseteq X \Rightarrow y_s \bar{q} u/A$

Hence, (A,t_A) is $FT_0(i)$. Proof of (b) is similar to proof of (a).

Now, we shall show that our notions satisfy the productive and projective properties.

Theorem 3.4. Let $(X_i, t_i), i \in \Lambda$ be fuzzy topological spaces and $X = \Pi_{i \in \Lambda} X_i$ and t be the product topology on X, then

(a) for all $i \in \Lambda, (X_i, t_i)$ is $FT_0(i)$ if and only if (X, t) is $FT_0(i)$.

(b) for all $i \in \Lambda, (X_i, t_i)$ is $FT_0(ii)$ if and only if (X, t) is $FT_0(ii)$.

Proof of (b) : Let for all $i \in \Lambda, (X_i, t_i)$ is $FT_0(ii)$ space. We have to prove that (X, t) is $FT_0(ii)$. Let x_r, y_s be fuzzy singletons in X with $x \neq y$. Then $(x_i)_r, (y_i)_s$ are fuzzy singletons with $x_i \neq y_i$ for some $i \in \Lambda$. Since (X_i, t_i) is $FT_0(ii)$, there exists $u_i \in t_i$ such that $(x_i)_r q u_i, (y_i)_s \cap u_i = 0$ or there exists $v_i \in t_i$ such that $(y_i)_s q v_i, (x_i)_r \cap v_i = 0$. But, we have $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$.

Now, $(x_i)_r q u_i \Rightarrow u_i(x_i) + r > 1, x \in X \Rightarrow u_i(\pi_i(x)) + r > 1$

$\Rightarrow (u_i \circ \pi_i)(x) + r > 1 \Rightarrow x_r q (u_i \circ \pi_i)$

and $(y_i)_s \cap u_i = 0 \Rightarrow u_i(y_i) = 0, y \in X \Rightarrow u_i(\pi_i(y)) = 0, y \in X$

$\Rightarrow (u_i \circ \pi_i)(y) = 0 \Rightarrow y_s \cap (u_i \circ \pi_i) = 0$

It follows that there exists $(u_i \circ \pi_i) \in t_i$ such that $x_r q (u_i \circ \pi_i), y_s \cap (u_i \circ \pi_i) = 0$. Hence, (X, t) is $FT_0(ii)$.

Conversely, let (X, t) be a fuzzy topological space and (X, t) is $FT_0(ii)$. We have to prove that $(X_i, t_i), i \in \Lambda$ is $FT_0(ii)$. Let a_i be a fixed element in X_i. Let $A_i = \{x \in X = \Pi_{i \in \Lambda} X_i : x_j = a_j \text{ for some } i \neq j\}$. Then A_i is a subset of X, and hence (A_i, t_{A_i}) is a subspace of (X, t). Since (X, t) is $FT_0(ii)$, so (A_i, t_{A_i}) is $FT_0(ii)$. Now, we have A_i is homeomorphic image of X_i. Hence, for all $i \in \Lambda, (X_i, t_i)$ is $FT_0(ii)$ space. Thus (b) holds. Proof of (a) is similar to proof of (b).

Now, we shall show that our notions satisfy the order preserving property.

Theorem 3.5. Let (X, t) and (Y, s) be two fuzzy topological spaces and $f : X \to Y$ be a one-one, onto and fuzzy open map then,

(a) (X, t) is $FT_0(i) \Rightarrow (Y, s)$ is $FT_0(i)$

(b) (X, t) is $FT_0(ii) \Rightarrow (Y, s)$ is $FT_0(ii)$.

Proof of (a) : Let (X, t) be a fuzzy topological space and (X, t) is $FT_0(i)$. Also let x'_r, y'_s be fuzzy singletons in Y with $x' \neq y'$. Since f is onto then there exist $x, y \in X$ with $f(x) = x', f(y) = y'$ and x_r, y_s are fuzzy singletons in X with $x \neq y$ as f is one-one. Again since (X, t) is $FT_0(i)$ space, there exists $u \in t$ such that $x_r q u, y_s \bar{q} u$ or there exists $v \in t$ such that $y_s q v, x_r \bar{q} v$.

Now, $x_r q u \Rightarrow u(x) + r > 1$ and, $y_s \bar{q} u \Rightarrow u(y) + s \leq 1$

Now, $f(u)(x') = \{\sup u(x) : f(x) = x'\} \Rightarrow f(u)(x') = u(x)$, for some x

and $f(u)(y') = \{\sup u(y) : f(y) = y'\} \Rightarrow f(u)(y') = u(y)$, for some y

Also, since f is fuzzy open map then $f(u) \in s$ as $u \in t$.

Again, $u(x) + r > 1 \Rightarrow f(u)(x') + r > 1 \Rightarrow x'_r q f(u)$

and, $u(y) + s \leq 1 \Rightarrow f(u)(y') + s \leq 1 \Rightarrow y'_s \bar{q} f(u)$.

It follows that there exists $f(u) \in s$ such that $x'_r q f(u), y'_s \bar{q} f(u)$. Hence, (Y, s) is $FT_0(i)$ space. Proof of (b) is similar to proof of (a).

Theorem 3.6. Let (X, t) and (Y, s) be two fuzzy topological spaces and $f : X \to Y$ be a one-one and fuzzy continuous map then,

(a) (Y, s) is $FT_0(i) \Rightarrow (X, t)$ is $FT_0(i)$

(b) (Y, s) is $FT_0(ii) \Rightarrow (X, t)$ is $FT_0(ii)$.

Proof of (b) : Let (Y, s) be a fuzzy topological space and (Y, s) is $FT_0(ii)$. Also let x_r, y_s be fuzzy singletons in X with $x \neq y$.Then $(f(x))_r, (f(y))_s$ are fuzzy singletons in Y with $f(x) \neq f(y)$ as f is one-one. Again, since (Y, s) is $FT_0(ii)$ space, there exists $u \in s$ such that $(f(x))_r q u, (f(y))_s \cap u = 0$ or there exists $v \in s$ such that $(f(y))_s q v, (f(x))_r \cap v = 0$.

Now, $(f(x))_r q u \Rightarrow u(f(x)) + r > 1 \Rightarrow f^{-1}(u(x)) + r > 1$

$\Rightarrow (f^{-1}(u))(x) + r > 1 \Rightarrow x_r q f^{-1}(u)$

and, $(f(y))_s \cap u = 0 \Rightarrow u(f(y)) = 0 \Rightarrow f^{-1}(u(y)) = 0$

$\Rightarrow (f^{-1}(u))(y) = 0 \Rightarrow y_s \cap f^{-1}(u) = 0$.

Now, since f is fuzzy continuous map and $u \in s$ then $f^{-1}(u) \in t$. It follows that there exists $f^{-1}(u) \in t$ such that $x_r q f^{-1}(u), y_s \cap f^{-1}(u) = 0$. Hence, (X, t) is $FT_0(ii)$ space. Proof of (a) is similar to proof of (b).

As our next work, here we introduce two theorems on our second notion. The idea of these theorems are taken from M. R. Amin and M. S. Hossain [4].

Theorem 3.7. If $\{(X_i, t_i)\}_i \in \Lambda$ is a family of $FT_0(ii)$ fts and $\{f_i : X \to (X_i, t_i)\}_i \in \Lambda$, a family of one-one and fuzzy continuous functions, then the initial fuzzy topology on X for the family $\{f_i\}_i \in \Lambda$ is $FT_0(ii)$.

Proof: Let t be the initial fuzzy topology on X for the family $\{f_i\}_i \in \Lambda$. Let x_r, y_s be fuzzy singletons in X with $x \neq y$. Then $f_i(x), f_i(y) \in X_i$ and $f_i(x) \neq f_i(y)$ as f_i is one-one. Since (X_i, t_i) is $FT_0(ii)$, then for every two distinct fuzzy singletons $(f_i(x))_r, (f_i(y))_s$ in X_i, there exist fuzzy sets u_i or $v_i \in t_i$ such that $(f_i(x))_r q u_i, (f_i(y))_s \cap u_i = 0$ or $(f_i(y))_s q v_i, (f_i(x))_r \cap v_i = 0$.
Now, $(f_i(x))_r q u_i$ and $(f_i(y))_s \cap u_i = 0$.
That is $u_i(f_i(x)) + r > 1$ and $u_i(f_i(y)) = 0$.
That is $f_i^{-1}(u_i)(x) + r > 1$ and $f_i^{-1}(u_i)(y) = 0$.
This is true for every $i \in \Lambda$. So, $\inf f_i^{-1}(u_i)(x) + r > 1$ and $\inf f_i^{-1}(u_i)(y) = 0$.
Let $u = \inf f_i^{-1}(u_i)$. Then $u \in t$ as f_i is fuzzy continuous. So, $u(x) + r > 1$ and $u(y) = 0$. Hence, $x_r q u$ and $y_s \cap u = 0$. Therefore, (X, t) is $FT_0(ii)$. Thus, the proof is complete.

Theorem 3.8. If $\{(X_i, t_i)\}_i \in \Lambda$ is a family of $FT_0(ii)$ fts and $\{f_i(X_i, t_i) \to X\}_i \in \Lambda$, a family of fuzzy open and bijective function, then the final fuzzy topology on X for the family $\{f_i\}_i \in \Lambda$ is $FT_0(ii)$.

Proof: Let t be the initial fuzzy topology on X for the family $\{f_i\}_i \in \Lambda$. Let x_r, y_s be fuzzy singletons in X with $x \neq y$. Then $f_i^{-1}(x), f_i^{-1}(y) \in X_i$ and $f_i^{-1}(x) \neq f_i^{-1}(y)$ as f_i is bijective. Since (X_i, t_i) is $FT_0(ii)$, then for every two distinct fuzzy singletons $(f_i^{-1}(x))_r, (f_i^{-1}(y))_s$ in X_i, there exist fuzzy sets u_i or $v_i \in t_i$ such that $(f_i^{-1}(x))_r q u_i$, $(f_i^{-1}(y))_s \cap u_i = 0$ or $(f_i^{-1}(y))_s q v_i$, $(f_i^{-1}(x))_r \cap v_i = 0$.
Now, $(f_i^{-1}(x))_r q u_i$ and $(f_i^{-1}(y))_s \cap u_i = 0$.
That is, $u_i(f_i^{-1}(x)) + r > 1$ and $u_i(f_i^{-1}(y)) = 0$.
That is, $f_i(u_i)(x) + r > 1$ and $f_i(u_i)(y) = 0$.
This is true for every $i \in \Lambda$. So, $\inf f_i(u_i)(x) + r > 1$ and $\inf f_i(u_i)(y) = 0$. Let $u = \inf f_i(u_i)$. Then $u \in t$ as f_i is fuzzy open. So, $u(x) + r > 1$ and $u(y) = 0$. Hence, $x_r q u$ and $y_s \cap u = 0$. Therefore, (X, t) is $FT_0(ii)$. This completes the proof.

4. Conclusion

Fuzzy topology is an important and a major area of mathematics. In this paper, we introduce and study notion of T_0 separation axiom in fts in quasi-coincidence sense. We have shown that all of our concepts are good extension of their counterparts and are stronger than other such notion [22]. Further, we have shown that hereditary, productive, projective and order preserving properties hold on our concepts. Finally, initial and final fuzzy topologies are studied on one of our notions. We hope that the findings of this paper will be helpful for researchers to carry out a general framework for more expansion of fuzzy mathematics.

Conflict of Interests

The authors declare that there is no conflict of interests.

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