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EVOLUTION OF SPHERICAL IMAGES AND SMARANDACHE CURVES OF A SPACE CURVE IN EUCLIDEAN 3-SPACE

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Abstract. We study the evolution of a regular space curves, spherical images and Smarandache curves. We derive dynamical equation of moving frame along the evolving curve and its curvatures, consequently we get the evolution equation for spherical images and Smarandache curves. Finally we shall display visualization of the evolving curve, spherical images and Smarandache curves.

Keywords: evolution of curves; spherical images; Serret-Frenet equations; Smarandache curves.

2010 AMS Subject Classification: 37L05.

1. Introduction

A lot of interesting phenomena in physics, chemistry, biology and engineering can be modeled as a curve evolving in space. The authors in [1] studied curves evolving in three dimensional space and obtained time evolution equation for curvatures. Nassar, et al [2, 3, 4] studied the motion of plane curves, the motion of hyper surfaces and the motion of space curves in R^n. Hasimoto showed in [5] the nonlinear Schrödinger equation describing the motion of an

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isolated non−stretching thin vortex filament. Lamb [6] obtained the mKdV and sine-Gordon equations by using Hasimoto transformation.

R. Mukherjee and R. Balakrishnan in [7] they provide a general scheme to mapping nonlinear differential equation to moving space curves.

Talat Körpinar and Essin Turhan in [8, 9] studied time-evolution equations for binromal indicatrix, they obtained partial differential equation which governing the evolution of these curves and surface generated by its motion on a sphere.

In this paper we study the evolution of a space curve and its spherical images by a method different from those in [7]. We derive the dynamical equation of the moving frame along the evolving curve and its curvatures, consequently we get the evolution equation of the spherical image for this curves and Smarandache curves. Finally we shall display geometric visualization of the evolving curve, spherical images and Smarandache curves.

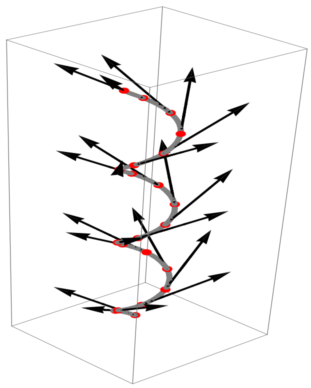
2. Preliminaries

Let us consider a curve embedded in three-dimensional space described in parametric form by a position vector $\zeta = \zeta(s)$, s being the usual arc length variable .The unit tangent vector $\mathbf{f_1}$, the principal normal $\mathbf{f_2}$ and the binormal $\mathbf{f_3}$ vary along the curve according to the S-F equations [10]

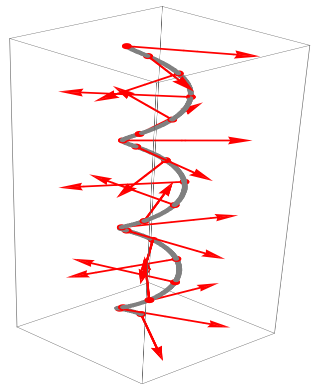
$$\frac{d}{ds}\begin{pmatrix} \mathbf{f_1} \\ \mathbf{f_2} \\ \mathbf{f_3} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{f_1} \\ \mathbf{f_2} \\ \mathbf{f_3} \end{pmatrix} \tag{1}$$

where κ and τ are the curvature and torsion of the curve.

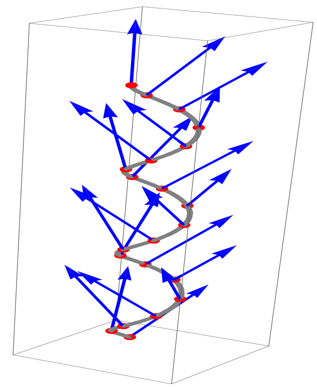
Definition 2.1. The image of a curve in the three-dimensional space under a mapping from the points of the curve onto the center of unit sphere keeping its direction in space by any of the following unit vectors: the principal normal, the tangent, or the binormal of this curve is called spherical image of a space curve as shown in figure 1[10]. let $\zeta = \zeta(s)$ be a unit speed regular curve in Euclidian space, $\mathbf{f_1}$ is the unit tangent vector, $\mathbf{f_2}$ is the principal normal and $\mathbf{f_3}$ is the binormal. Now we can define the following Smarandache curves which are linear combinations of tangent , normal, binormal[11].



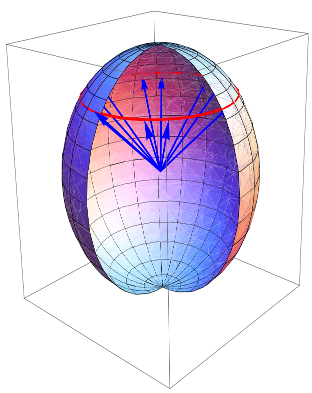
(A) Tangent vecor field



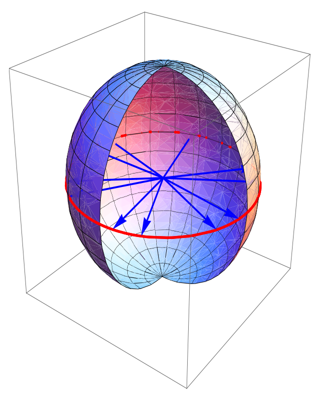
(B) Normal vecor field



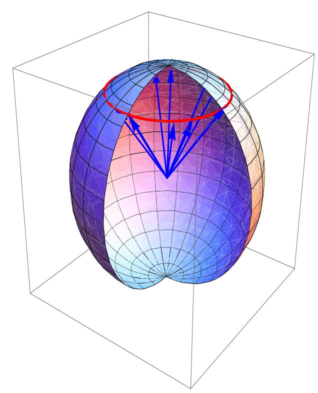
(C) Binormal vecor field



(D) Spherical image of Tangent



(E) Spherical image of Normal



(F) Spherical image of binormal

FIGURE 1. Frenet fram field and their spherical images for circular helix

Definition 2.2. Smarandache curves $\mathbf{f_1f_2}$ is defined by

$$S_1(s_\zeta) = \frac{1}{\sqrt{2}}(\mathbf{f_1}(s) + \mathbf{f_2}(s)). \tag{2}$$

Definition 2.3. Smarandache curve $\mathbf{f_2f_3}$ is defined by

$$S_2(s_\zeta) = \frac{1}{\sqrt{2}}(\mathbf{f_2}(s) + \mathbf{f_3}(s)). \tag{3}$$

Definition 2.4. Smarandache curve $\mathbf{f_1f_2f_3}$ is defined by

$$S_3(s_\zeta) = \frac{1}{\sqrt{3}}(\mathbf{f_1}(s) + \mathbf{f_2}(s) + \mathbf{f_3}(s)). \tag{4}$$

3. Time-evolution equations (TEE) of curves and their spherical image

In this section we study TEE which governing the motion of space curves in the three dimensional space. let $\zeta = \zeta(s,t)$ denote the position vector of a point on the curve at time t. Any flow of $\zeta(s,t)$ can be represented as

$$\frac{\partial \zeta(s,t)}{\partial t} = \alpha \mathbf{f_1} + \beta \mathbf{f_2} + \gamma \mathbf{f_3}, \tag{5}$$

where α, β, γ are smooth functions of arc length.

The new evolving moving frame $(\mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3})$ is related with Frenet frame $(\mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3})$ by

$$\begin{pmatrix} \mathbf{h_1} \\ \mathbf{h_2} \\ \mathbf{h_3} \end{pmatrix} = \begin{pmatrix} L_1^1 & L_1^2 & L_1^3 \\ L_2^1 & L_2^2 & L_2^3 \\ L_3^1 & L_3^2 & L_3^3 \end{pmatrix} \begin{pmatrix} \mathbf{f_1} \\ \mathbf{f_2} \\ \mathbf{f_3} \end{pmatrix} \tag{6}$$

. where, $\mathbf{h_1}$ is the tangent vector to the evolving curve, $\mathbf{h_2}$ normal vector $\mathbf{h_3}$ binormal vector.

The quantities L_i^j, $i = 1,3$,$j = 1,3$ are given by

$$\begin{aligned} L_1^1 &= \frac{\eta_{11}}{\sqrt{\eta_{11}^2 + \eta_{12}^2 + \eta_{13}^2}}, \\ L_1^2 &= \frac{\eta_{12}}{\sqrt{\eta_{11}^2 + \eta_{12}^2 + \eta_{13}^2}}, \\ L_1^3 &= \frac{\eta 13}{\sqrt{\eta_{11}^2 + \eta_{12}^2 + \eta_{13}^2}}, \\ L_2^1 &= \frac{(\eta_{12}\eta_{23} - \eta_{13}\eta_{22})}{\sqrt{(\eta_{11}\eta_{22} - \eta_{12}\eta_{21})^2}}, \\ L_2^2 &= \frac{(\eta_{13}\eta_{21} - \eta_{11}\eta_{23})}{\sqrt{(\eta_{11}\eta_{22} - \eta_{12}\eta_{21})^2}}, \\ L_2^3 &= \frac{(\eta_{11}\eta_{22} - \eta_{12}\eta_{21})}{\sqrt{(\eta_{11}\eta_{22} - \eta_{12}\eta_{21})^2}}, \end{aligned} \tag{7}$$

$$L_3^1 = \frac{\eta_{21}\eta_{12}^2 - \eta_{11}\eta_{22}\eta_{12} + \eta_{13}^2\eta_{21} - \eta_{11}\eta_{13}\eta_{23}}{\sqrt{a(\eta_{11}^2 + \eta_{12}^2 + \eta_{13}^2)}},$$

$$L_3^2 = \frac{\eta_{11}^2\eta_{22} - \eta_{11}\eta_{12}\eta_{21} - \eta_{12}\eta_{13}\eta_{23} + \eta_{13}^2\eta_{22}}{\sqrt{a(\eta_{11}^2 + \eta_{12}^2 + \eta_{13}^2)}},$$

$$L_3^3 = \frac{\eta_{23}\eta_{11}^2 - \eta_{13}\eta_{21}\eta_{11} + \eta_{12}\eta_{13}\eta_{22} + \eta_{12}^2\eta_{23}}{\sqrt{a(\eta_{11}^2 + \eta_{12}^2 + \eta_{13}^2)}},$$

$$\begin{aligned} a &= \eta_{12}^2\eta_{21}^2 + \eta_{13}^2\eta_{21}^2 - 2\eta_{11}\eta_{12}\eta_{22}\eta_{21} \\ &- 2\eta_{11}\eta_{13}\eta_{23}\eta_{21} + \eta_{11}^2\eta_{22}^2 + \eta_{13}^2\eta_{22}^2 \\ &+ \eta_{11}^2\eta_{23}^2 + \eta_{12}^2\eta_{23}^2 - 2\eta_{12}\eta_{13}\eta_{22}\eta_{23}, \end{aligned}$$

where, η_i^j, $i = 1,3$,$j = 1,3$ are given by

(8)

$$\eta_{11} = 1 - t\beta\kappa + t\alpha',$$

$$\eta_{12} = t\alpha\kappa - t\gamma + \mathtt{t}\beta',$$

$$\eta_{13} = t\beta + \mathtt{t}\gamma',$$

$$\eta_{21} = -t\alpha\kappa^2 + t\gamma\kappa\tau - 2t\kappa\beta' - t\beta\kappa' + t\alpha'',$$

$$\eta_{22} = \kappa - t\beta\kappa^2 + 2t\kappa\alpha' - 2t\tau\gamma' + t\alpha\kappa' - t\gamma\tau' + t\beta'',$$

$$\eta_{23} = t\alpha\kappa\tau + 2t\tau\beta' + t\beta\tau' + t\gamma'' - t\gamma\tau^2,$$

$$\eta_{31} = -\kappa^2 + t\alpha^{(3)} - 3t\kappa^2\alpha' - 3t\alpha\kappa\kappa' - 3t\kappa\beta'' - 3t\beta'\kappa' - t\beta\kappa'' + t\beta\kappa\tau^2 + t\beta\kappa^3 + 3t\kappa\tau\gamma' + t\gamma\tau\kappa' + 2t\gamma\kappa\tau'.$$

$$\begin{aligned} \eta_{32} = \kappa' + 3t\kappa\alpha'' + 3t\alpha'\kappa' + t\alpha\kappa' - t\alpha\kappa\tau^2 - t\alpha\kappa^3 + t\beta^{(3)} - 3t\kappa^2\beta' - 3t\tau^2\beta' - 3t\beta\kappa\kappa' - 3t\beta\tau\tau' \\ - 3t\tau\gamma'' - 3t\gamma'\tau' + t\gamma\kappa^2\tau - t\gamma\tau'' + t\gamma\tau^3, \end{aligned}$$

$$\eta_{33} = \kappa\tau + 3t\kappa\tau\alpha' + 2t\alpha\tau\kappa' + t\alpha\kappa\tau' + 3t\tau\beta'' + 3t\beta'\tau' - t\beta\kappa^2\tau + t\beta\tau'' - t\beta\tau^3 + t\gamma^{(3)} - 3t\tau^2\gamma' - 3t\gamma\tau\tau'.$$

The curvature κ_1 and torsion τ_1 of the evolving curve are,

$$\kappa_1(s,t) = \frac{\sqrt{(\eta_{12}\eta_{21} - \eta_{11}\eta_{22})^2 + (\eta_{13}\eta_{21} - \eta_{11}\eta_{23})^2 + (\eta_{13}\eta_{22} - \eta_{12}\eta_{23})^2}}{(\eta_{11}^2 + \eta_{12}^2 + \eta_{13}^2)^{(3/2)}},$$

(9)

$$\tau_1(s,t) = \frac{(-\eta_{13}\eta_{22}\eta_{31} + \eta_{12}\eta_{23}\eta_{12}\eta_{21}\eta_{33} + \eta_{11}\eta_{22}\eta_{33})}{(\eta_{12}\eta_{21} - \eta_{11}\eta_{22})^2 + (\eta_{13}\eta_{21} - \eta_{11}\eta_{23})^2 + (\eta_{13}\eta_{22} - \eta_{12}\eta_{23})^2},$$

TEE of the new moving frame is given by,

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{h_1}(s,t) \\ \mathbf{h_2}(s,t) \\ \mathbf{h_3}(s,t) \end{pmatrix} = \begin{pmatrix} \frac{\partial L_1^1}{\partial t} & \frac{\partial L_1^2}{\partial t} & \frac{\partial L_1^3}{\partial t} \\ \frac{\partial L_2^1}{\partial t} & \frac{\partial L_2^2}{\partial t} & \frac{\partial L_2^3}{\partial t} \\ \frac{\partial L_3^1}{\partial t} & \frac{\partial L_3^2}{\partial t} & \frac{\partial L_3^3}{\partial t} \end{pmatrix} \begin{pmatrix} \mathbf{f_1} \\ \mathbf{f_2} \\ \mathbf{f_3} \end{pmatrix}. \tag{10}$$

TEE of the curvatures are

$$\begin{aligned} \frac{\partial \kappa_1}{\partial t} &= \frac{z_2 \frac{\partial z_1}{\partial t} - z_1 \frac{\partial z_2}{\partial t}}{z_1^2}, \\ \frac{\partial \tau_1}{\partial t} &= \frac{z_4 \frac{\partial z_3}{\partial t} - z_3 \frac{\partial z_4}{\partial t}}{z_2^2}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} z_1 &= \sqrt{(\eta_{12}\eta_{21} - \eta_{11}\eta_{22})^2 + (\eta_{13}\eta_{21} - \eta_{11}\eta_{23})^2 + (\eta_{13}\eta_{22} - \eta_{12}\eta_{23})^2}, \\ z_2 &= (\eta_{11}^2 + \eta_{12}^2 + \eta_{13}^2)^{(3/2)}, \\ z_3 &= (-\eta_{13}\eta_{22}\eta_{31} + \eta_{12}\eta_{23}\eta_{12}\eta_{21}\eta_{33} + \eta_{11}\eta_{22}\eta_{33}), \\ z_4 &= (\eta_{12}\eta_{21} - \eta_{11}\eta_{22})^2 + (\eta_{13}\eta_{21} - \eta_{11}\eta_{23})^2 + (\eta_{13}\eta_{22} - \eta_{12}\eta_{23})^2. \end{aligned} \tag{12}$$

Now after the curve $\zeta = \zeta(s)$ moves with time t the parametric representations of Smarandache curves are given as follows

$$\begin{pmatrix} S_1(s,t) \\ S_2(s,t) \\ S_3(s,t) \end{pmatrix} = \begin{pmatrix} L_1^1 + L_2^1 & L_1^2 + L_2^2 & L_1^3 + L_2^3 \\ L_2^1 + L_3^1 & L_2^2 + L_3^2 & L_2^3 + L_3^3 \\ L_1^1 + L_2^1 + L_3^1 & L_1^2 + L_2^2 + L_3^2 & L_1^3 + L_2^3 + L_3^3 \end{pmatrix} \begin{pmatrix} \mathbf{f_1} \\ \mathbf{f_2} \\ \mathbf{f_3} \end{pmatrix} \tag{13}$$

4. Visualization of evolving curves

In this section we shall display the evolving curve, the spherical image and the Smarandache curves. Let the curve is given by

$$\zeta(s) = (s\cos(s), s\sin(s), s), \tag{14}$$

then, the curvature and torsion are given by

$$\kappa = \frac{\sqrt{8 + 5s^2 + s^4}}{(2 + s^2)^{3/2}}, \tau = \frac{6 + s^2}{\sqrt{8 + 5s^2 + s^4}}. \tag{15}$$

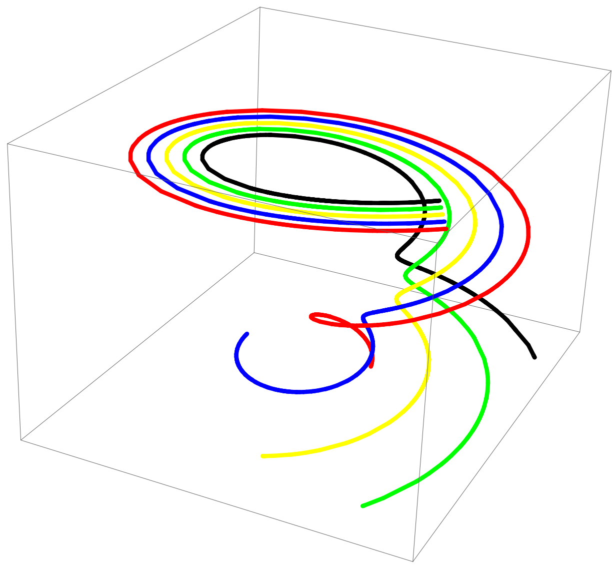
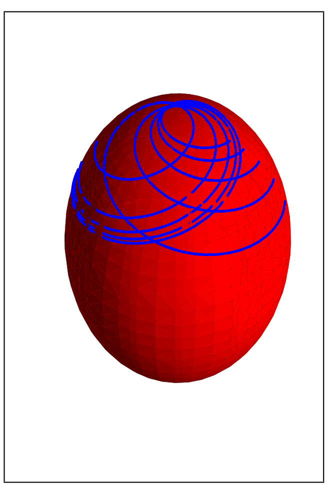
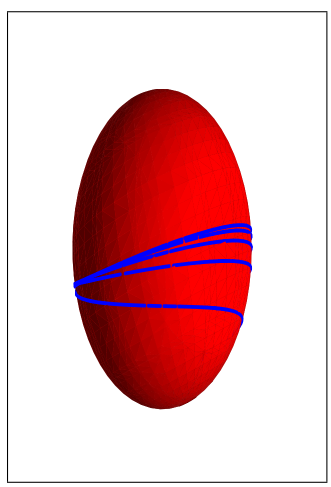


FIGURE 2. Time evolution of the curve in space corresponding to κ_1, τ_1 .

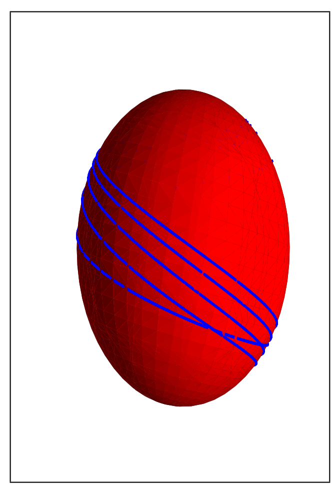
Figure 2 represent the evolution of the curve in space, figure 3 represent the evolution of the spherical image of space curve, figure 4 represent the evolution of Smarandache curves and t runs from $t = 0, \ldots 5$, $\alpha = 0, \beta = \kappa, \gamma = 0$.



(A) Spherical image of the tangent.

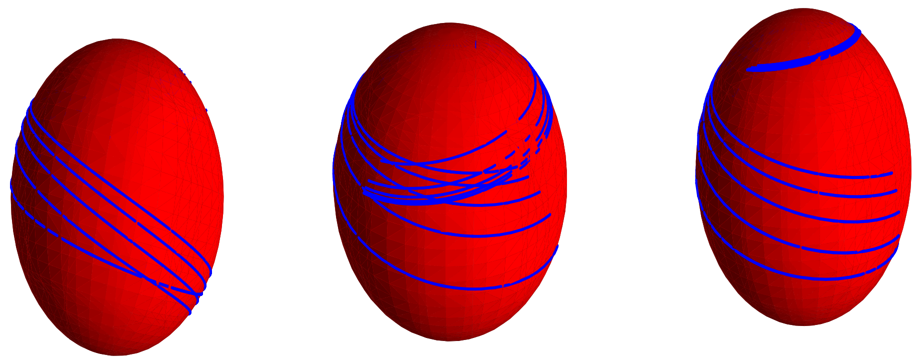


(B) Spherical image of the normal.



(C) Spherical image of the binormal.

FIGURE 3. Time evolution of spherical images



(A) Smarandache curves $f_1 f_2$

(B) Smarandache curves $f_2 f_3$

(C) Smarandache curves $f_1 f_2 f_3$

FIGURE 4. Time evolution of Smarandache curves

5. Conclusion

We presented new geometrical models for a curve evolving in space by a method different those in [7]. In addition to, the evolution equation for spherical images, Smarandache curves are obtained. Finally we displayed geometric visualization of the evolving curve, spherical images and Smarandache curves.

Conflict of Interests

The author declare that there is no conflict of interests.

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