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OPTIMALITY CONDITIONS OF SECOND-ORDER RADIAL EPIDERIVATIVES

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Abstract. In this paper, we introduce the concepts of second-order radial epiderivative and second-order generalized radial epiderivative for nonconvex set-valued maps. We give existence theorems for the second-order generalized radial epiderivatives. We also establish the second-order optimality conditions by using second-order radial epiderivatives.

Keywords: second-order radial set, second-order radial epiderivative, second-order optimality condition

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1. Introduction

In the last years, the second-order optimality conditions have a great deal of attention in scalar and vector-optimization problems and have been widely investigated [2,3,4,5,8,9,10,11,12,13,14, 15,16,17,19, 22, 24, 26]. It can be seen that a second-order contingent set, introduced by Aubin and Frankowska [1], and a second-order asymptotic contingent cone, introduced by Penot [24], play a important role in establishing second-order optimality conditions.Jahn et al. proposed the second-order contingent derivative and the second-order contingent epiderivative in terms of the

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second-order contingent set [15], introduced by Aubin and Frankowska [1]. They obtained the second-order optimality conditions by using these derivatives in set-valued optimization. In [22], Khan and Tammer gave new second-order optimality conditions in set-valued optimization. They presented an extension of the well-known Dubovitski-Milutin approach to set-valued optimization. In [3], Anh and Khanh introduced the higher-order radial sets and corresponding derivatives. They established both necessary and sufficient higher-order conditions for weak efficiency in set-valued vector optimization problem . In [4], Anh and Khanh gave both necessary and sufficient higher-order conditions for various kinds of proper solutions to nonsmooth vector optimization problem in terms of higher-order radial sets and radial derivatives. In [18], İnceoğlu introduce the concepts of second-order radial epiderivative and second-order generalized radial epiderivative for nonconvex set-valued maps. They also investigate in [18] some of their properties and give existence theorems for the second-order generalized radial epiderivatives.

Motivated by the work above, we study the second-order radial epiderivatives and the second-order generalized radial epiderivative. We also propose second-order optimality conditions by using second-order radial epiderivatives. This paper is divided into four sections. In Section 2, we recall some basic concepts. In Section 3, we introduce the second-order radial epiderivative and the second-order generalized radial epiderivative and give the existence theorems and some of their basic properties. In Section 4, we establish the second-optimality conditions for weak minimizers.

2. Preliminaries

Throughout this paper, let $(X,\|.\|_X)$ and $(Y,\|.\|_Y)$ be real normed spaces and let Y be partially ordered by a closed convex pointed cone $C \subset Y$. Let $F : X \to 2^Y$ be a set-valued map, let $(\bar{x},\bar{y}) \in graph(F)$, let $(\bar{u},\bar{v}) \in X \times Y$.

We recall the concept of the radial epiderivative and the generalized radial epiderivative introduced by Kasımbeyli [20], and Kasımbeyli and İnceoğlu [21], respectively, together with some standard notions.

Definition 2.1. Let U be a nonempty subset of a real normed space $(Z,\|.\|_Z)$, and let $\bar{z}\in cl(U)$ (closure of U) be a given element. The closed radial cone $R(U,\bar{z})$ of U at $\bar{z}\in cl(U)$ is the set of all $z\in Z$ such that there are $\lambda_n>0$ and a sequence $(z_n)_{n\in\mathbb{N}}\subset Z$ with $\lim_{n\to\infty} z_n = z$ so that $\bar{z}+\lambda_n z_n\in U$, for all $n\in\mathbb{N}$ [6], [20,21], [25].

It follows from this definitions that $R(U,\bar{z}) = cl(cone(U-\bar{z}))$, where cone denotes the conic hull of a set, which is the smallest cone containing $U-\bar{z}$ [6], [7], [20,21].

Definition 2.2. Let $(X,\|.\|_X)$ and $(Y,\|.\|_Y)$ be real normed spaces, let $F: X\to 2^Y$ be a set-valued map.

(i) The set

$$graph(F) = \{(x,y)\in X\times Y \mid y\in F(x)\}$$

is called the graph of F;

(ii) The set

$$dom(F) = \{x\in X \mid F(x)\neq\emptyset\}$$

is called the domain of F;

(iii) Let Y be partially ordered by a proper, convex, and pointed cone $C\subset Y$. The set

$$epi(F) = \{(x,y)\in X\times Y \mid y\in F(x)+C\}$$

is called the epigraph of F,

(iv) Let $C\subset Y$ a proper, convex and pointed cone. The profile map $P_F: X\to 2^Y$ is defined by

$$P_F(x) = F(x)+C,$$

for every $x\in dom(F)$.

(v) Let $(\bar{x},\bar{y})\in graph(F)$. A set valued map $D_R F(\bar{x},\bar{y}): X\to 2^Y$ whose graph coincides with the contingent cone to graph of F at $(\bar{x},\bar{y})$, that is

$$graph(D_R F(\bar{x},\bar{y})) = R(graph(F),(\bar{x},\bar{y})),$$

is called radial derivative of F at $(\bar{x},\bar{y})$,[6],[25].

Now, we give the definition of the radial epiderivative given by Kasımbeyli without convexity and boundedness [20].

Definition 2.3. Let Y be partially ordered by a convex cone $C \subset Y$, let S be a nonempty subset of X and let $F : S \to 2^Y$ be a set-valued map. Let a pair $(\bar{x}, \bar{y}) \in graph(F)$ be given. A single-valued map $D_r F(\bar{x}, \bar{y}) : X \to Y$ whose epigraph equals the radial cone to the epigraph of F at $(\bar{x}, \bar{y})$, i.e.

$$epi(D_r F(\bar{x}, \bar{y})) = R(epi(F), (\bar{x}, \bar{y})),$$

is called radial epiderivative of F at $(\bar{x}, \bar{y})$.

To give the definition of the generalized radial epiderivative, we recall the minimality concept [23].

Definition 2.4. Let $(Y, \|.\|_Y)$ be a real normed space partially ordered by a convex cone $C \subset Y$. Let D be a subset of Y and let $\bar{y} \in D$.

(i) The element $\bar{y}$ is said to be a minimal element of D, if $D \cap (\{\bar{y}\} - C) = \{\bar{y}\}$.

(ii) Let the ordering cone have a nonempty interior $int(C)$. The element $\bar{y}$ is said to be a weakly minimal element of D, if $D \cap (\{\bar{y}\} - int(C)) = \emptyset$. The set of all minimal, weakly minimal elements of D with respect to the ordering cone C is denoted by $MinD$, $W - MinD$, respectively.

Now, we recall the generalized radial epiderivative for set-valued maps given by Kasımbeyli and İnceoğlu in [21].

Definition 2.5. A set valued map $D_{gr}F(\bar{x}, \bar{y}) : X \to 2^Y$ is called the generalized radial epiderivative of F at $(\bar{x}, \bar{y})$ if

$$D_{gr}F(\bar{x}, \bar{y})(x) = Min(G(x), C),$$

where $G : X \to 2^Y$ is the set-valued map given by

$$G(x) = \{y \in Y \mid (x, y) \in R(epi(F), (\bar{x}, \bar{y}))\}, \forall x \in X.$$

3.Second-Order Radial Set and Second-Order Radial Epiderivatives

In this section, we propose the definitions of the second-order radial epiderivatives. By using these definitions, we prove existence theorem and give some of their properties and optimality conditions.

Anh and Khanh defined m-th-order radial set and m-th-order radial derivative [4]. Based on this, we give the following definitions of second-order radial set and second-order radial derivative.

Definition 3.1. Let $(X, \|.\|_X)$ be a real normed space, let S be a nonempty subset of X, let $\bar{x} \in cl(S)$ and let $w \in X$ The second-order radial set of S at $\bar{x}$ with respect to w is

$$R^2(S,\bar{x},w) = \left\{x \in X \mid \exists t_n > 0, \exists x_n \to x, \forall n, \bar{x} + t_n w + t_n^2 x_n \in S\right\}.$$

It is also clear that $R^2(S,\bar{x},0_X) = R(S,\bar{x})$, 0_X the zero element of X.

The following definition was presented by Ha in [13].

Definition 3.2. Let $F : X \to 2^Y$ be a set-valued map, let $(\bar{x},\bar{y}) \in graph(F)$, let $(\bar{u},\bar{v}) \in X \times Y$. The second-order radial derivative of F at $(\bar{x},\bar{y})$ with respect to $(\bar{u},\bar{v})$ is the set-valued map $D_R^2F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \to 2^Y$ whose graph is

$$graph\left(D_R^2F(\bar{x},\bar{y},\bar{u},\bar{v})\right) = R^2\left(graph(F),(\bar{x},\bar{y}),(\bar{u},\bar{v})\right). \tag{1}$$

The relation (1) can be expressed equivalently by

$$D_R^2F(\bar{x},\bar{y},\bar{u},\bar{v})(x) = \left\{\begin{array}{c} y \in Y \mid \exists t_n > 0, \exists x_n \to x, \exists y_n \to y, \forall n, \\ \bar{y} + t_n\bar{v} + t_n^2 y_n \in F\left(\bar{x} + t_n\bar{u} + t_n^2 x_n\right) \end{array}\right\}.$$

The following definition is a generalization given by Kasımbeyli and Kasımbeyli and İnceoğlu, respectively [20],[21].

Definition 3.3. [18] Let $F : X \to 2^Y$ be a set-valued map, let $(\bar{x},\bar{y}) \in graph(F)$, let $(\bar{u},\bar{v}) \in X \times Y$.

(i) A single-valued map $D_r^2F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \to Y$ whose epigraph equals the second-order radial set to the epigraph of F at $(\bar{x},\bar{y})$ with respect to $(\bar{u},\bar{v})$, i.e.,

$$epi\left(D_r^2F(\bar{x},\bar{y},\bar{u},\bar{v})\right) = R^2\left(epi(F),(\bar{x},\bar{y}),(\bar{u},\bar{v})\right),$$

is called the second-order radial epiderivative.

(ii) A set-valued map $D^2_{gr}F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \to 2^Y$ is called the second-order generalized radial epiderivative of F at $(\bar{x},\bar{y})$ with respect to $(\bar{u},\bar{v})$ if

$$D^2_{gr}F(\bar{x},\bar{y},\bar{u},\bar{v})(x) = Min\left(G^2(x), C\right), x \in dom\left(G^2(x)\right),$$

where $G^2 : X \to 2^Y$ is a set-valued map defined by

$$G^2(x) = \left\{ \ y \in Y \mid (x,y) \in R^2(epi(F), (\bar{x},\bar{y}), (\bar{u},\bar{v})) \ \right\}.$$

Example 3.1. Let $F : \mathbb{R} \to 2^{\mathbb{R}}$ be a set-valued map given by

$$F(x) = \{y \in \mathbb{R} \mid y \geq x\}, \text{for all } x \in \mathbb{R}.$$

Let $(\bar{x},\bar{y}) = (0,0)$ and let $(\bar{u},\bar{v}) = (1,0)$. Then

$$R^2(epi(F), (0,0), (1,0)) = \left\{cz \in \mathbb{R}^2 \mid \exists t_n > 0, \exists (z_n) \to z, \text{for all } n, t_n(1,0) + t_n^2 z_n \in epiF\right\}.$$

The condition

$$t_n(1,0) + t_n^2 z_n \in epi(F)$$

is equivalent to

$$t_n^2 z_{n_2} \geq t_n + t_n^2 z_{n_1};$$

hence,

$$z_{n_2} \geq (1 + t_n z_{n_1})^2$$

Since $t_n > 0$ and $z_{n_2} \to z_2$, $z_{n_1} \to z_1$, we obtain that

$$R^2(epi(F), (0,0), (1,0)) = \mathbb{R} \times [1,0)$$

Consequently, we have

$$G^2(x) = [1,0),$$

for every $x \in \mathbb{R}$. On the other hand,

$$D^2_r F(0,0,1,0)(x) = \{1\}, f o r e v e r y x \in \mathbb{R}$$

.

$$D^2_{gr}F(0,0,1,0)(x) = Min\left(G^2(x), \mathbb{R}_+\right) = \{1\},$$

for every $x \in \mathbb{R}$.

Proposition 3.1. For every $x \in dom\left(D_R^2 F\left(\bar{x},\bar{y},\bar{u},\bar{v}\right)\right)$, the following inclusion holds:

$$D_R^2 F\left(\bar{x},\bar{y},\bar{u},\bar{v}\right)(x) + C_Y \subseteq D_R^2 P_F\left(\bar{x},\bar{y},\bar{u},\bar{v}\right)(x).$$

Corollary 3.1. For every $x \in dom\left(D_R^2 P_F\left(\bar{x},\bar{y},\bar{u},\bar{v}\right)\right)$, the following inclusion holds:

$$D_R^2 P_F\left(\bar{x},\bar{y},\bar{u},\bar{v}\right)(x) + C_Y = D_R^2 P_F\left(\bar{x},\bar{y},\bar{u},\bar{v}\right)(x).$$

The following existence theorem for second-order generalized radial epiderivative is proved in [18].

Theorem3.1. Let the convex cone $C \subset Y$ be regular. For every $x \in dom\left(G^2\right)$, let the set $D_{gr}^2 F\left(\bar{x},\bar{y},\bar{u},\bar{v}\right)(x)$ have a C-lower bound. Then for every $x \in dom\left(G^2\right)$, $D_{gr}^2 F\left(\bar{x},\bar{y},\bar{u},\bar{v}\right)(x)$ exists. Moreover, the following equality holds:

$$epi\left(D_{gr}^2 F\left(\bar{x},\bar{y},\bar{u},\bar{v}\right)\right) = R^2\left(epi\left(F\right),\left(\bar{x},\bar{y}\right),\left(\bar{u},\bar{v}\right)\right).$$

Propositon3.2. Let the convex cone $C \subset Y$ be regular. Let $F : X \to 2^Y$ be a set-valued map, let $\left(\bar{x},\bar{y}\right) \in graph\left(F\right)$, let $\left(\bar{u},\bar{v}\right) \in X \times Y$. For every $x \in dom\left(G^2\left(x\right)\right)$, let the set $G^2\left(x\right)$ have a $C-$ lower bound. The following assertion is satisfied:

$$epi\left(D_{gr}^2 F\left(\bar{x},\bar{y},\bar{u},\bar{v}\right)\right) \subset R^2\left(dom\left(F\right),\bar{x},\bar{u}\right) \times Y.$$

Proof. Let $\left(\bar{x},\bar{y}\right) \in epi\left(D_{gr}^2\left(\bar{x},\bar{y},\bar{u},\bar{v}\right)\right)$. Then $\left(\bar{x},\bar{y}\right) \in R^2\left(epi\left(f\right),\left(\bar{x},\bar{y}\right),\left(\bar{u},\bar{v}\right)\right)$ It follows from the definition of the second-order generalized radial epiderivative that there exist sequences $t_n > 0$ and $\left(x_n,y_n\right)$ with $\left(x_n,y_n\right) \to \left(x,y\right)$ such that

$$\left(\bar{x},\bar{y}\right) + t_n\left(\bar{u},\bar{v}\right) + t_n^2\left(x_n,y_n\right) \in epi\left(F\right), \text{for all} n \in \mathbb{N},$$

$$\bar{y} + t_n\bar{v} + t_n^2 \in F\left(\bar{x} + t_n\bar{u} + t_n^2 x_n\right) + C, \text{for all} n \in \mathbb{N}.$$

Therefore we have $\bar{x} + t_n\bar{u} + t_n^2 x_n \in dom\left(F\right)$. This implies that $\left(x,y\right) \in R^2\left(dom\left(F\right),\bar{x},\bar{u}\right) \times Y$.

Propositon3.3. Let $A \subset X$ be nonempty set and let $C \subset Y$ be a convex cone with $int\left(C\right) \neq \emptyset$. Let $F : A \to 2^Y$ be a set-valued map, let $E = dom\left(D_{gr}^2 F\left(\bar{x},\bar{y},\bar{u},\bar{v}\right)\right)$. Then

$$\bigcup_{x \in E} D_{gr}^2 F\left(\bar{x},\bar{y},\bar{u},\bar{v}\right) \subset R^2\left(F\left(A\right) + C,\bar{y},\bar{v}\right)$$

Proof. Let $y \in D^2_{gr}F(\bar{x},\bar{y},\bar{u},\bar{v})(E)$ and let $x \in E$ be the corresponding element such that $y \in D^2_{gr}F(\bar{x},\bar{y},\bar{u},\bar{v})(x)$.Then, $(x,y) \in R^2(epi(F),(\bar{x},\bar{y}),(\bar{u},\bar{v})$. There exist $t_n > 0$, $(x_n,y_n) \to (x,y)$ such that ,for all $n \in \mathbb{N}$,

$$\bar{y}+t_n\bar{v}+t_n^2y_n \in F\left(\bar{x}+t_n\bar{u}+t_n^2x_n\right)+C \subset F(A)+C$$

Since $\lambda_n > 0$ and $y_n \to y$, we get $y \in R^2(F(A)+C,\bar{y},\bar{v})$. Because y is chosen arbitrarily, we have $D^2_{gr}F(\bar{x},\bar{y},\bar{u},\bar{v})(E) \subset R^2(F(A)+C,\bar{y},\bar{v})$.

The following proposition shows that relationship between second-order radial epiderivative and second-order generalized radial epiderivative.

Propositon3.4. [18] Assume that the second-order radial epiderivative $D^2_rF(\bar{x},\bar{y},\bar{u},\bar{v})$ of $F : X \to 2^Y$ at $(\bar{x},\bar{y}) \in graph(F)$ with respect to $(\bar{u},\bar{v}) \in X \times Y$ exist. Then

$$D^2_{gr}F(\bar{x},\bar{y},\bar{u},\bar{v})(x) = Min\left(D^2_rF(\bar{x},\bar{y},\bar{u},\bar{v}),C_Y\right),$$

for all $x \in dom\left(D^2_rF(\bar{x},\bar{y},\bar{u},\bar{v})\right)$.

Proof. It follows from the Definition 3.3 that $D^2_rF(\bar{x},\bar{y},\bar{u},\bar{v})$

$$epi\left(D^2_rF(\bar{x},\bar{y},\bar{u},\bar{v})\right) = R^2(epi(F),(\bar{x},\bar{y}),(\bar{u},\bar{v})) = graph\left(D^2_RP_F(\bar{x},\bar{y},\bar{u},\bar{v})\right).$$

Hence,

$$\left\{D^2_rF(\bar{x},\bar{y},\bar{u},\bar{v})(x)\right\}+C_Y = D^2_RP_F(\bar{x},\bar{y},\bar{u},\bar{v})(x),$$

for every $x \in dom\left(D^2_RP_F(\bar{x},\bar{y},\bar{u},\bar{v})\right)$. In view of the Definition **??** and the () equality, the second-order generalized radial epiderivative $D^2_{gr}F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \to 2^Y$ is given by

$$D^2_{gr}F(\bar{x},\bar{y},\bar{u},\bar{v})(x) = Min\left(D^2_rF(\bar{x},\bar{y},\bar{u},\bar{v}),C_Y\right).$$

4.Optimality Conditions

Now, we obtain the optimality conditions for set-valued maps in terms of second-order radial epiderivativatives. Let $F : S \to 2^Y$ be a set-valued map.

Consider the following set-valued optimization problem:

$$(P)\begin{cases} min\, F(x) \\ \text{s.t. } x \in S \end{cases}$$

Definition 4.1. Let the ordering cone C have a nonempty interior $int(C)$. A pair $(\bar{x},\bar{y}) \in graph(F)$ is called weak minimizer of (), if $\bar{y}$ is a weakly minimal element of the set $F(S)$ where

$$F(S) = \bigcup_{x\in S} F(x).$$

Here we present a second-order optimality condition by using the second-order radial derivative.

Theorem 4.1. Let $(\bar{x},\bar{y}) \in graph(F)$ be a weak minimizer of the problem (P) and let $\bar{u} \in dom(DP_F(\bar{x},\bar{y}))$ be arbitrary. Then, for every $\bar{v} \in D_R P_F(\bar{x},\bar{y})(\bar{u}) \cap (-\partial C)$,

for every $x \in dom\left(D_R^2 P_F(\bar{x},\bar{y},\bar{u},\bar{v})\right)$,

$$D_{gr}^2 F(\bar{x},\bar{y},\bar{u},\bar{v})(x) \notin (-int(C) - \{\bar{v}\}).$$

Proof. Let $(\bar{x},\bar{y}) \in graph(F)$ and let $\bar{y} \in W-Min(F(S),C)$. Assume to the contrary that there exist an element $x \in dom\left(D_R^2 P_F(\bar{x},\bar{y},\bar{u},\bar{v})\right)$ with

$$y \in D_R^2 P_F(\bar{x},\bar{y},\bar{u},\bar{v})(x) \cap (-int(C) - \{\bar{v}\}).$$

By the definition of the second-order radial epiderivative

$$(x,y) \in R^2(epi(F),(\bar{x},\bar{y}),(\bar{u},\bar{v})).$$

Then $\exists t_n > 0$, $\exists (x_n,y_n) \subset epi(F)$, with

$$(x_n,y_n) \to (x,y) \ni \forall n, (\bar{x},\bar{y}) + t_n(\bar{u},\bar{v}) + t_n^2(x_n,y_n) \in epi(F).$$

By the definition of $epi(F)$, we get

$$\bar{y} + t_n\bar{v} + t_n^2 y_n \in F\left(\bar{x} + t_n\bar{u} + t_n^2 x_n\right) + C. \tag{2}$$

Since

$$y + \bar{v} \in (-int(C)), y_n \to y,$$

there exist $n_0 \in \mathbb{N}$ such that

$$\bar{v} + t_n^2 y_n \in (-int(C)), \text{for every} \quad n \geq n_0.$$

From $t_n > 0$, we get

$$t_n \bar{v} + t_n^2 y_n \in (-int(C)), \text{ for every} \quad n \geq n_0. \tag{3}$$

By using the above equality (2), we have

$$\bar{y} + t_n \bar{v} + t_n^2 y_n \in (\bar{y} - int(C)), \text{ for every} \quad n \geq n_0. \tag{4}$$

We set

$$\begin{aligned} \alpha_n &= \bar{x} + t_n \bar{u} + t_n^2 x_n, \\ \beta_n &= \bar{y} + t_n \bar{v} + t_n^2 y_n. \end{aligned} \tag{5}$$

Because of the equalities (5) and (2), we have

$$\beta_n \in F(\alpha_n) + C.$$

Therefore, there exists some $\vartheta_n \in F(\alpha_n) + C$ with $\beta_n \in \vartheta_n + C$. From here

$$\vartheta_n \in \beta_n - C.$$

Because of the inclusion $int(C) + C \subset int(C)$ and the equality
$\beta_n \in (\bar{y} - int(C))$, we have

$$\vartheta_n \in (\bar{y} - int(C)), \text{ for every} \quad n \geq n_0.$$

Therefore, we have shown that

$$F(\alpha_n) \cap (\bar{y} - int(C)) \neq \emptyset,$$

which is a contradiction to the assumption that $(\bar{x}, \bar{y})$ is a weak minimizer.

Now we propose some important properties of the second-order radial epiderivative.

Lemma 4.1. $F : S \to 2^Y$ be a set-valued map and $(\bar{x}, \bar{y}) \in graph(F)$, $\bar{u} \in S$ and $\bar{v} \in F(\bar{u}) + C$. If the second-order radial epiderivative $D_r^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$ exists, then

$$F(x) - \{\bar{y}\} + C \subset D_r^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x - \bar{x} - \bar{u}), \text{for all} \quad x \in S.$$

Proof. Let $x \in S$, $y \in F(x) - \{\bar{y}\} + C$. Then $y + \bar{y} \in F(x) + C$. By setting $t_n = 1$, $x_n = x - \bar{x} - \bar{u}$, $y_n = y - \bar{v}$, for all $n \in \mathbb{N}$, we have $\exists t_n > 0$, $\exists (x_n, y_n) \to (x - \bar{x} - \bar{u}, y - \bar{v}) \ni \bar{y} + t_n \bar{v} + t_n^2 y_n \in F\left(\bar{x} + t_n \bar{u} + t_n^2 x_n\right) + C$, for all $n \in \mathbb{N}$. Consequently, we get $y + \bar{y} \in D_r^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x - \bar{x} - \bar{u})$.

The following sufficient optimality condition for the weak minimizer will be proved by using the Lemma 4.1.

Theorem 4.2. Let the set-valued optimization problem (P) be given, let $(\bar{x}, \bar{y}) \in graph(F)$. If for every $\bar{u} \in X$ with $\bar{v} \in D_r F(\bar{x}, \bar{y})(\bar{u}) \cap (-\partial C)$

$$D_r^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x - \bar{x} - \bar{u}) \cap (-int(C)) = \emptyset,$$

for every $x \in S$, then $(\bar{x}, \bar{y})$ is a weak minimizer of the problem (P).

Proof. By the Lemma4.1

$$(F(x) - \{\bar{y}\} + C) \cap (-int(C)) = \emptyset,$$

for every $x \in S$. This implies that

$$(F(x) - \{\bar{y}\}) \cap (-int(C) - C) = \emptyset,$$

for every $x \in S$. We obtain with the equality $int(C) + C = int(C)$

$$(F(x) - \{\bar{y}\}) \cap (-int(C)) = \emptyset,$$

for every $x \in S$. Consequently, $\bar{y}$ is a weakly minimal element of the set $F(S)$; that is $(\bar{x}, \bar{y})$ is a weak minimizer of the problem (P).

5. Conclusion

In this paper two new concept of second-order epiderivative are presented. The relationship between the second-order radial epiderivative and the second-order generalized radial epiderivative are discussed. Some of their properties are investigated also. In set-valued optimization, second-order optimality conditions are obtained by using these epiderivatives.

Conflict of Interests

The author declare that there is no conflict of interests.

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