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FUZZY n-CONTINUOUS AND n-BOUNDED LINEAR OPERATORS

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Abstract. In this paper, we define three types of fuzzy n-continuous linear operators (strongly, weakly, and sequentially) and research the relation between three. Also strongly, weakly fuzzy n-bounded linear operators and the closed graph theorem are defined.

Keywords: fuzzy n-continuous linear operator; fuzzy n-bounded linear operator; strongly; weakly; sequentially.

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1. Introduction

In 1984, Katsaras [4] introduced the concept of fuzzy norm. In 1992, Felbin [5] introduced an idea of fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear space, so that corresponding metric associated to this fuzzy norm is a Kaleva type fuzzy metric. In 2005, Narayanan and Vijayabalaji [6] extended the notion of n-normed linear space to fuzzy n-normed linear space. In 2012, Hakan Efe [7] defined various types of continuities of operators and boundedness of linear operators. In2012, A.L.Soenjaya [9] defined n-bounded and n-continuous linear operators in n-normed linear space. In 2012, B.S.Reddy [8]

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introduced the concept of fuzzy-anti-n-continuous linear operator and three types of fuzzy-anti-n-continuous linear operators,also introduced the concept of fuzzy-anti-n-bounded linear operator and two types of fuzzy-anti-n-bounded linear operators. In 2015, Parijat Sinha [11] defined two types of fuzzy 2-bounded linear operators.

In this paper, we extend the notion of three types of fuzzy n-continuous linear operators (strongly, weakly, and sequentially) and research the relation between three. Also strongly , weakly fuzzy n-bounded linear operators and the closed graph theorem are defined.

2. Preliminaries

Definition 2.1. [2,3] Let X be a real linear space of dimension greater than $n-1$ and let $\|\cdot,\cdots,\cdot\|$ be a real valued function on X^n satisfying the following condition:

(1) $\|x_1,x_2,\cdots,x_n\| = 0$ if and only if $x_1,x_2,\cdots,x_n$ are linearly dependent;

(2) $\|x_1,x_2,\cdots,x_n\|$ is invariant under any permutation;

(3) $\|\alpha x_1,x_2,\cdots,x_n\| = |\alpha|\|x_1,x_2,\cdots,x_n\|$ for any $\alpha \in \mathbb{R}$;

(4) $\|x_0+x_1,x_2,\cdots,x_n\| \leq \|x_0,x_2,\cdots,x_n\| + \|x_1,x_2,\cdots,x_n\|$ for all $x_0,x_1,\cdots,x_n \in X$.

$\|\cdot,\cdots,\cdot\|$ is called an *n-norm* on X and the pair $(X,\|\cdot,\cdots,\cdot\|)$ is called an *n-normed linear space*.

Definition 2.2. [1] Let X be a linear space over K(field of real or complex numbers). A fuzzy subset N of $X^n \times R$ (R, the set of real numbers) is called a fuzzy n-norm on X if and only if:

(N1) For all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1,x_2,\cdots,x_n,t) = 0$;

(N2) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1,x_2,\cdots,x_n,t) = 1$ if and only if $x_1,x_2,\cdots,x_n$ are linearly dependent;

(N3) $N(x_1,x_2,\cdots,x_n,t)$ is invariant under any permutation of $x_1,x_2,\cdots,x_n$;

(N4) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1,x_2,\cdots,\lambda x_n,t) \doteq N(x_1,x_2,\cdots,x_n,\frac{t}{|\lambda|})$, if $\lambda \neq 0$;

(N5) For all $s,t \in \mathbb{R}$, $N(x_1,x_2,\cdots,x_n+x_n^{'},s+t) \geq \min\,\{N(x_1,x_2,\cdots,x_n,s), N(x_1,x_2,\cdots, x_n^{'},t)\}$;

(N6) $N(x_1,x_2,\cdots,x_n,t)$ is a non-decreasing function of $t \in \mathbb{R}$ and $\lim\limits_{t\to\infty} N(x_1,x_2,\cdots,x_n,t) = 1$.

Then (X,N) is called a fuzzy n-normed linear spaces or f-n-NLS in short.

Definition 2.3. [10] A sequence $\{x_n\}$ in a f-n-NLS (X,N) is said to be converge to x if given $0 < r < 1, t > 0$, there exists an integer $n_0 \in N$ such that $N(x_1, x_2, \cdots, x_{n-1}, x_n, t) > 1 - r$ for all $n \geq n_0$.

Theorem 2.4. [10] In a f-n-NLS (X,N) a sequence $\{x_n\}$ converges to x if and only if

$$\lim_{n\to\infty} N(x_1, x_2, \cdots, x_{n-1}, x_n - x, t) = 1$$

3. Fuzzy n-Continuous Linear Operators

Let (X,N_1) and (Y,N_2) are fuzzy n-normed linear spaces defined on the same field.

Definition 3.1. T is a mapping from $X_1 \times X_2 \times \cdots \times X_n$ to $Y_1 \times Y_2 \times \cdots \times Y_n$ where $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ are subspaces of (X,N_1) , (Y,N_2) respectively. Then T is said to be a fuzzy n-linear operator if

$$T(\sum_{i_n=1}^{n} x_1^{(i_n)}, \sum_{i_{n-1}=1}^{n} x_2^{(i_{n-1})}, \cdots, \sum_{i_1=1}^{n} x_n^{(i_1)}) = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} T(x_1^{(i_n)}, x_2^{(i_{n-1})}, \cdots, x_n^{(i_1)})$$

and

$$T(\alpha_1 x_1, \alpha_2 x_2 \cdots, \alpha_n x_n) = \alpha_1 \alpha_2 \cdots \alpha_n T(x_1, x_2 \cdots, x_n), \forall (x_1, \cdots, x_n) \in X_1 \times \cdots \times X_n$$

Definition 3.2. Let T be a fuzzy n-linear mapping from $X_1 \times X_2 \times \cdots \times X_n$ to $Y_1 \times Y_2 \times \cdots \times Y_n$, $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ are subspaces of (X,N_1) , (Y,N_2) respectively. Then T is called fuzzy n-continuous at $(x_0^{(1)}, x_0^{(2)}, \cdots, x_0^{(n)}) \in X_1 \times X_2 \times \cdots \times X_n$ if given $\varepsilon > 0, \alpha \in (0,1), \exists \delta = \delta(\alpha,\varepsilon) > 0, \beta = \beta(\alpha,\varepsilon) \in (0,1)$,such that for all$(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \in X_1 \times X_2 \times \cdots \times X_n$

$$N_1[(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) - (x_0^{(1)}, x_0^{(2)}, \cdots, x_0^{(n)}), \delta] > \beta$$
$$\Rightarrow N_2[T(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, \cdots, x_0^{(n)}), \varepsilon] > \alpha$$

If T is fuzzy n-continuous(briefly f-n-continuous) at every point of T: $X_1 \times X_2 \times \cdots \times X_n \to Y_1 \times Y_2 \times \cdots \times Y_n$, then T is fuzzy n-continuous on $X_1 \times X_2 \times \cdots \times X_n$.

Definition 3.3. Let T: $X_1 \times X_2 \times \cdots \times X_n \to Y_1 \times Y_2 \times \cdots \times Y_n$ be a fuzzy n-linear mapping, $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ are subspaces of (X,N_1) , (Y,N_2) respectively. Then T is called sequentially fuzzy n-continuous(briefly Sq-f-n-continuous) at $(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})\in X_1\times X_2\times\cdots\times X_n$ if

$$\forall k,(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)})\to(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})$$

$$\Rightarrow T(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)})\to T(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})$$

$$\lim_{k\to\infty}N_1[(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)})-(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),t]=1,\forall t>0$$
$$\Rightarrow\lim_{k\to\infty}N_2[T(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)})-T(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),t]=1,\forall t>0$$

If T is Sq-f-n-continuous at every point of $X_1\times X_2\times\cdots\times X_n$, then T is called Sq-f-$n$-continuous on $X_1\times X_2\times\cdots\times X_n$.

Definition 3.4. Let T: $X_1\times X_2\times\cdots\times X_n\to Y_1\times Y_2\times\cdots\times Y_n$ be a fuzzy n-linear mapping, $X_1,X_2,\cdots,X_n$ and $Y_1,Y_2,\cdots,Y_n$ are subspaces of (X,N_1) , (Y,N_2) respectively. Then T is called strongly fuzzy n-continuous(briefly St-f-n-continuous)at $(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})\in X_1\times X_2\times\cdots\times X_n$, if for each $\varepsilon>0,\exists\delta>0$ such that $\forall(x^{(1)},x^{(2)},\cdots,x^{(n)})\in X_1\times X_2\times\cdots\times X_n$

$$N_2[T(x^{(1)},x^{(2)},\cdots,x^{(n)})-T(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),\varepsilon]\geq$$
$$N_1[(x^{(1)},x^{(2)},\cdots,x^{(n)})-(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),\delta]$$

Definition 3.5. Let T: $X_1\times X_2\times\cdots\times X_n\to Y_1\times Y_2\times\cdots\times Y_n$ be a fuzzy n-linear mapping, $X_1,X_2,\cdots,X_n$ and $Y_1,Y_2,\cdots,Y_n$ are subspaces of (X,N_1) , (Y,N_2) respectively. Then T is called weakly fuzzy n-continuous (briefly Wk-f-n-continuous)at $(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})\in X_1\times X_2\times\cdots\times X_n$, if for given $\varepsilon>0,\alpha\in(0,1)\exists\delta=\delta(\alpha,\varepsilon)>0$, such that $\forall(x^{(1)},x^{(2)},\cdots,x^{(n)})\in X_1\times X_2\times\cdots\times X_n$

$$N_1[(x^{(1)},x^{(2)},\cdots,x^{(n)})-(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),\delta]\geq\alpha$$
$$\Rightarrow N_2[T(x^{(1)},x^{(2)},\cdots,x^{(n)})-T(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),\varepsilon]\geq\alpha$$

Theorem 3.6. Let T: $X_1\times X_2\times\cdots\times X_n\to Y_1\times Y_2\times\cdots\times Y_n$ be a fuzzy n-linear mapping, $X_1,X_2,\cdots,X_n$ and $Y_1,Y_2,\cdots,Y_n$ are subspaces of (X,N_1) , (Y,N_2) respectively. If T is St-f-n-continuous then T is Sq-f-n-continuous.

Proof. Let us assume that T is St-f-n-continuous $(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}) \in X_1 \times X_2 \times \cdots \times X_n$, then for each $\varepsilon > 0, \exists \delta = \delta(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)},\varepsilon) > 0$, such that for all $(x^{(1)},x^{(2)},\cdots,x^{(n)}) \in X_1 \times X_2 \times \cdots \times X_n$.

$$N_2[T(x^{(1)},x^{(2)},\cdots,x^{(n)}) - T(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),\varepsilon] \geq$$

$$N_1[(x^{(1)},x^{(2)},\cdots,x^{(n)}) - (x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),\delta] \qquad (a)$$

Let $(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)})$ be a sequence in $X_1 \times X_2 \times \cdots \times X_n$, such that

$$(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)}) \to (x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})$$

i.e

$$\lim_{k\to\infty} N_1[(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)}) - (x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),t] = 1, \forall t > 0 \qquad (b)$$

Now from (a),by (b) we have

$$\begin{aligned}
&N_2[T(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)}) - T(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),\varepsilon] \geq \\
&\qquad N_1(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)}) - (x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),\delta] \\
\Rightarrow &\lim_{k\to\infty} N_2[T(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)}) - T(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),\varepsilon] \geq \\
&\qquad \lim_{k\to\infty} N_1(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)}) - (x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),\delta] \\
\Rightarrow &\lim_{k\to\infty} N_2[T(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)}) - T(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),\varepsilon] = 1
\end{aligned}$$

Since ε is arbitrary, it follows that $T(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)}) \to T(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})$. Therefore T is Sq-f-$n$-continuous.

Theorem 3.7. Let T: $X_1 \times X_2 \times \cdots \times X_n \to Y_1 \times Y_2 \times \cdots \times Y_n$ be a fuzzy n-linear mapping, $X_1,X_2,\cdots,X_n$ and $Y_1,Y_2,\cdots,Y_n$ are subspaces of (X,N_1) , (Y,N_2) respectively. If T is f-n-continuous if and only if T is Sq-f-n-continuous.

Proof. Let us assume that T is f-n-continuous at $(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}) \in X_1 \times X_2 \times \cdots \times X_n$. Let $(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)})$ be a sequence in $X_1 \times X_2 \times \cdots \times X_n$, such that $(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)}) \to (x_0^{(1)},x_0^{(2)},$ $\cdots,x_0^{(n)})$. Let $\varepsilon > 0$ be given, choose $\alpha \in (0,1)$, since T is f-n-continuous at $(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})$, then $\exists\delta=\delta(\alpha,\varepsilon)>0,\beta=\beta(\alpha,\varepsilon)\in(0,1)$, such that for all$(x^{(1)},x^{(2)},\cdots,x^{(n)})\in X_1\times X_2\times\cdots\times X_n$,

$$N_1[(x^{(1)},x^{(2)},\cdots,x^{(n)})-(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),\delta]>\beta$$
$$\Rightarrow N_2[T(x^{(1)},x^{(2)},\cdots,x^{(n)})-T(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),\varepsilon]>\alpha$$

Since $(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)})\to(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})$ in (X,N_1),$\exists$ a positive integer n_0, such that

$$N_1(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)})-(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),\delta]>\beta,\forall n\geq n_0$$
$$\Rightarrow N_2[T(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)})-T(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),\varepsilon]>\alpha,\forall n\geq n_0$$
$$\Rightarrow N_2[T(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)})-T(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)}),\varepsilon]=1$$

Since ε is arbitrary, thus $T(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)})\to T(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})$ in $Y_1\times Y_2\times\cdots\times Y_n$. Therefore T is Sq-f-$n$-continuous.

Next let us assume T is Sq-f-n-continuous at $(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})\in X_1\times X_2\times\cdots\times X_n$. If possible suppose that T is not f-n-continuous at $(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})$. Thus $\exists\varepsilon>0$ and $\alpha>0$ such that for any $\delta>0$ and $\beta\in(0,1),\exists(y^{(1)},y^{(2)},\cdots,y^{(n)})$ (depending on δ,β), such that

$$N_1[(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})-(y^{(1)},y^{(2)},\cdots,y^{(n)}),\delta]>\beta$$

but

$$N_2[T(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})-T(y^{(1)},y^{(2)},\cdots,y^{(n)}),\varepsilon]\leq\alpha$$

Thus for $\beta=1-\frac{1}{k+1},\delta=\frac{1}{k+1},k=1,2,3,\cdots,\exists(y_k^{(1)},y_k^{(2)},\cdots,y_k^{(n)})$, such that

$$N_1[(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})-(y_k^{(1)},y_k^{(2)},\cdots,y_k^{(n)}),\frac{1}{k+1}]>1-\frac{1}{k+1} \qquad (c)$$

but

$$N_2[T(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})-T(y_k^{(1)},y_k^{(2)},\cdots,y_k^{(n)}),\varepsilon]\leq\alpha$$

Taking $\delta > 0, \exists k_0$, such that $(1 - \frac{1}{k+1}) < \delta, \forall k \geq k_0$. Then,

$$N_1[(x_0^{(1)}, x_0^{(2)}, \cdots, x_0^{(n)}) - (y_k^{(1)}, y_k^{(2)}, \cdots, y_k^{(n)}), \delta]$$

$$\geq N_1[(x_0^{(1)}, x_0^{(2)}, \cdots, x_0^{(n)}) - (y_k^{(1)}, y_k^{(2)}, \cdots, y_k^{(n)}), \frac{1}{k+1}] > 1 - \frac{1}{k+1}$$

$$\lim_{k \to \infty} N_1[(x_0^{(1)}, x_0^{(2)}, \cdots, x_0^{(n)}) - (y_k^{(1)}, y_k^{(2)}, \cdots, y_k^{(n)}), \delta] \geq 1$$

$$\Rightarrow (y_k^{(1)}, y_k^{(2)}, \cdots, y_k^{(n)}) \to (x_0^{(1)}, x_0^{(2)}, \cdots, x_0^{(n)})$$

But from (c), $N_2[T(x_0^{(1)}, x_0^{(2)}, \cdots, x_0^{(n)}) - T(y_k^{(1)}, y_k^{(2)}, \cdots, y_k^{(n)}), \varepsilon] \leq \alpha$. So

$$N_2[T(x_0^{(1)}, x_0^{(2)}, \cdots, x_0^{(n)}) - T(y_k^{(1)}, y_k^{(2)}, \cdots, y_k^{(n)}), \varepsilon] \nrightarrow 1$$

as $k \to \infty$. Thus $T(y_k^{(1)}, y_k^{(2)}, \cdots, y_k^{(n)})$ does not converges to $T(x_0^{(1)}, x_0^{(2)}, \cdots, x_0^{(n)})$, where as $(y_k^{(1)}, y_k^{(2)}, \cdots, y_k^{(n)}) \to (x_0^{(1)}, x_0^{(2)}, \cdots, x_0^{(n)})$(with respect to N_1). This would be contradiction to above assumption. Therefore T is f-n-continuous at $(x_0^{(1)}, x_0^{(2)}, \cdots, x_0^{(n)})$.

4. Fuzzy n-Bounded Linear Operators

Definition 4.1. Let T: $X_1 \times X_2 \times \cdots \times X_n \to Y_1 \times Y_2 \times \cdots \times Y_n$ be a fuzzy n-linear mapping, $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ are subspaces of (X, N_1) , (Y, N_2) respectively. Then T is said to be strongly fuzzy n-bounded (briefly St-f-n-bounded) on $X_1, X_2, \cdots, X_n$ if and only if $\exists$ a positive real number M, such that for all $(x_1, x_2, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$ and $\forall t \in R$

$$N_2[T(x_1, x_2 \cdots, x_n), t)] \geq N_1[(x_1, x_2 \cdots, x_n), \frac{t}{M})$$

Example 4.2. Let $(X, \|\cdot, \cdots, \cdot\|)$ be a n-normed linear space. Let $N_1, N_2 : X \times X \times \cdots \times X \times R^+ \to [0,1]$ be defined by

$$N_1(x_1, x_2, \cdots, x_n, t) = \begin{cases} \dfrac{t}{t + k_1\|x_1, x_2, \cdots, x_n\|}, & t > 0, \\ 0 & , t \leq 0. \end{cases}$$

and

$$N_2(x_1, x_2, \cdots, x_n, t) = \begin{cases} \dfrac{t}{t + k_2\|x_1, x_2, \cdots, x_n\|}, & t > 0, \\ 0 & , t \leq 0. \end{cases}$$

Clearly (X,N_1) and (Y,N_2) are fuzzy n-normed linear spaces.

Consider the mapping T: $X_1 \times X_2 \times \cdots \times X_n \to Y_1 \times Y_2 \times \cdots \times Y_n$ defined by $T(x_1,x_2\cdots,x_n) = r(x_1,x_2\cdots,x_n)$, where $r(\neq 0) \in R$ is fixed.

Clearly T is a linear operator. Let us choose an arbitrary but fixed $M > 0$ such that $M \geq |r|$ and $(x_1,x_2,\cdots,x_n) \in X_1 \times X_2 \times \cdots \times X_n$. Now for all $t > 0$

$$M \geq |r|$$

$$\Rightarrow k_1 M\|x_1,x_2,\cdots,x_n\| \geq k_2|r|\|x_1,x_2,\cdots,x_n\|$$

$$\Rightarrow t + k_1 M\|x_1,x_2,\cdots,x_n\| \geq t + k_2|r|\|x_1,x_2,\cdots,x_n\|$$

$$\Rightarrow \frac{t}{t+k_2|r|\|x_1,x_2,\cdots,x_n\|} \geq \frac{t}{t+k_1 M\|x_1,x_2,\cdots,x_n\|}$$

$$\Rightarrow \frac{t}{t+k_2\|r(x_1,x_2,\cdots,x_n)\|} \geq \frac{\frac{t}{M}}{\frac{t}{M}+k_1\|x_1,x_2,\cdots,x_n\|}$$

$$\Rightarrow N_2[r(x_1,x_2,\cdots,x_n),t] \geq N_1[(x_1,x_2,\cdots,x_n),\frac{t}{M}]$$

i.e.

$$\Rightarrow N_2[T(x_1,x_2,\cdots,x_n),t] \geq N_1[(x_1,x_2,\cdots,x_n),\frac{t}{M}]$$

If $t \leq 0$ then above relation holds for all $(x_1,x_2,\cdots,x_n) \in X_1 \times X_2 \times \cdots \times X_n$. Therefore T is St-f-$n$-bounded.

Definition 4.3. Let T: $X_1 \times X_2 \times \cdots \times X_n \to Y_1 \times Y_2 \times \cdots \times Y_n$ be a fuzzy n-linear mapping, $X_1,X_2,\cdots,X_n$ and $Y_1,Y_2,\cdots,Y_n$ are subspaces of (X,N_1) , (Y,N_2) respectively. Then T is said to be weakly fuzzy n-bounded (briefly Wk-f-n-bounded) on $X_1,X_2,\cdots,X_n$ if and only if for any $\alpha \in (0,1), \exists M_\alpha > 0$, such that for all $(x_1,x_2,\cdots,x_n) \in X_1 \times X_2 \times \cdots \times X_n$and $\forall t \in R$

$$N_1[(x_1,x_2\cdots,x_n),\frac{t}{M}] \geq \alpha \Rightarrow N_2[T(x_1,x_2\cdots,x_n),t)] \geq \alpha$$

Theorem 4.4. Let T: $X_1 \times X_2 \times \cdots \times X_n \to Y_1 \times Y_2 \times \cdots \times Y_n$ be a fuzzy n-linear mapping, $X_1,X_2,\cdots,X_n$ and $Y_1,Y_2,\cdots,Y_n$ are subspaces of (X,N_1) , (Y,N_2) respectively. If T is St-f-n-bounded, then T is Wk-f-n-bounded but not conversely.

Proof. Let us assume that T is St-f-n-bounded. Then $\exists M > 0$, such that for all $(x_1, x_2, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$and $\forall t \in R$,

$$N_2[T(x_1, x_2 \cdots, x_n), t)] \geq N_1[(x_1, x_2 \cdots, x_n), \frac{t}{M}]$$

Thus for any $\alpha \in (0,1), \exists M_\alpha (= M) > 0$, such that

$$N_1[(x_1, x_2 \cdots, x_n), \frac{t}{M_\alpha}] \geq \alpha \Rightarrow N_2[T(x_1, x_2 \cdots, x_n), t)] \geq \alpha$$

This implies that T is Wk-f-n-bounded.

For the converse result we consider the following example.

Example 4.5. Let $(X, \|\cdot, \cdots, \cdot\|)$ be a n-normed linear space. Let $N_1, N_2 : X \times X \times \cdots \times X \times R^+ \to [0,1]$ be defined by

$$N_1(x_1, x_2, \cdots, x_n, t) = \begin{cases} \dfrac{t^2 - \|x_1, x_2, \cdots, x_n\|^2}{t^2 + \|x_1, x_2, \cdots, x_n\|^2}, & t > \|x_1, x_2, \cdots, x_n\| \\ 0 & , t \leq \|x_1, x_2, \cdots, x_n\| \end{cases}$$

and

$$N_2(x_1, x_2, \cdots, x_n, t) = \begin{cases} \dfrac{t}{t + \|x_1, x_2, \cdots, x_n\|}, & t > 0, \\ 0 & , t \leq 0. \end{cases}$$

We know that (X, N_2) is a fuzzy n-normed linear space. Now we want to show that (X, N_1) is a fuzzy n-normed linear space.

(1)$\forall t \in R$ with $t \leq 0$ and by definition $N_1(x_1, x_2 \cdots, x_n, t) = 0$

(2)$\forall t \in R$ with $t > 0$, we have

$$N_1(x_1, x_2 \cdots, x_n, t) = 1 \Leftrightarrow \frac{t^2 - \|x_1, x_2, \cdots, x_n\|^2}{t^2 + \|x_1, x_2, \cdots, x_n\|^2} = 1$$

$$\Leftrightarrow t^2 - \|x_1, x_2, \cdots, x_n\|^2 = t^2 + \|x_1, x_2, \cdots, x_n\|^2$$

$$\Leftrightarrow \|x_1, x_2, \cdots, x_n\| = 0 \Leftrightarrow x_1, x_2, \cdots, x_n \textit{ are linearly dependent}.$$

(3)As $\|x_1, x_2, \cdots, x_n\|$ is invariant under any permutation of $x_1, x_2, \cdots, x_n$, it follows that $N_1(x_1, x_2 \cdots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \cdots, x_n$.

(4)$\forall t \in R$ with $t > 0$ and $c \neq 0, c \in K$

$$N_1(x_1,x_2,\cdots,x_n,\frac{t}{|c|}) = \frac{t^2-|c|^2\|x_1,x_2,\cdots,x_n\|^2}{t^2+|c|^2\|x_1,x_2,\cdots,x_n\|^2}$$
$$= \frac{t^2-\|x_1,x_2,\cdots,cx_n\|^2}{t^2+\|x_1,x_2,\cdots,cx_n\|^2} = N_1(x_1,x_2,\cdots,cx_n,t)$$

(5)$\forall s,t \in \mathbb{R}$ and$x_1,x_2,\cdots,x_n,x_n^{'} \in X$, we have to show that

$$N_1(x_1,x_2,\cdots,x_n+x_n^{'},s+t) \geq min\{N_1(x_1,x_2,\cdots,x_n,s),N_1(x_1,x_2,\cdots,x_n^{'},t)\}$$

If $s \leq \|x_1,x_2,\cdots,x_n\|$ or $t \leq \|x_1,x_2,\cdots,x_n^{'}\|$, then relation is obvious.

Suppose $s > \|x_1,x_2,\cdots,x_n\|$ and $t > \|x_1,x_2,\cdots,x_n^{'}\|$

without loss of generality assume,$N_1(x_1,\cdots,x_n^{'},t) \geq N_1(x_1,\cdots,x_n,s)$ then

$$t^2\|x_1,x_2,\cdots,x_n\|^2 - s^2\|x_1,x_2,\cdots,x_n^{'}\|^2 \geq 0 \qquad (d)$$

Now

$$\Rightarrow s+t > \|x_1,x_2,\cdots,x_n\| + \|x_1,x_2,\cdots,x_n^{'}\|$$

$$\Rightarrow s+t > \|x_1,x_2,\cdots,x_n+x_n^{'}\|$$

So

$$\begin{aligned} N_1(x_1,\cdots,x_n+x_n^{'},s+t) &= \frac{(s+t)^2-\|x_1,\cdots,x_n+x_n^{'}\|^2}{(s+t)^2+\|x_1,\cdots,x_n+x_n^{'}\|^2} \\ &\geq \frac{(s+t)^2-(\|x_1,\cdots,x_n\|+\|x_1,\cdots,x_n^{'}\|)^2}{(s+t)^2+(\|x_1,\cdots,x_n\|+\|x_1,\cdots,x_n^{'}\|)^2} \end{aligned}$$

Again

$$\frac{(s+t)^2-(\|x_1,\cdots,x_n\|+\|x_1,\cdots,x_n^{'}\|)^2}{(s+t)^2+(\|x_1,\cdots,x_n\|+\|x_1,\cdots,x_n^{'}\|)^2} - \frac{s^2-\|x_1,\cdots,x_n\|^2}{s^2+\|x_1,\cdots,x_n\|^2}$$
$$= \frac{2(s+t)^2\|x_1,\cdots,x_n\|^2 - 2s^2(\|x_1,\cdots,x_n\|+\|x_1,\cdots,x_n^{'}\|)^2}{\{(s+t)^2+(\|x_1,\cdots,x_n\|+\|x_1,\cdots,x_n^{'}\|)^2\}\{s^2+\|x_1,\cdots,x_n\|^2\}}$$
$$= \frac{2}{A}[(s+t)^2\|x_1,\cdots,x_n\|^2 - s^2(\|x_1,\cdots,x_n\|+\|x_1,\cdots,x_n^{'}\|)^2]$$

where $A = \{(s+t)^2+(\|x_1,\cdots,x_n\|+\|x_1,\cdots,x_n^{'}\|)^2\}\{s^2+\|x_1,\cdots,x_n\|^2\}$

$$= \frac{2}{A}[t^2\|x_1,\cdots,x_n\|^2 - s^2\|x_1,\cdots,x_n^{'}\|^2 + 2s\|x_1,\cdots,x_n\|(t\|x_1,\cdots,x_n\| - s\|x_1,\cdots,x_n^{'}\|)]$$

> 0 $[by(d)]$

Thus

$$N_1(x_1,\cdots,x_n+x_n^{'},s+t)\geq N_1(x_1,\cdots,x_n,s) \quad if\ N_1(x_1,\cdots,x_n^{'},t)\geq N_1(x_1,\cdots,x_n,s)$$

Similarly

$$N_1(x_1,\cdots,x_n+x_n^{'},s+t)\geq N_1(x_1,\cdots,x_n^{'},t) \quad if\ N_1(x_1,\cdots,x_n,s)\geq N_1(x_1,\cdots,x_n^{'},t)$$

Thus $N_1(x_1,x_2,\cdots,x_n+x_n^{'},s+t)\geq min\{N_1(x_1,x_2,\cdots,x_n,s),N_1(x_1,x_2,\cdots,x_n^{'},t)\}$

(6)$\forall t_1,t_2\in\mathbb{R}$, if $t_1<t_2\leq\|x_1,\cdots,x_n\|$, then by definition

$$N_1(x_1,\cdots,x_n,t_1)=N_1(x_1,\cdots,x_n^{'},t_2)=0$$

suppose $t_2>t_1>\|x_1,\cdots,x_n\|$ then

$$\frac{t_2^2-\|x_1,\cdots,x_n\|^2}{t_2^2+\|x_1,\cdots,x_n\|^2}-\frac{t_1^2-\|x_1,\cdots,x_n\|^2}{t_1^2+\|x_1,\cdots,x_n\|^2}$$

$$=\frac{(t_2^2-\|x_1,\cdots,x_n\|^2)(t_1^2+\|x_1,\cdots,x_n\|^2)-(t_1^2-\|x_1,\cdots,x_n\|^2)(t_2^2+\|x_1,\cdots,x_n\|^2)}{(t_2^2+\|x_1,\cdots,x_n\|^2)(t_1^2+\|x_1,\cdots,x_n\|^2)}$$

$\geq 0\Rightarrow N_1(x_1,x_2,\cdots,x_n,t_2)\geq N_1(x_1,x_2,\cdots,x_n,t_1)$

for all $(x_1,x_2,\cdots,x_n)\in X_1\times X_2\times\cdots\times X_n$.

Thus $N_1(x_1,x_2,\cdots,x_n,t)$ is non-decreasing function.

Also

$$\begin{aligned}\lim_{t\to\infty}N_1(x_1,\cdots,x_n,t) &= \lim_{t\to\infty}\frac{t^2-\|x_1,\cdots,x_n\|^2}{t^2+\|x_1,\cdots,x_n\|^2}\\ &= \lim_{t\to\infty}\frac{t^2(1-\frac{\|x_1,\cdots,x_n\|^2}{t^2})}{t^2(1+\frac{\|x_1,\cdots,x_n\|^2}{t^2})} = 1\end{aligned}$$

Therefore (X,N_1) is a fuzzy n-normed linear space. Now let us consider the mapping T: $X_1\times X_2\times\cdots\times X_n\to Y_1\times Y_2\times\cdots\times Y_n$ defined by

$$T(x_1,x_2,\cdots,x_n)=(x_1,x_2,\cdots,x_n)\ \forall(x_1,x_2,\cdots,x_n)\in X_1\times X_2\times\cdots\times X_n$$

Let $\alpha \in (0,1)$ and $t \in R^+$ and choose $M_\alpha = \frac{1}{1-\alpha}$

We now prove that

$$N_1[(x_1,x_2\cdots,x_n),\frac{t}{M_\alpha}] \geq \alpha \Rightarrow N_2[T(x_1,x_2\cdots,x_n),t)] \geq \alpha$$

$$N_1[(x_1,x_2\cdots,x_n),\frac{t}{M_\alpha}] \geq \alpha \Rightarrow \frac{\frac{t^2}{M_\alpha^2} - \|x_1,\cdots,x_n\|^2}{\frac{t^2}{M_\alpha^2} + \|x_1,\cdots,x_n\|^2} \geq \alpha$$

$$\Leftrightarrow \frac{t^2(1-\alpha)^2 - \|x_1,\cdots,x_n\|^2}{t^2(1-\alpha)^2 + \|x_1,\cdots,x_n\|^2} \geq \alpha$$

$$\begin{aligned}
&\Rightarrow \quad t^2(1-\alpha)^2 - \|x_1,\cdots,x_n\|^2 \geq \alpha t^2(1-\alpha)^2 + \alpha\|x_1,\cdots,x_n\|^2 \\
&\Rightarrow \quad t^2(1-\alpha)^2 - \alpha t^2(1-\alpha)^2 \geq \alpha\|x_1,\cdots,x_n\|^2 + \|x_1,\cdots,x_n\|^2 \\
&\Rightarrow \quad t^2(1-\alpha)^2(1-\alpha) \geq (1+\alpha)\|x_1,\cdots,x_n\|^2 \\
&\Rightarrow \quad \|x_1,\cdots,x_n\|^2 \leq \frac{t^2(1-\alpha)^2(1-\alpha)}{(1+\alpha)} \\
&\Rightarrow \quad \|x_1,\cdots,x_n\| \leq \frac{t(1-\alpha)\sqrt{(1-\alpha)}}{\sqrt{(1+\alpha)}} \quad (Since\ \alpha \neq 1) \\
&\Rightarrow \quad t + \|x_1,\cdots,x_n\| \leq \frac{t(1-\alpha)\sqrt{(1-\alpha)}}{\sqrt{(1+\alpha)}} + t \ = \ \frac{t\{(1-\alpha)\sqrt{(1-\alpha)} + \sqrt{(1+\alpha)}\}}{\sqrt{(1+\alpha)}} \\
&\Rightarrow \quad \frac{t}{t+\|x_1,\cdots,x_n\|} \geq \frac{\sqrt{(1+\alpha)}}{(1-\alpha)\sqrt{(1-\alpha)} + \sqrt{(1+\alpha)}} \qquad (e)
\end{aligned}$$

Now

$$\frac{\sqrt{(1+\alpha)}}{(1-\alpha)\sqrt{(1-\alpha)} + \sqrt{(1+\alpha)}} \geq \alpha$$

$$\begin{aligned}
&\Leftrightarrow \quad \sqrt{(1+\alpha)} \geq \alpha(1-\alpha)\sqrt{(1-\alpha)} + \alpha\sqrt{(1+\alpha)} \\
&\Leftrightarrow \quad (1-\alpha)\sqrt{(1+\alpha)} \geq \alpha(1-\alpha)\sqrt{(1-\alpha)} \\
&\Leftrightarrow \quad \sqrt{(1+\alpha)} \geq \alpha\sqrt{(1-\alpha)} \quad (Since\ \alpha \neq 1) \\
&\Leftrightarrow \quad 1+\alpha \geq \alpha^2(1-\alpha) \\
&\Leftrightarrow \quad 1+\alpha+\alpha^3 \geq \alpha^2
\end{aligned}$$

which is true for all $\alpha \in (0,1)$. Thus from(e).

We get $\frac{t}{t+\|x_1,\cdots,x_n\|} \geq \alpha \Rightarrow N_2[T(x_1,x_2\cdots,x_n),t)] \geq \alpha$ if $t > \|x_1,x_2,\cdots,x_n\|$.

Again since for $t \leq \|x_1,x_2,\cdots,x_n\|$,

$$\frac{t^2-\|x_1,x_2,\cdots,x_n\|^2}{t^2+\|x_1,x_2,\cdots,x_n\|^2} = 0$$

It follows that $N_1[(x_1,x_2\cdots,x_n),\frac{t}{M\alpha}] \geq \alpha \Rightarrow N_2[T(x_1,x_2\cdots,x_n),t)] \geq \alpha$

$\forall \alpha \in (0,1)$ Hence T is Wk-f-n-bounded.

Now conversely, let T be St-f-n-bounded

$$N_2[T(x_1,x_2\cdots,x_n),t)] \geq N_1[(x_1,x_2\cdots,x_n),\frac{t}{M}]$$

$$\begin{aligned}
&\Leftrightarrow \quad \frac{t}{t+\|x_1,\cdots,x_n\|} \geq \frac{\frac{t^2}{M^2}-\|x_1,\cdots,x_n\|^2}{\frac{t^2}{M^2}+\|x_1,\cdots,x_n\|^2}\\
&\Leftrightarrow \quad \frac{t}{t+\|x_1,\cdots,x_n\|} \geq \frac{t^2-M^2\|x_1,\cdots,x_n\|^2}{t^2+M^2\|x_1,\cdots,x_n\|^2}\\
&\Rightarrow \quad 2tM^2\|x_1,\cdots,x_n\|^2 \geq t^2\|x_1,\cdots,x_n\| + M^2\|x_1,\cdots,x_n\|\|x_1,\cdots,x_n\|^2\\
&\Rightarrow \quad M^2\|x_1,\cdots,x_n\|^2(2t+\|x_1,\cdots,x_n\|) \geq t^2\|x_1,\cdots,x_n\|\\
&\Leftrightarrow \quad M^2 \geq \frac{t^2}{(2t+\|x_1,\cdots,x_n\|)(\|x_1,\cdots,x_n\|)}\\
&\Leftrightarrow \quad M \geq \frac{t}{\sqrt{(2t+\|x_1,\cdots,x_n\|)(\|x_1,\cdots,x_n\|)}}\\
&\Leftrightarrow \quad M = \infty \; as \; t \to \infty
\end{aligned}$$

This would be contradiction to above assumption. Therefore T is not St-f-n-bounded

Definition 4.6. (closed graph theorem) Let T be a fuzzy n-linear operator from fuzzy n-Banach space (X,N_1) to fuzzy n-Banach space (Y,N_2). Suppose for every $(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)})$ $\in (X,N_1)$ such that$(x_k^{(1)},x_k^{(2)},\cdots,x_k^{(n)}) \to (x^{(1)},x^{(2)},\cdots,x^{(n)})$ and $(Tx_k^{(1)},Tx_k^{(2)},\cdots,Tx_k^{(n)}) \to$ $(y^{(1)},y^{(2)},\cdots,y^{(n)})$ for some $x^{(1)},x^{(2)},\cdots,x^{(n)} \in X, y^{(1)},y^{(2)},\cdots,y^{(n)} \in Y$, it follows $T(x^{(1)},x^{(2)},$ $\cdots,x^{(n)}) = (y^{(1)},y^{(2)},\cdots,y^{(n)})$. Then T is f-$n$-continuous.

Conflict of Interests

The authors declare that there is no conflict of interests.

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