Available online at http://scik.org

J. Math. Comput. Sci. 8 (2018), No. 2, 196-215

https://doi.org/10.28919/jmcs/3446

ISSN: 1927-5307

# ON BOUNDED $n$-LINEAR OPERATORS

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**Abstract.** In this paper, the various concepts of bounded and continuous $n$-linear operators in normed spaces as well as in $n$-normed spaces are discussed. A sufficient condition for the space of $n$-linear operators to be a Banach space is given.Further, we prove the equality of two different formulae of norms of an $n$-linear operator .Also we introduce an $n$-norm on the dual space of a real linear space.

**Keywords:** norm; $n$-norm; bounded $n$-linear operator; Banach space.

**2010 AMS Subject Classification:** 46B20, 46B99, 46E15, 47A30, 47L05.

## 1. Introduction

Let $X$ be a real linear space of dimension greater than 1 and $\|.,.\|$ be a real valued function on $X \times X$ satisfying the following conditions:

$(2N_1)$ $\|x,y\| = 0$ if and only if $x$ and $y$ are linearly dependent.

$(2N_2)$ $\|x,y\| = \|y,x\|$.

$(2N_3)$ $\|\alpha x,y\| = |\alpha|\|x,y\| \;\; \forall\, x,y \in X$ and $\alpha \in \mathbb{R}$.

$(2N_4)$ $\|x+y,z\| \leq \|x,z\| + \|y,z\| \;\; \forall\, x,y,z \in X$.

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Received July 31, 2017

Then, $\|.,.\|$ is called a 2-norm on $X$ and $(X,\|.,.\|)$ is called a linear 2-normed space.2-norms are non-negative and $\|x,y+\alpha x\| = \|x,y\|$ for every $x,y \in X$ and $\alpha \in \mathbb{R}$.

The concept of 2-normed spaces was initially investigated and developed by S.G*ä*hler in 1960s and has been extensively developed by C. Diminnie, S.G*ä*hler, A. White and many others [1, 2, 3, 4, 5, 10, 15].

Let $X$ be a real vector space with dim$X \geq n$ where $n$ is a positive integer. A real valued function $\|.,...,.\| : X^n \to \mathbb{R}$ is called an $n$-norm on $X$ if the following conditions hold:

(1) $\|x_1,\ldots,x_n\| = 0$ iff $x_1,\ldots,x_n$ are linearly dependent.

(2) $\|x_1,\ldots,x_n\|$ remains invariant under permutations of $x_1,\ldots,x_n$.

(3) $\|\alpha x_1,x_2,\ldots,x_n\| = |\alpha|\|x_1,\ldots,x_n\| \, \forall x_1,\ldots,x_n \in X$ and $\alpha \in \mathbb{R}$.

(4) $\|x_0+x_1,x_2,\ldots,x_n\| \leq \|x_0,\ldots,x_n\| + \|x_1,\ldots,x_n\|$ for all $x_0,x_1,\ldots,x_n \in X$.

The pair $(X,\|.,\ldots,.\|)$ is called an $n$-normed space.

Let $X$ be a real vector space with dim$X \geq n$, $n$ is a poitive integer and be equipped with an inner product $\langle .,. \rangle$.Then the standard $n$-norm on $X$ is given by

$$\|x_1,\ldots,x_n\|_S := \sqrt{\det[\langle x_i,x_j \rangle]}.$$

A standard example of an $n$-normed space is $X = \mathbb{R}^n$ equipped with the Euclidean $n$-norm:

$$\|x_1,\ldots,x_n\|_E = \text{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right)$$

where $x_i = (x_{i1},...,x_{in}) \in \mathbb{R}^n$ for each $i = 1,2,...,n$.

Note that the value of $\|x_1,...,x_n\|_s$ represents the volume of $n$-dimensional parallelepiped s-panned by $x_1,...,x_n$.

The theories of 2-normed spaces and $n$-normed spaces were initially developed by Gähler [2]-[5] in 1960s . The theory of $n$-normed space was much developed later by Misiak [10]. Notions of boundedness in 2-normed space was then introduced by White [15]. Related works can also be found in [6, 9].

Gozali et al. also introduced the notion of bounded $n$-linear functionals in $n$-normed spaces in [6]. Zofia Lewandowska introduced notions of 2-linear operators on 2-normed sets in [9]. Agus L. Soenjaya then introduced the notions of continuity and boundedness of $n$-linear operators in [14]. Notions of different formulae of $n$-norms can be seen in [6, 12, 13].

The above papers motivate us to write this paper. In this paper we introduce the notion of bounded $n$-linear operators as further extensions of the corresponding notions in [14] and a new $n$-norm as a further work of the notions in [12, 13].

## 2. Preliminaries

Let $(X, \|.,\ldots,.\|)$ be an $n$-normed space and $(Y, \|.\|)$ be a normed space.

**Definition 2.1.** An operator $T : X^n \to Y$ is an $n$-linear operator on $X$ if $T$ is linear in each of the variables.

**Definition 2.2.** An $n$-linear operator $T$ is bounded if there is a constant $K$ such that

$$\|T(x_1,\ldots,x_n)\| \leq K\|x_1,\ldots,x_n\| \text{ for all } (x_1,\ldots,x_n) \in X^n.$$

If $T$ is bounded,

$$\|T\| := \sup_{\|x_1,\ldots,x_n\|\neq 0} \frac{\|T(x_1,\ldots,x_n)\|}{\|x_1,\ldots,x_n\|}.$$

**Proposition 2.1.** [14, Proposition 2.3] Let $T : X^n \to Y$ be an $n$-linear operator on $X$, where $(X, \|.,\ldots,.\|)$ is an $n$-normed space and $(Y, \|.\|)$ is a normed space. $T$ is bounded if and only if for all $(x_1,\ldots,x_n), (y_1,\ldots,y_n) \in X^n$,

$$\|T(x_1,\ldots,x_n) - T(y_1,\ldots,y_n)\| \leq K(\|x_1 - y_1, x_2,\ldots,x_n\| + \|y_1, x_2 - y_2,\ldots,x_n\| + \cdots + \|y_1,\ldots,y_{n-1}, x_n - y_n\|). \tag{2.1.1}$$

**Proposition 2.2.** [14, Proposition 2.4] Let $T$ be a bounded $n$-linear operator. Then

$$\begin{aligned}\|T\| &= \inf\{K : \|T(x_1,\dots,x_n)\| \le K\|x_1,\dots,x_n\|\}\\ &= \inf\{K : \|T(x_1,\dots,x_n)-T(y_1,\dots,y_n)\| \le K(\|x_1-y_1,x_2,\dots,x_n\|\\ &\qquad + \|y_1,x_2-y_2,\dots,x_n\| + \cdots + \|y_1,\dots,y_{n-1},x_n-y_n\|)\} \qquad (2.2.1)\\ &= \sup_{\|x_1,\dots,x_n\|\le 1} \|T(x_1,\dots,x_n)\|.\end{aligned}$$

If $X$ is an $n$-normed space with dual $X^{'}$, the following formula-formulated by G$\ddot{a}$hler [3]

$$\|x_1,...,x_n\|^G = \sup_{f_j\in X^{'},\|f_j\|\le 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1)\\ \vdots & \ddots & \vdots\\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}$$

defines an $n$-norm on $X$.

**Proposition 2.3.** [12, Proposition 2.1] Let $X$ be a normed space of dim$X \ge n$ with dual $X'$.Then the function

$$\|x_1,...,x_n\| = Abs\left(\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1)\\ \vdots & \ddots & \vdots\\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}\right)$$

defines an n-norm on $X$ for fixed linearly independent n funtionals
$f_1, f_2, ..., f_n \in X'$

The above concepts motivate us to have the following results.

## 3. Main results

**Proposition 3.1.** Let $T : X^n \longrightarrow Y$ be a bounded $n$-linear operator where $(X, \|.,\dots,.\|)$ is an

$n$-normed space and $(Y, \|.\|)$ is a normed space. Then

$$\|T\| = \sup_{\Sigma \neq 0} \frac{\|T(x_1, \dots, x_n) - T(y_1, \dots, y_n)\|}{\Sigma},$$

where $\Sigma = \|x_1 - y_1, x_2, \dots, x_n\| + \|y_1, x_2 - y_2, \dots, x_n\| + \cdots + \|y_1, \dots, y_{n-1}, x_n - y_n\|$.

**Proof.** Let

$$\|T\|_n = \sup_{\Sigma \neq 0} \frac{\|T(x_1, \dots, x_n) - T(y_1, \dots, y_n)\|}{\Sigma}.$$

Since $T$ is bounded, by definition

$$\|T\| = \sup_{\|x_1, \dots, x_n\| \neq 0} \frac{\|T(x_1, \dots, x_n)\|}{\|x_1, \dots, x_n\|}.$$

It is enough to prove that $\|T\|_n = \|T\|$.

Let

$$\mathscr{P} = \left\{K : K = \frac{\|T(x_1, \dots, x_n) - T(y_1, \dots, y_n)\|}{\Sigma} \text{ and } \Sigma \neq 0\right\},$$
$$\mathscr{Q} = \left\{K : K = \frac{\|T(x_1, \dots, x_n)\|}{\|x_1, \dots, x_n\|} \text{ and } \|x_1, \dots, x_n\| \neq 0\right\}.$$

Clearly,

$$\mathscr{Q} \subseteq \mathscr{P}$$
$$\Rightarrow \sup \mathscr{Q} \leq \sup \mathscr{P}$$
$$\Rightarrow \|T\| \leq \|T\|_n. \tag{3.1.1}$$

Let

$$\mathscr{W} = \{K : K = \|T(x_1, \dots, x_n) - T(y_1, \dots, y_n)\| \leq K(\Sigma)\}.$$

For each $K \in \mathscr{W}$,

$$
\begin{aligned}
& \|T(x_1,\ldots,x_n) - T(y_1,\ldots,y_n)\| \leq K(\Sigma) \\
\Rightarrow\ & \frac{\|T(x_1,\ldots,x_n) - T(y_1,\ldots,y_n)\|}{\Sigma} \leq K, \text{which is true for all } K \ \&\ \Sigma \neq 0 \\
\Rightarrow\ & \frac{\|T(x_1,\ldots,x_n) - T(y_1,\ldots,y_n)\|}{\Sigma} \leq \inf \mathscr{W} \text{ for all } \Sigma \neq 0 \\
\Rightarrow\ & \frac{\|T(x_1,\ldots,x_n) - T(y_1,\ldots,y_n)\|}{\Sigma} \leq \|T\| \text{ for all } \Sigma \neq 0 \ \text{, using (2.2.1)} \\
\Rightarrow\ & \sup_{\Sigma \neq 0} \frac{\|T(x_1,\ldots,x_n) - T(y_1,\ldots,y_n)\|}{\Sigma} \leq \|T\| \\
\Rightarrow\ & \sup \mathscr{P} \leq \|T\| \\
\Rightarrow\ & \|T\|_n \leq \|T\|. \qquad (3.1.2)
\end{aligned}
$$

Conclusion follows (3.1.1) and (3.1.2).

The following is an extension of the notions of bounded $n$-linear functionals, operators found in [6, 9, 10].

Let $(X, \|.\|)$ and $(Y, \|.\|)$ be normed spaces.

**Definition 3.1.** An operator $T : X^n \to Y$ is an $n$-linear operator on $X$ if $T$ is linear in each variable.

**Definition 3.2.** An $n$-linear operator $T$ is bounded if there exists a positive constant $K$ such that

$$\|T(x_1,...,x_n)\| \leq K\|x_1\|...\|x_n\| \text{ for all } (x_1,...,x_n) \in X^n.$$

When $n = 1$, it is reduced to the usual notion of bounded operator in a normed space.

If $T$ is bounded, norm of $T$ is defined to be

$$\|T\| = \sup_{\substack{x_i \in X \\ \|x_i\| \neq 0}} \frac{\|T(x_1,...,x_n)\|}{\|x_1\|...\|x_n\|}.$$

**Proposition 3.2.** Let $T$ be a bounded $n$-linear operator. Then,

$$\begin{aligned}\|T\| &= \sup_{x_i\in X,\|x_i\|=1} \|T(x_1,\ldots,x_n)\| \\ &= \sup_{x_i\in X,\|x_i\|\leq 1} \|T(x_1,\ldots,x_n)\| \\ &= \inf\{K : \|T(x_1,\ldots,x_n)\| \leq K\|x_1\|...\|x_n\|\}\end{aligned}$$

**Proof.** Let

$$\|T\|_a = \sup_{x_i\in X,\|x_i\|=1} \|T(x_1,\ldots,x_n)\|,$$

$$\|T\|_b = \sup_{x_i\in X,\|x_i\|\leq 1} \|T(x_1,\ldots,x_n)\|,$$

$$\|T\|_c = \inf\{K : \|T(x_1,\ldots,x_n)\| \leq K\|x_1\|...\|x_n\|\}.$$

Since $T$ is bounded $n$-linear operator,

$$\|T\| = \sup_{\substack{x_i\in X \\ \|x_i\|\neq 0}} \frac{\|T(x_1,\ldots,x_n)\|}{\|x_1\|\cdots\|x_n\|}$$

Clearly, $\|T\|_a \leq \|T\|$.

Conversely, define $y_i = \frac{x_i}{\|x_i\|}; i = 1,2,...,n$.

Now,

$$\begin{aligned}\frac{\|T(x_1,\ldots,x_n)\|}{\|x_1\|...\|x_n\|} &= \|T(y_1,\ldots,y_n)\| \\ &\leq \sup_{\|y_i\|=1} \|T(y_1,\ldots,y_n)\| \\ &\leq \|T\|_a \; \forall\, x_i \in X \;\&\; x_i \neq 0\end{aligned}$$

$$\Rightarrow \|T\| \leq \|T\|_a$$

$$\text{Therefore, } \|T\| = \|T\|_a. \qquad (3.2.1)$$

Again, $\|T\|_a \leq \|T\|_b$ is obvious.

Conversely, define $y_i = \frac{x_i}{\|x_i\|}$ ; $0 < \|x_i\| \leq 1$.

Then,

$$\begin{aligned}\|T(x_1,\ldots,x_n)\| &= \|x_1\|...\|x_n\|\|T(y_1,\ldots,y_n)\| \\ &\leq \|T(y_1,\ldots,y_n)\|, \text{ because } \|x_i\| \leq 1\ \forall\, i \\ &\leq \sup_{\|y_i\|=1} \|T(y_1,\ldots,y_n)\| \\ &= \|T\|_a\ \forall\, x_i \in X\ \&\ 0 < \|x_i\| \leq 1\end{aligned}$$

$$\Rightarrow \|T\|_b \leq \|T\|_a$$

$$\text{Therefore, } \|T\|_b = \|T\|_a. \tag{3.2.2}$$

Again, let $\mathscr{L} = \{K : \|T(x_1,...,x_n)\| \leq K\|x_1\|...\|x_n\|\}$.

For each $K \in \mathscr{L}$,

$$\begin{aligned}\frac{\|T(x_1,\ldots,x_n)\|}{\|x_1\|...\|x_n\|} &\leq K\ \forall\, x_i\ \&\ \|x_i\| \neq 0 \\ \Rightarrow \sup_{\|x_i\|\neq 0} \frac{\|T(x_1,\ldots,x_n)\|}{\|x_1\|...\|x_n\|} &\leq K\end{aligned}$$

$$\Rightarrow \|T\| \leq K, \text{which is true for all } k \in \mathscr{L}.$$

$$\Rightarrow \|T\| \leq \inf \mathscr{L}$$

$$\Rightarrow \|T\| \leq \|T\|_c.$$

Conversely,

$$\|T\| = \sup_{\|x_i\|\neq 0} \frac{\|T(x_1,\ldots,x_n)\|}{\|x_1\|...\|x_n\|}$$

$$\Rightarrow \|T(x_1,\ldots,x_n)\| \leq \|T\|\|x_1\|...\|x_n\|$$

$$\Rightarrow \|T\| \in \mathscr{L}$$

$$\Rightarrow \|T\| \geq \inf \mathscr{L}$$

$$\Rightarrow \|T\| \geq \|T\|_c$$

$$\text{Thus,} \quad \|T\| = \|T\|_c. \tag{3.2.3}$$

Hence, $\|T\| = \|T\|_a = \|T\|_b = \|T\|_c$ using $(3.2.1), (3.2.2)$ and $(3.2.3)$.

**Proposition 3.3.** Let $T : X^n \to Y$ be an $n$-linear operator where $n$ is a positive integer and $X$, $Y$ are normed spaces. Then, $T$ is bounded iff there exists a positive constant $K$ such that for all $(x_1,...,x_n),(y_1,...,y_n) \in X^n$,

$$\begin{aligned} \|T(x_1,\ldots,x_n) - T(y_1,\ldots,y_n)\| \quad &\leq \quad K(\|x_1 - y_1\|\|x_2\|...\|x_n\| + \|y_1\|\|x_2 - y_2\|...\|x_n\| + \\ &\cdots + \|y_1\|...\|y_{n-1}\|\|x_n - y_n\|). \end{aligned}$$

**Proof.** Suppose the inequality holds. We take $(y_1,\ldots,y_n) = (0,\ldots,0)$.

Then, $T(y_1,\ldots,y_n) = 0$.

Using the inequality, we have

$$\begin{aligned} \|T(x_1,\ldots,x_n) - 0\| \quad &\leq \quad K(\|x_1\|...\|x_n\| + 0 + ... + 0) \\ \Rightarrow \|T(x_1,\ldots,x_n)\| \quad &\leq \quad K\|x_1\|...\|x_n\|. \end{aligned}$$

Therefore, $T$ is bounded.

Conversely, suppose $T$ is bounded. Then by $n$-linearity, we have

$$\begin{aligned} &\|T(x_1,\ldots,x_n) - T(y_1,\ldots,y_n)\| \\ &= \|T(x_1 - y_1, x_2,\ldots,x_n) + T(y_1, x_2 - y_2,\ldots,x_n) + \\ &\qquad \cdots + T(y_1, y_2,\ldots,y_{n-1}, x_n - y_n)\| \\ &\leq \|T(x_1 - y_1, x_2,\ldots,x_n)\| + \|T(y_1, x_2 - y_2,\ldots,x_n)\| + \\ &\qquad \cdots + \|T(y_1, y_2,\ldots,y_{n-1}, x_n - y_n)\| \end{aligned}$$

(by triangle inequality)

$$\leq K(\|x_1-y_1\|\|x_2\|...\|x_n\|+\|y_1\|\|x_2-y_2\|...\|x_n\|+ \cdots+\|y_1\|...\|y_{n-1}\|\|x_n-y_n\|).$$

It completes the proof.

**Proposition 3.4.** Let $T$ be a bounded $n$-linear operator from a normed space $X$ into a normed space $Y$. Then,

$$\begin{aligned}
\|T\| &= \inf\{K : \|T(x_1,\ldots,x_n)\| \leq K\|x_1\|...\|x_n\|\} \\
&= \inf\{K : \|T(x_1,\ldots,x_n)-T(y_1,\ldots,y_n)\| \leq K(\|x_1-y_1\|\|x_2\|...\|x_n\| \\
&\qquad +\|y_1\|\|x_2-y_2\|...\|x_n\|+...+\|y_1\|...\|y_{n-1}\|\|x_n-y_n\|)\} \\
&= \sup_{\Sigma\neq 0}\frac{\|T(x_1,\ldots,x_n)-T(y_1,\ldots,y_n)\|}{\Sigma} \\
&\quad \text{where } \Sigma = \|x_1-y_1\|\|x_2\|...\|x_n\|+\|y_1\|\|x_2-y_2\|...\|x_n\|+\cdots \\
&\qquad +\|y_1\|...\|y_{n-1}\|\|x_n-y_n\|
\end{aligned}$$

**Proof.** By definition,

$$\|T\| = \sup_{\substack{x_i\in X \\ \|x_i\|\neq 0}} \frac{\|T(x_1,\ldots,x_n)\|}{\|x_1\|...\|x_n\|}$$

Let

$$\begin{aligned}
\|T\|_c &= \inf\{K : \|T(x_1,\ldots,x_n)\| \leq K\|x_1\|...\|x_n\|\}, \\
\|T\|_d &= \inf\{K : \|T(x_1,\ldots,x_n)-T(y_1,\ldots,y_n)\| \leq K(\|x_1-y_1\|\|x_2\|...\|x_n\| \\
&\qquad +\|y_1\|\|x_2-y_2\|...\|x_n\|+\cdots+\|y_1\|...\|y_{n-1}\|\|x_n-y_n\|)\} \\
\text{and } \|T\|_e &= \sup_{\Sigma\neq 0}\frac{\|T(x_1,\ldots,x_n)-T(y_1,\ldots,y_n)\|}{\Sigma}.
\end{aligned}$$

Also, let

$$\begin{aligned}\mathscr{K} &= \{K : \|T(x_1,\dots,x_n) - T(y_1,\dots,y_n)\| \le K(\|x_1-y_1\|\|x_2\|...\|x_n\| \\ &\qquad + \|y_1\|\|x_2-y_2\|...\|x_n\| + \cdots + \|y_1\|...\|y_{n-1}\|\|x_n-y_n\|)\} \\ \text{and } \mathscr{C} &= \{K : \|T(x_1,\dots,x_n)\| \le K\|x_1\|...\|x_n\|\}.\end{aligned}$$

Clearly, $\mathscr{C} \subseteq \mathscr{K}$.

$$\begin{aligned}\text{Therefore, } \inf\mathscr{K} &\le \inf\mathscr{C}. \\ \Rightarrow \|T\|_d &\le \|T\|_c \qquad (3.4.1) \\ \Rightarrow \|T\|_d &\le \|T\|, \text{ because } \|T\| = \|T\|_c \text{ by proposition(3.2).}\end{aligned}$$

Let

$$\begin{aligned}\mathscr{A} &= \left\{K : K = \frac{\|T(x_1,\dots,x_n)\|}{\|x_1\|...\|x_n\|}, \|x_i\| \ne 0\right\}, \\ \mathscr{B} &= \left\{K : K = \frac{\|T(x_1,\dots,x_n) - T(y_1,\dots,y_n)\|}{\Sigma}, \Sigma \ne 0\right\}.\end{aligned}$$

Obviuosly,

$$\begin{aligned}\mathscr{A} &\subseteq \mathscr{B} \\ \Rightarrow \sup\mathscr{A} &\le \sup\mathscr{B} \qquad (3.4.2) \\ \Rightarrow \|T\| &\le \|T\|_e.\end{aligned}$$

For each $K \in \mathscr{K}$,

$$\begin{aligned}&\tfrac{\|T(x_1,\dots,x_n) - T(y_1,\dots,y_n)\|}{\Sigma} \le K \ \forall \ \Sigma \ne 0 \\ &\Rightarrow \sup_{\Sigma\ne 0} \frac{\|T(x_1,\dots,x_n) - T(y_1,\dots,y_n)\|}{\Sigma} \le K \\ &\Rightarrow \|T\|_e \le K \text{ , which is true for all } K. \\ &\Rightarrow \|T\|_e \le \inf\mathscr{K} \qquad (3.4.3) \\ &\Rightarrow \|T\|_e \le \|T\|_d.\end{aligned}$$

From $(3.4.1), (3.4.2), (3.4.3)$ and proposition 3.2, it follows that

$$\|T\| \le \|T\|_e \le \|T\|_d \le \|T\| \text{ and } \|T\| = \|T\|_c.$$

It completes the proof.

## 4. Continuity of $n$-linear operators

**Definition 4.1.** An $n$-linear operator $T : X^n \to Y$ where $X$ and $Y$ are normed linear spaces is continuous at $(x_1, ..., x_n) \in X^n$ if for $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|T(x_1, ..., x_n) - T(y_1, ..., y_n)\| < \varepsilon$$

whenever

$$\|x_1 - y_1\|\|x_2\|...\|x_n\| < \delta, \qquad \|y_1\|\|x_2 - y_2\|...\|x_n\| < \delta, \quad \dots \quad \text{and}$$
$$\|y_1\|\|y_2\|...\|y_{n-1}\|\|x_n - y_n\| < \delta$$

or

$$\|x_1 - y_1\|\|y_2\|...\|y_n\| < \delta, \qquad \|x_1\|\|x_2 - y_2\|...\|y_n\| < \delta, \quad \dots \quad \text{and}$$
$$\|x_1\|\|x_2\|...\|x_{n-1}\|\|x_n - y_n\| < \delta,$$

where $(y_1, \dots, y_n) \in X^n$.

**Definition 4.2.** $T$ is continuous on $X^n$ if it is continuous at every $(x_1, \dots, x_n) \in X^n$.
When $n = 1$, it reduces to the usual notion of continuity in the normed space.

The above definition is similar to the definition given by A. L. Soenjaya in [14] as an extension of that given by White in [15]. In Soenjaya's paper, the $n$-linear operator is from an $n$-normed space to a normed space. But, in this paper the $n$-linear operator is from a normed space to a normed space.

**Theorem 4.1.** Let $T : X^n \to Y$ be an $n$-linear operator where $(X, \|.\|)$ and $(Y, \|.\|)$ are normed spaces. Then the followings are equivalent:

(a) $T$ is continuous.

(b) $T$ is continuous at $(0, \dots, 0) \in X^n$.

(c) $T$ is bounded.

**Proof.** $(a) \Rightarrow (b)$ is obvious.
$(b) \Rightarrow (c)$ : If $T = 0$, the proof is through. Suppose $T \neq 0$. Let $(x_1, ..., x_n) \in X^n$ with $\|x_i\| \neq 0 \, \forall \, i$. Set

$$u_i = \left( \frac{\delta}{2\|x_1\| \cdots \|x_n\|} \right)^{\frac{1}{n}} x_i.$$

Then,

$$\|u_1\|...\|u_n\| = \frac{\delta}{2} < \delta. \tag{4.1.1}$$

Now,

$$\begin{aligned} \|T(u_1,\ldots,u_n)\| &= \tfrac{\delta}{2\|x_1\|...\|x_n\|}\|T(x_1,\ldots,x_n)\| \\ \Rightarrow \|T(x_1,\ldots,x_n)\| &= \tfrac{2}{\delta}\|x_1\|...\|x_n\|\|T(u_1,...,u_n)\|. \end{aligned} \tag{4.1.2}$$

Since $T$ is continuous at $(0,\ldots,0)\in X^n$, for $\varepsilon=1$ there exists $\delta>0$ such that

$$\|T(u_1,\ldots,u_n)\|<1 \text{ whenever } \|u_1\|...\|u_n\|<\delta.$$

Therefore,

$$\begin{aligned} \|T(x_1,\ldots,x_n)\| &< \frac{2}{\delta}\|x_1\|...\|x_n\|.1, \text{ using (4.1.1) and (4.1.2)} \\ &= \frac{2}{\delta}\|x_1\|\cdots\|x_n\| \\ \Rightarrow \|T(x_1,\ldots,x_n)\| &< \frac{2}{\delta}\|x_1\|\cdots\|x_n\| \end{aligned}$$

$\Rightarrow T$ is bounded.

$(c)\Rightarrow(a)$ : Suppose $T$ is bounded. If $T=0$, the statement is trivial. Let $T\neq 0$. Consider any point $(x_1^{'},\ldots,x_n^{'})\in X^n$.

Let $\varepsilon>0$ be given. For every $(x_1,\ldots,x_n)\in X^n$ such that

$$\begin{aligned} &\|x_1-x_1^{'}\|\|x_2\|...\|x_n\|<\delta, \quad \|x_1^{'}\|\|x_2-x_2^{'}\|...\|x_n\|<\delta\,, \quad \ldots \quad \text{and} \\ &\|x_1^{'}\|\|x_2^{'}\|...\|x_{n-1}^{'}\|\|x_n-x_n^{'}\|<\delta, \end{aligned} \tag{4.1.3}$$

where $\delta=\frac{\varepsilon}{n\|T\|}$,

$$\begin{aligned} \|T(x_1,\ldots,x_n)-T(x_1^{'},\ldots,x_n^{'})\| &\leq \|T\|(\|x_1-x_1^{'}\|\|x_2\|...\|x_n\|+\|x_1^{'}\|\|x_2-x_2^{'}\|...\|x_n\|+ \\ &\quad \cdots+\|x_1^{'}\|...\|x_{n-1}^{'}\|\|x_n-x_n^{'}\|) \\ &< \|T\|(\underbrace{\delta+\delta+\cdots+\delta}_{n \text{ terms}}), \text{ using (4.1.3)} \\ &= n\|T\|\delta \\ &= \varepsilon \end{aligned}$$

$\Rightarrow T$ is continuous at $(x_1^{'},\ldots,x_n^{'})$. But, $(x_1^{'},\ldots,x_n^{'})$ is arbitrary. Therefore, $T$ is continuous. This completes the proof.

Let $B(X^n, Y)$ denote the space of all bounded $n$-linear operators from $X^n$ into $Y$, where $X$ and $Y$ are normed spaces. Then we have the following theorems.

**Theorem 4.2.** $(B(X^n,Y), \|.\|)$ is a normed space with respect to the norm given by

$$\|T\| = \sup_{\substack{x_i \in X \\ \|x_i\| \neq 0}} \frac{\|T(x_1,\dots,x_n)\|}{\|x_1\|\dots\|x_n\|}.$$

**Proof.** (i)

$$\begin{aligned}
\|\alpha T\| &= \sup_{\|x_i\|\neq 0} \frac{\|\alpha T(x_1,\dots,x_n)\|}{\|x_1\|\dots\|x_n\|} \\
&= |\alpha| \sup_{\|x_i\|\neq 0} \frac{\|T(x_1,\dots,x_n)\|}{\|x_1\|\dots\|x_n\|} \\
&= |\alpha|\|T\|.
\end{aligned}$$

(ii)

$$\begin{aligned}
\|T_1+T_2\| &= \sup_{\|x_i\|\neq 0} \frac{\|(T_1+T_2)(x_1,\dots,x_n)\|}{\|x_1\|\dots\|x_n\|} \\
&= \sup_{\|x_i\|\neq 0} \frac{\|(T_1(x_1,\dots,x_n)+T_2(x_1,\dots,x_n)\|}{\|x_1\|\dots\|x_n\|} \\
&\leq \sup_{\|x_i\|\neq 0} \frac{\|(T_1(x_1,\dots,x_n)\|}{\|x_1\|\dots\|x_n\|} + \sup_{\|x_i\|\neq 0} \frac{\|(T_2(x_1,\dots,x_n)\|}{\|x_1\|\dots\|x_n\|} \\
&= \|T_1\| + \|T_2\|.
\end{aligned}$$

(iii) Clearly, $\|T\| \geq 0$. Lastly, it is to show that $\|T\| = 0 \Longleftrightarrow T = 0$.

$$\begin{aligned}
\|T\| = 0 &\Leftrightarrow \sup_{\|x_i\|\neq 0} \frac{\|T(x_1,\dots,x_n)\|}{\|x_1\|\dots\|x_n\|} = 0 \\
&\Leftrightarrow \|T(x_1,\dots,x_n)\| = 0 \text{ for } x_i \neq 0 \text{ for each i} \\
&\Leftrightarrow T(x_1,\dots,x_n) = 0 \text{ for all } (x_1,\dots,x_n) \in X^n \text{ and each } x_i \neq 0.
\end{aligned}$$

If $\|x_1\|...\|x_n\| = 0$, at least one of $x_i s$ is 0. Since $T$ is bounded $n$-linear operator, $T(x_1,\ldots,x_n) = 0$ if $\|x_1\|...\|x_n\| = 0$. So,

$$\|T\| = 0 \Leftrightarrow T(x_1,\ldots,x_n) = 0 \text{ for all } (x_1,\ldots,x_n) \in X^n$$

$$\Leftrightarrow T \equiv 0.$$

Therefore, $\|.\|$ is a norm and it completes the proof.

**Theorem 4.3.** If $(Y,\|.\|)$ is a Banach space, $(B(X^n,Y),\|.\|)$ is a Banach space.

**Proof.** Let $\{T_k\}$ be a cauchy sequence in $B(X^n,Y)$. Let $\varepsilon > 0$ be given. Then there exists $N > 0$ such that $\|T_k - T_m\| < \frac{\varepsilon}{2}\ \forall\ k,m > N$.

Now,

$$\|T_k(x_1,\ldots,x_n) - T_m(x_1,\ldots,x_n)\| = \|(T_K - T_m)(x_1,\ldots,x_n)\|$$
$$\leq \|T_k - T_m\|\|x_1\|...\|x_n\|.$$

This shows that $\{T_k(x_1,\ldots,x_n)\}$ is a cauchy sequence in $Y$. Since $Y$ is complete, there exists a vector in $Y$, say $T(x_1,\ldots,x_n)$ such that $T_k(x_1,\ldots,x_n) \to T(x_1,\ldots,x_n)$. Clearly, $T$ is an $n$-linear operator from $X^n$ into $Y$. To conclude the proof, we have to show that $T$ is bounded and that $T_k \to T$.

Now,

$$\|T(x_1,\ldots,x_n)\| = \lim\|T_k(x_1,\ldots,x_n)\| \leq \sup(\|T_k\|\|x_1\|...\|x_n\|)$$
$$= (\sup\|T_k\|)(\|x_1\|...\|x_n\|). \tag{4.3.1}$$

For $k,m > N$,

$$\|T_k(x_1,\ldots,x_n) - T_m(x_1,\ldots,x_n)\| \leq \|T_k - T_m\|\|x_1\|...\|x_n\|$$
$$< \frac{\varepsilon}{2} \text{ if } \|x_1\|...\|x_n\| \leq 1.$$

Now, holding $k$ fixed and allowing $m \to \infty$, we have

$$\|T_k(x_1,\ldots,x_n) - T(x_1,\ldots,x_n)\| < \varepsilon\ \forall k > N \text{and}\ \forall\ (x_1,\ldots,x_n) \in X^n : \|x_1\|...\|x_n\| \leq 1$$
$$\Rightarrow \|(T_k - T)(x_1,\ldots,x_n)\| < \varepsilon\ \forall\ k > N \text{ and } \forall\ (x_1,\ldots,x_n) \in X^n : \|x_1\|...\|x_n\| \leq 1$$
$$\Rightarrow \sup_{\|x_i\|\leq 1} \|(T_k - T)(x_1,\ldots,x_n)\| \leq \frac{\varepsilon}{2}$$
$$\Rightarrow \|T_k - T\| < \varepsilon\ \ \forall\ k > N$$
$$\Rightarrow T_k \longrightarrow T \ \text{ as } \ k \longrightarrow \infty \text{ and by (4.3.1), } T \in B(X^n, Y).$$

This completes the proof.

**Theorem 4.4.** Let $X$ be a real vector space with $\dim X = d$ where $d \geq n$ and $n$ is a positive integer. Let $(Y, \|.\|)$ be a normed space and $T : X^n \longrightarrow Y$ be an $n$-linear operator. If $X$ is equipped with a norm $\|.\|$ and an $n$-norm $\|., ..., .\|$, define

$$\|T\|_1 = sup_{x_i \in X, \|x_1,...,x_n\| \neq 0} \frac{\|T(x_1,...,x_n)\|}{\|x_1,...,x_n\|}$$
$$\text{and } \ \|T\|_2 = sup_{x_i \in X, \|x_i\| \neq 0} \frac{\|T(x_1,...,x_n)\|}{\|x_1\|...\|x_n\|}.$$

Then, $\|.\|_1$ and $\|.\|_2$ are identical.

**Proof.** Define

$$y_i = \frac{\|y_i\| x_i}{\sqrt[n]{\|x_1,...,x_n\|}}$$
$$= \frac{\|y_i\| x_i}{\gamma} \ \ ; \gamma = \sqrt[n]{\|x_1,...,x_n\|} \neq 0$$

Now,

$$\begin{aligned}
T(x_1,\ldots,x_n) &= T\left(\frac{\gamma y_1}{\|y_1\|},\ldots,\frac{\gamma y_n}{\|y_n\|}\right) \\
&= \frac{\gamma^n T(y_1,\ldots,y_n)}{\|y_1\|,\ldots,\|y_n\|} \\
\Rightarrow \frac{T(x_1,\ldots,x_n)}{\gamma^n} &= \frac{T(y_1,\ldots,y_n)}{\|y_1\|,\ldots,\|y_n\|} \\
\Rightarrow \frac{T(x_1,\ldots,x_n)}{\|x_1,\ldots,x_n\|} &= \frac{T(y_1,\ldots,y_n)}{\|y_1\|,\ldots,\|y_n\|} \qquad (4.4.1)
\end{aligned}$$

Taking supremum on the right side of $(4.4.1)$ over $y_i s \in X$ with $\|y_i\| \neq 0$, we have

$$\begin{aligned}
\frac{T(x_1,\ldots,x_n)}{\|x_1,\ldots,x_n\|} &\leq \sup_{y_i\in X,\|y_i\|\neq 0} \frac{T(y_1,\ldots,y_n)}{\|y_1\|,\ldots,\|y_n\|} \\
&= \|T\|_2
\end{aligned}$$

It is true for all $x_i \in X$ with $\|x_1,\ldots,x_n\| \neq 0$.

Therefore

$$\begin{aligned}
&\sup_{x_i\in X,\|x_1,\ldots,x_n\|\neq 0} \frac{T(x_1,\ldots,x_n)}{\|x_1,\ldots,x_n\|} \leq \|T\|_2 \\
&\Rightarrow \|T\|_1 \leq \|T\|_2 \qquad (4.4.2)
\end{aligned}$$

Again,Taking supremum on the left side of $(4.4.1)$ over $\{(x_1,\ldots,x_n) \in X^n : \|x_1,\ldots,x_n\| \neq 0\}$, we have

$$\begin{aligned}
\sup_{x_i\in X,\|x_1,\ldots,x_n\|\neq 0} \frac{T(x_1,\ldots,x_n)}{\|x_1,\ldots,x_n\|} &\geq \frac{T(y_1,\ldots,y_n)}{\|y_1\|,\ldots,\|y_n\|} \\
\Rightarrow \|T\|_1 &\geq \frac{T(y_1,\ldots,y_n)}{\|y_1\|,\ldots,\|y_n\|}
\end{aligned}$$

It is true for all $y_i \in X$ with $\|y_i\| \neq 0$. Therefore,

$$\|T\|_1 \geq \sup_{y_i \in X, \|y_i\| \neq 0} \frac{T(y_1,...,y_n)}{\|y_1\|,...,\|y_n\|}$$
$$\Rightarrow \|T\|_1 \geq \|T\|_2 \tag{4.4.3}$$

Conclusion follows from (4.4.2) and (4.4.3).

## 5. NEW $n$-NORM

The idea of the following formula of $n$-norm is derived from [12, 13]. Its similar formula in Proposition 2.3 is defined on a real vector space with dimension $\geq n$ but the forthcoming formula is defined on a dual space.

**Theorem 5.1.** Let $X$ be a real vector space with $\dim(X) \geq n$ where $n$ is a positive integer and $X'$ be the dual of $X$. Then, the function $\|.,...,.\| : (X')^n \longrightarrow \mathbb{R}$ given by

$$\|f_1,...,f_n\| = abs\left(\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}\right) \text{ for fixed linearly independent } n$$
$$\text{elements } x_1,...,x_n \in X,$$

defines an $n$-norm on $X'$.

**Proof.** $(i)$

$$f_1, f_2, ..., f_n \text{ are linearly dependent.}$$
$$\Leftrightarrow \text{columns of the matrix } [f_i(x_j)] \text{ are linearly dependent.}$$
$$\Leftrightarrow \text{value of } det[f_i(x_j)] = 0.$$

$$\Leftrightarrow abs\left(\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}\right) = 0$$

$$\Leftrightarrow \|f_1,...,f_n\| = 0.$$

$(ii)$ By the properties of determinant and definition of absolute (abs), $\|f_1,...,f_n\|$ remains invariant under the permutations of $f_1, f_2, ..., f_n$.

$(iii)$ For $\alpha \in \mathbb{R}$,

$$\|\alpha f_1,...,f_n\| = abs\left(\begin{vmatrix} \alpha f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ \alpha f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}\right)$$

$$= |\alpha| abs\left(\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ \alpha f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}\right)$$

$$= |\alpha| \|f_1,...,f_n\|$$

$(iv)$

$$\|f_0 + f_1,...,f_n\| = abs\left(\begin{vmatrix} (f_0+f_1)(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ (f_0+f_1)(x_n) & \cdots & f_n(x_n) \end{vmatrix}\right)$$

$$= abs\left(\begin{vmatrix} f_0(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_0(x_n) & \cdots & f_n(x_n) \end{vmatrix} + \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}\right)$$

$$\leq abs\left(\begin{vmatrix} f_0(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_0(x_n) & \cdots & f_n(x_n) \end{vmatrix}\right) + abs\left(\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}\right)$$

$$= \|f_0,...,f_n\| + \|f_1,...,f_n\|.$$

It completes the proof.

## Conflict of Interests

The authors declare that there is no conflict of interests.

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