Available online at http://scik.org
J. Math. Comput. Sci. 7 (2017), No. 6, 1100-1114
https://doi.org/10.28919/jmcs/3474
ISSN: 1927-5307

BINOMINAL TRANSFORMS OF K-JACOBSTHAL SEQUENCES

ŞÜKRAN UYGUN*, ARZUM ERDOĞDU

Department of Mathematics, Science and Art Faculty, Gaziantep University, 27310, Gaziantep, Turkey

Abstract. In this paper, we define the binomial, $k-$binomial, rising, and falling transforms for $k-$Jacobsthal sequence. We investigate some properties of these sequences such as recurrence relations, Binet's formula, generating functions and in the sequel of this paper denote Pascal Jacobsthal triangle for all binomial transformation sequences.

Keywords: Jacobsthal numbers; binomial transforms; Binet's formula; generating function; Pascal triangle.

2010 AMS Subject Classification: 11B83, 11B37.

1. Introduction

Special integer sequences such as Fibonacci, Lucas, Jacobsthal, Pell, Horadam are very popular in the last decade. We can see abundant applications in Physics, Engineering, Architecture, Nature and Art. For instance, the ratio of two consecutive elements of Fibonacci sequence is called golden ratio. You can encounter it almost every area of science and art. And specially computers use conditional directives to change the flow of execution of a program. In addition to branch instructions, some microcontrollers use skip instructions which conditionally bypass

*Corresponding author
E-mail address: suygun@gantep.edu.tr
Received August 17, 2017

the next instruction.This brings out being useful for one case out of the four possibilities on 2 bits, 3 cases on 3 bits, 5 cases on 4 bits, 11 cases on 5 bits, 21 cases on 6 bits,..., which are exactly the Jacobsthal numbers [1]. The Jacobsthal numbers j_n are terms of the sequence $\{0,1,1,3,5,11,...\}$, defined by the recurrence relation

$$j_n = j_{n-1} + 2j_{n-2}.$$

for $n \geq 2$, beginning with the values $j_0 = 0,\ j_1 = 1$. Because of the importance of special integer sequences, the researchers generalize them by the different methods. In this paper we introduce k-Jacobsthal sequence,depending only on one positive integer parameter k. k-Jacobsthal sequence, $\{j_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by

$$j_{k,n} = kj_{k,n-1} + 2j_{k,n-2},\ \ j_{k,0} = 0,\ j_{k,1} = 1. \tag{1.1}$$

The Binet formula for the k-Jacobsthal sequence is denoted by $j_{k,n} = \frac{x_1^n - x_2^n}{x_1 - x_2}$, where $x_1 = \frac{k+\sqrt{k^2+8}}{2}$, $x_2 = \frac{k-\sqrt{k^2+8}}{2}$. x_1 and x_2 are the characteristic polynomial equation of recurrence formula (1.1). You can see detailed information about k-Jacobsthal sequence in [2]. The main goal of this paper is to apply different binomial transforms to the k-Jacobsthal sequence and find some relations and properties of these new binomial transform sequences. In the literature Prodinger gave some information about the binomial transformation in [3]. Chen investigated identities from the binomial transform in [4]. Falcon and Plaza investigated the properties of k-Fibonacci sequence [5,6], and binomial transform of k-Fibonacci sequence in [7]. The authors found the properties of the binomial transform of the k-Lucas sequence in [8].

2. Binomial Transform of k-Jacobsthal Sequences

Definition 2.1. The binomial transform of k-Jacobsthal sequence $\{j_{k,n}\}_{n\in\mathbb{N}}$ is indicated as $\{b_{k,n}\}_{n\in\mathbb{N}}$ where $b_{k,n}$ is given by

$$b_{k,n} = \sum_{i=0}^{n} \binom{n}{i} j_{k,i}. \tag{2.1}$$

for any positive integer parameter k.

Lemma 2.1. *Let $n \geq 1$ positive integer, then the binomial transform of k-Jacobsthal sequence* *verifies the following relation*

$$b_{k,n+1}=\sum_{i=0}^{n}\binom{n}{i}(j_{k,i}+j_{k,i+1}).$$

Proof. We use the addition property of binomial numbers as $\binom{n+1}{i}=\binom{n}{i}+\binom{n}{i-1}$ and $\binom{n}{n+1}=0$ for the proof. Then

$$\begin{aligned}
b_{k,n+1} &= \sum_{i=0}^{n+1}\binom{n+1}{i}j_{k,i}=0+\sum_{i=1}^{n+1}\left[\binom{n}{i}+\binom{n}{i-1}\right]j_{k,i}\\
&= \sum_{i=1}^{n}\binom{n}{i}j_{k,i}+\sum_{i=0}^{n}\binom{n}{i}j_{k,i+1}\\
&= \sum_{i=0}^{n}\binom{n}{i}(j_{k,i}+j_{k,i+1}).
\end{aligned}$$

Now we denote the recurrence relation for the binomial transform of k-Jacobsthal sequence.

Theorem 2.1. *The following recurrence relation is verified by the binomial transform of k-Jacobsthal sequence*

$$b_{k,n+1}=(k+2)b_{k,n}+(1-k)b_{k,n-1}. \tag{2.2}$$

Proof. The initial conditions are found by the definition as $b_{k,0}=0$ and $b_{k,1}=1$. By using Lemma 2.1 and (1.1), we obtain

$$\begin{aligned}
b_{k,n+1} &= \sum_{i=0}^{n}\binom{n}{i}(j_{k,i}+j_{k,i+1})=1+\sum_{i=1}^{n}\binom{n}{i}(j_{k,i}+j_{k,i+1})\\
&= 1+\sum_{i=1}^{n}\binom{n}{i}(j_{k,i}+kj_{k,i}+2j_{k,i-1})\\
&= 1+(k+1)\sum_{i=1}^{n}\binom{n}{i}j_{k,i}+2\sum_{i=1}^{n}\binom{n}{i}j_{k,i-1}
\end{aligned}$$

$$b_{k,n+1}=(k+1)b_{k,n}+2\sum_{i=1}^{n}\binom{n}{i}j_{k,i-1}+1. \tag{2.3}$$

If we write the last equality again for n in place of $n+1$,we get

$$\begin{aligned} b_{k,n} &= (k+1)b_{k,n-1}+2\sum_{i=1}^{n-1}\binom{n-1}{i}j_{k,i-1}+1 \\ &= kb_{k,n-1}+\left[\sum_{i=0}^{n-1}\binom{n-1}{i}j_{k,i}\right]+2\sum_{i=1}^{n-1}\binom{n-1}{i}j_{k,i-1}+1 \\ &= kb_{k,n-1}+\left[\sum_{i=1}^{n}\binom{n-1}{i-1}j_{k,i-1}\right]+2\sum_{i=1}^{n-1}\binom{n-1}{i}j_{k,i-1}+1 \end{aligned}$$

If we take into account that $\binom{n-1}{n}=0$, we obtain

$$\begin{aligned} b_{k,n} &= kb_{k,n-1}+\sum_{i=1}^{n}\left[2\binom{n-1}{i}+\binom{n-1}{i-1}\right]j_{k,i-1}+1 \\ &= kb_{k,n-1}+\sum_{i=1}^{n}\left[2\binom{n-1}{i}+\binom{n-1}{i-1}+2\binom{n-1}{i-1}-2\binom{n-1}{i-1}\right]j_{k,i-1}+1 \\ &= kb_{k,n-1}+\sum_{i=1}^{n}\left[-\binom{n-1}{i-1}+2\binom{n}{i}\right]j_{k,i-1}+1 \\ &= kb_{k,n-1}+2\sum_{i=1}^{n}\binom{n}{i}j_{k,i-1}-\sum_{i=0}^{n-1}\binom{n-1}{i}j_{k,i}+1. \end{aligned}$$

$$b_{k,n}=(k-1)b_{k,n-1}+2\sum_{i=1}^{n}\binom{n}{i}j_{k,i-1}+1. \tag{2.4}$$

By substituting the above equality (2.3) into (2.4), we get

$$\begin{aligned} b_{k,n} &= (k-1)b_{k,n-1}+2\sum_{i=1}^{n}\binom{n}{i}j_{k,i-1}+1 \\ &= (k-1)b_{k,n-1}+b_{k,n+1}-(k+1)b_{k,n}-1+1 \end{aligned}$$

The proof is completed by some simple calculations.

Theorem 2.2. **(Binet Formula)** *Any terms of the binomial transform of k-Jacobsthal sequence can be calculated by means of Binet formula. It is demonstrated by*

$$b_{k,n}=\frac{b_1^n-b_2^n}{b_1-b_2}$$

where $b_1=\frac{k+2+\sqrt{k^2+8}}{2}$, $b_2=\frac{k+2-\sqrt{k^2+8}}{2}$.

Proof. The characteristic polynomial equation of recurrence formula (2.2) is $x^2-(k+2)x+(k-1)=0$, whose solutions are b_1 and b_2. For obtaining Binet formula let us write $b_{k,n}=$

$c_1 b_1^n + c_2 b_2^n$. We know that $b_{k,0} = 0$ and $b_{k,1} = 1$. For $n = 0$ and $n = 1$, it is deduced that $c_1 = -c_2 = \frac{1}{b_1 - b_2}$.

Theorem 2.3. (Generating function) *The generating function of the binomial transform of the k-Jacobsthal sequence is a power series centered at the origin whose coefficients are the binomial transform of the k-Jacobsthal sequence. Let us demonstrate the generating function as* $b_k(x)$ *. So,* $b_k(x) = b_{k,0} + b_{k,1}x + b_{k,2}x^2 + \ldots$ *And then, if we multiply the equality with* -$(k+2)x$ *and* $(1-k)x^2$*, we get*

$$\begin{aligned} -(k+2)xb_k(x) &= -(k+2)xb_{k,0} - (k+2)x^2 b_{k,1} + \ldots \\ (1-k)x^2 b_k(x) &= (k-1)x^2 b_{k,0} + (k-1)x^3 b_{k,1} + \ldots \end{aligned}$$

From these three equalities and the recurrence relation (2.2), we obtain

$$\begin{aligned} [1-(k+2)x+(1-k)x^2]b_k(x) &= b_{k,0} + x(b_{k,1} - (k+2)b_{k,0}) \\ &\quad + x^2(b_{k,2} - (k+2)b_{k,1} + (k-1)b_{k,0}) + \ldots \\ &= 0 + x(1-0) + x^2(0) \end{aligned}$$

The generating function is denoted by

$$b_k(x) = \frac{x}{1-(k+2)x+(1-k)x^2}. \tag{2.5}$$

Triangle of the binomial transform of the k-Jacobsthal sequence

In this part we introduce a new triangle of numbers for each k by using the following rules:

- The elements of the left diagonal of the triangle consist of the elements of the k- Jacobsthal numbers.
- Any number off the left diagonal is the sum of the number to its left and the number diagonally above it to the left.
- On the right diagonal is the binomial transform of the k-Jacobsthal sequence.

For example following triangle is for 3-Jacobsthal sequence and its binomial transform.

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 3 & & 4 & & 5 & & \\ & 11 & & 14 & & 18 & & 23 & \\ 39 & & 50 & & 64 & & 82 & & 105 \end{array}$$

3. The k-Binomial Transforms Of The k-Jacobsthal Sequence

Definition 3.1. The k-binomial transform of the $k-$Jacobsthal sequence $\left\{w_{k,n}\right\}_{n\in N}$ is given by the following formula

$$w_{k,n} = \sum_{i=0}^{n} \binom{n}{i} k^n j_{k,i} \tag{3.1}$$

You can see easily $\left\{w_{1,n}\right\}_{n\in N} = \left\{b_{k,n}\right\}_{n\in N}$, and $w_{k,n} = k^n b_{k,n}$.

Lemma 3.1. The $k-$*binomial transform of the* $k-$*Jac*obsthal sequence verifies the relation

$$w_{k,n+1} = \sum_{i=0}^{n} \binom{n}{i} k^{n+1} (j_{k,i} + j_{k,i+1}). \tag{3.2}$$

Proof.

$$\begin{aligned} w_{k,n+1} &= \sum_{i=0}^{n+1} \binom{n+1}{i} k^{n+1} j_{k,i} = 0 + \sum_{i=1}^{n+1} \left[\binom{n}{i} + \binom{n}{i-1} \right] k^{n+1} j_{k,i} \\ &= \sum_{i=1}^{n} k^{n+1} \binom{n}{i} j_{k,i} + \sum_{i=0}^{n} k^{n+1} \binom{n}{i} j_{k,i+1} \\ &= \sum_{i=0}^{n} \binom{n}{i} k^{n+1} (j_{k,i} + j_{k,i+1}). \end{aligned}$$

Theorem 3.1. The following recurrence relation is verified by the $k-$binomial transform of the $k-$Jacobsthal sequence

$$w_{k,n+1} = k(k+2) w_{k,n} - k^2 (k-1) w_{k,n-1}. \tag{3.3}$$

Proof. The initial conditions are found by (3.1) as $w_{k,0}=0$ and $w_{k,1}=k$. By using Lemma (3.1), we obtain

$$\begin{aligned} w_{k,n+1} &= \sum_{i=0}^{n}\binom{n}{i}k^{n+1}(j_{k,i}+j_{k,i+1}) = k^{n+1}+\sum_{i=1}^{n}\binom{n}{i}k^{n+1}(j_{k,i}+j_{k,i+1}) \\ &= k^{n+1}+\sum_{i=1}^{n}\binom{n}{i}k^{n+1}(j_{k,i}+kj_{k,i}+2j_{k,i-1}) \\ &= k^{n+1}+(k+1)\sum_{i=1}^{n}\binom{n}{i}k^{n+1}j_{k,i}+2\sum_{i=1}^{n}\binom{n}{i}k^{n+1}j_{k,i-1}. \end{aligned}$$

$$w_{k,n+1}=k^{n+1}+k(k+1)w_{k,n}+2\sum_{i=1}^{n}\binom{n}{i}k^{n+1}j_{k,i-1}. \tag{3.4}$$

If we write th equality (3.4) again for n in place of $n+1$

$$\begin{aligned} w_{k,n} &= k(k+1)w_{k,n-1}+2\sum_{i=1}^{n-1}\binom{n-1}{i}k^{n}j_{k,i-1}+k^{n} \\ &= k^{2}w_{k,n-1}+k\left[\sum_{i=0}^{n-1}\binom{n-1}{i}k^{n-1}j_{k,i}\right]+2\sum_{i=1}^{n-1}\binom{n-1}{i}k^{n}j_{k,i-1}+k^{n} \\ &= k^{2}w_{k,n-1}+\left[\sum_{i=1}^{n}\binom{n-1}{i-1}k^{n}j_{k,i-1}\right]+2\sum_{i=1}^{n-1}\binom{n-1}{i}k^{n}j_{k,i-1}+k^{n} \end{aligned}$$

$$\begin{aligned} w_{k,n} &= k^{2}w_{k,n-1}+\sum_{i=1}^{n}\left[2\binom{n-1}{i}+\binom{n-1}{i-1}\right]k^{n}j_{k,i-1}+k^{n} \\ &= k^{2}w_{k,n-1}+\sum_{i=1}^{n}\left[2\binom{n-1}{i}+\binom{n-1}{i-1}+2\binom{n-1}{i-1}-2\binom{n-1}{i-1}\right]k^{n}j_{k,i-1}+k^{n} \\ &= k^{2}w_{k,n-1}+\sum_{i=1}^{n}\left[-\binom{n-1}{i-1}+2\binom{n}{i}\right]k^{n}j_{k,i-1}+k^{n} \\ &= k^{2}w_{k,n-1}+2\sum_{i=1}^{n}\binom{n}{i}k^{n}j_{k,i-1}-\sum_{i=0}^{n-1}\binom{n-1}{i}k^{n}j_{k,i}+k^{n} \end{aligned}$$

$$w_{k,n}=k(k-1)w_{k,n-1}+2\sum_{i=1}^{n}\binom{n}{i}k^{n}j_{k,i-1}+k^{n} \tag{3.5}$$

By substituting the above equality (3.4) into (3.5), we get

$$\begin{aligned} w_{k,n} &= k(k-1)w_{k,n-1} + 2\sum_{i=1}^{n}\binom{n}{i}k^n j_{k,i-1} + k^n \\ &= k(k-1)w_{k,n-1} + (w_{k,n+1} - k(k+1)w_{k,n} - k^{n+1})/k + k^n \\ w_{k,n} &= k(k-1)w_{k,n-1} + w_{k,n+1}/k - (k+1)w_{k,n} \end{aligned}$$

The proof is found by doing some simple calculations as

$$w_{k,n+1} = k(k+2)w_{k,n} - k^2(k-1)w_{k,n-1}.$$

Theorem 3.2. **(Binet Formula)** *The characteristic polynomial equation of recurrence formula (3.3) is* $x^2 - k(k+2)x + k^2(k-1) = 0$*, whose solutions are* w_1 *and* w_2*. Any terms of the* $k-$*binomial transform of* k*-Jacobsthal sequence can be calculated by means of Binet formula. It is demonstrated by*

$$w_{k,n} = k\frac{w_1^n - w_2^n}{w_1 - w_2}. \tag{3.6}$$

where $w_1 = k\frac{k+2+\sqrt{k^2+8}}{2}$, $w_2 = k\frac{k+2-\sqrt{k^2+8}}{2}$.

Theorem 3.3. (**Generating function**) *Let us demonstrate the generating function as* $w_k(x) = w_{k,0} + w_{k,1}x + w_{k,2}x^2 + \ldots$ *The generating function for the* $k-$*binomial transform of* k*-Jacobsthal sequence is obtained as*

$$w_k(x) = \frac{kx}{1 - k(k+2)x + k^2(1-k)x^2} \tag{3.7}$$

. ***Proof.*** *If we multiply the equality* $w_k(x)$ *with* $k(k+2)x$ *and* $k^2(1-k)x^2$*, we get*

$$\begin{aligned} -k(k+2)xw_k(x) &= -k(k+2)xw_{k,0} - k(k+2)x^2w_{k,1} + \ldots \\ k^2(1-k)x^2w_k(x) &= k^2(k-1)x^2w_{k,0} + k^2(k-1)x^3w_{k,1} + \ldots \end{aligned}$$

From these equalities and the recurrence relation (3.3), we have

$$w_k(x) = \frac{kx}{1 - k(k+2)x + k^2(1-k)x^2}$$

Triangle of the k− binomial transform of the k-Jacobsthal sequence

In this part we introduce a new triangle of numbers for each k by using the following rules:

- The left diagonal of the triangle consists of the elements of the k- Jacobsthal numbers.
- Any number of the left diagonal is k times the sum of the number to its left and the number diagonally above it to the left.
- On the right diagonal is the $k-$binomial transform of the k-Jacobsthal sequence.

For example following triangle is for 3-Jacobsthal sequence and its 3- binomial transform.

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 3 & & & \\ & & 3 & & 12 & & 45 & & \\ & 11 & & 42 & & 162 & & 621 & \\ 39 & & 150 & & 576 & & 2214 & & 8505 \end{array}$$

4. The Rising k-Binomial Transform of the k-Jacobsthal Sequence

Definition 4.1. The rising k-binomial transform of the $k-$Jacobsthal sequence $\left\{r_{k,n}\right\}_{n\in N}$ is efined as the following formula

$$r_{k,n}=\sum_{i=0}^{n}\binom{n}{i}k^{i}j_{k,i}. \tag{4.1}$$

Theorem 4.1. (Binet Formula) *The Binet formula for the rising k-binomial transform of the $k-$Jacobsthal sequence is*

$$r_{k,n}=\frac{\left(x_1^2-1\right)^n-\left(x_2^2-1\right)^n}{x_1-x_2} \tag{4.2}$$

where $x_1=\frac{k+\sqrt{k^2+8}}{2}$, $x_2=\frac{k-\sqrt{k^2+8}}{2}$.

Proof.

$$\begin{aligned} r_{k,n} &= \sum_{i=0}^{n}\binom{n}{i}k^i j_{k,i}=\sum_{i=0}^{n}\binom{n}{i}k^i\frac{x_1^i-x_2^i}{x_1-x_2} \\ &= \frac{1}{x_1-x_2}\left(\sum_{i=0}^{n}\binom{n}{i}(kx_1)^i-\sum_{i=0}^{n}\binom{n}{i}(kx_2)^i\right) \\ &= \frac{1}{x_1-x_2}((kx_1+1)^n-(kx_2+1)^n=\frac{(x_1^2-1)^n-(x_2^2-1)^n}{x_1-x_2} \end{aligned}$$

since $kx_1+2=x_1^2$ and $kx_2+2=x_2^2$.

Theorem 4.2. *For $n\geqslant 1$, the rising k-binomial transform of the $k-$Jacobsthal sequence is a recurrence sequence such that*

$$r_{k,n+1}=(k^2+2)r_{k,n}-(1-k^2)r_{k,n-1}. \tag{4.3}$$

with initial conditions $r_{k,0}=0$ and $r_{k,1}=k$.

Proof. By using Binet Formulas

$$\begin{aligned} (k^2+2)r_{k,n}-(1-k^2)r_{k,n-1} &= (k^2+2)\frac{\left(x_1^2-1\right)^n-\left(x_2^2-1\right)^n}{x_1-x_2} \\ &\quad -(1-k^2)\frac{\left(x_1^2-1\right)^{n-1}-\left(x_2^2-1\right)^{n-1}}{x_1-x_2} \\ &= \frac{1}{x_1-x_2}\left\{\begin{array}{c} (kx_1+1)^{n-1}\left[(k^2+2)(kx_1+1)-(1-k^2)\right] \\ -(kx_2+1)^{n-1}\left[(k^2+2)(kx_2+1)-(1-k^2)\right] \end{array}\right\} \\ &= \frac{1}{x_1-x_2}\left\{\begin{array}{c} (kx_1+1)^{n-1}\left[k^2(kx_1+2)+2kx_1+1\right] \\ -(kx_2+1)^{n-1}\left[k^2(kx_2+2)+2kx_2+1\right] \end{array}\right\} \\ &= \frac{1}{x_1-x_2}\left\{\begin{array}{c} (kx_1+1)^{n-1}\left[(kx_1+1)^2\right] \\ -(kx_2+1)^{n-1}\left[(kx_2+1)^2\right] \end{array}\right\} \\ &= r_{k,n+1}. \end{aligned}$$

Theorem 4.3. **(Generating function)** *The generating function of rising k-binomial transform of the $k-$Jacobsthal sequence is denoted by $R_k(x)$. So,*

$$R_k(x)=r_{k,0}+r_{k,1}x+r_{k,2}x^2+\dots.$$

and then the generating function is obtained as

$$R_k(x) = \frac{kx}{1-(k^2+2)x+(1-k^2)x^2}. \tag{4.4}$$

Proof. By following same procedure with Theorem (3.3), we have

$$\begin{aligned} -(k^2+2)xR_k(x) &= -x(k^2+2)r_{k,0} - x^2(k^2+2)r_{k,1} + ... \\ (1-k^2)x^2R_k(x) &= (1-k^2)x^2r_{k,0} + (1-k^2)x^3r_{k,1} + ... \\ \left[1-(k^2+2)x+(1-k^2)x^2\right]R_k(x) &= r_{k,0} - x(k^2+2)r_{k,0} + r_{k,1}x = 0 + x(k-0) \end{aligned}$$

By the above computations, we obtain the generating function as $R_k(x) = \frac{kx}{1-(k^2+2)x+(1-k^2)x^2}$.

Triangle of the rising k$-$ binomial transform of the k-Jacobsthal sequence

In this part we introduce a new triangle of numbers for each k by using the following rules:

- The left diagonal of the triangle consists of the elements of the k- Jacobsthal numbers.
- Any number of the left diagonal is the sum of the number diagonally above it to the left and k- times the number to its left.
- On the right diagonal is the rising $k-$binomial transform of the k-Jacobsthal sequence.

For example following triangle is for 3-Jacobsthal sequence and its rising 3- binomial transform.

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 3 & & & \\ & & 3 & & 10 & & 33 & & \\ & 11 & & 36 & & 118 & & 387 & \\ 39 & & 128 & & 420 & & 1378 & & 4521 \end{array}$$

5. The Falling k-Binomial Transform of the k-Jacobsthal Sequence

Definition 5.1. Let k is any positive integer. The falling k-binomial transform of the $k-$Jacobsthal sequence $\left\{f_{k,n}\right\}_{n\in N}$ is given by

$$f_{k,n} = \sum_{i=0}^{n} \binom{n}{i} k^{n-i} j_{k,i}. \tag{5.1}$$

Lemma 5.1. *The falling $k-$binomial transform of the $k-$Jacobsthal sequence verifies the relation*

$$f_{k,n+1} = \sum_{i=0}^{n} \binom{n}{i} k^{n-i}(j_{k,i} + j_{k,i+1}). \tag{5.2}$$

with initial conditions $f_{k,0}$*= 0 and* $f_{k,1}$*= 1.*

Proof.

$$\begin{aligned} f_{k,n+1} &= \sum_{i=0}^{n+1} \binom{n+1}{i} k^{n+1-i} j_{k,i} = 0 + \sum_{i=1}^{n+1} \left[\binom{n}{i} + \binom{n}{i-1} \right] k^{n+1-i} j_{k,i} \\ &= \sum_{i=1}^{n} k^{n+1-i} \binom{n}{i} j_{k,i} + \sum_{i=0}^{n} k^{n-i} \binom{n}{i} j_{k,i+1} \\ &= \sum_{i=0}^{n} \binom{n}{i} k^{n-i}(k j_{k,i} + j_{k,i+1}). \end{aligned}$$

Theorem 5.1. *The following recurrence relation is verified by the falling $k-$binomial transform of the $k-$Jacobsthal sequence*

$$f_{k,n+1} = 3k f_{k,n} - 2(k^2-1) f_{k,n-1}. \tag{5.3}$$

Proof. The initial conditions are found by the definition (5.1) as $f_{k,0} = 0$ and $f_{k,1} = 1$. By using Lemma 5.1 and (1.1), we obtain

$$\begin{aligned} f_{k,n+1} &= \sum_{i=0}^{n} \binom{n}{i} k^{n-i}(k j_{k,i} + j_{k,i+1}) = k^n + \sum_{i=1}^{n} \binom{n}{i} k^{n-i}(k j_{k,i} + j_{k,i+1}) \\ &= k^n + \sum_{i=1}^{n} \binom{n}{i} k^{n-i}(k j_{k,i} + k j_{k,i} + 2 j_{k,i-1}) \\ &= k^n + 2k \sum_{i=1}^{n} \binom{n}{i} k^{n-i} j_{k,i} + 2 \sum_{i=1}^{n} \binom{n}{i} k^{n-i} j_{k,i-1} \end{aligned}$$

$$f_{k,n+1} = k^n + 2k f_{k,n} + 2 \sum_{i=1}^{n} \binom{n}{i} k^{n-i} j_{k,i-1}. \tag{5.4}$$

If we write this equality again for n in place of $n+1$

$$\begin{aligned}
f_{k,n} &= 2kf_{k,n-1}+2\sum_{i=1}^{n-1}\binom{n-1}{i}k^{n-1-i}j_{k,i-1}+k^{n-1} \\
&= kf_{k,n-1}+\left[\sum_{i=0}^{n-1}\binom{n-1}{i}k^{n-i}j_{k,i}\right] \\
&\quad +2\sum_{i=1}^{n-1}\binom{n-1}{i}k^{n-1-i}j_{k,i-1}+k^{n-1} \\
&= kf_{k,n-1}+\left[\sum_{i=1}^{n}\binom{n-1}{i-1}k^{n+1-i}j_{k,i-1}\right] \\
&\quad +2\sum_{i=1}^{n-1}\binom{n-1}{i}k^{n-1-i}j_{k,i-1}+k^{n-1}
\end{aligned}$$

If we take into account that $\binom{n-1}{n}=0$,

$$\begin{aligned}
f_{k,n} &= kf_{k,n-1}+\sum_{i=1}^{n}\left[2\binom{n-1}{i}+k^2\binom{n-1}{i-1}\right]k^{n-1-i}j_{k,i-1}+k^{n-1} \\
&= kf_{k,n-1}+\sum_{i=1}^{n}\left[2\binom{n-1}{i}+k^2\binom{n-1}{i-1}+2\binom{n-1}{i-1}-2\binom{n-1}{i-1}\right]k^{n-1-i}j_{k,i-1}+k^{n-1} \\
&= kf_{k,n-1}+\sum_{i=1}^{n}\left[(k^2-2)\binom{n-1}{i-1}+2\binom{n}{i}\right]k^{n-1-i}j_{k,i-1}+k^{n-1} \\
&= kf_{k,n-1}+2\sum_{i=1}^{n}\binom{n}{i}k^{n-1-i}j_{k,i-1}+(k^2-2)\sum_{i=0}^{n-1}\binom{n-1}{i}k^{n-2-i}j_{k,i}+k^{n-1}
\end{aligned}$$

$$f_{k,n}=kf_{k,n-1}+2\sum_{i=1}^{n}\binom{n}{i}k^{n-1-i}j_{k,i-1}+(k^2-2)f_{k,n-1}/k+k^{n-1} \tag{5.5}$$

By substituting the above equality (5.4 into (5.5), we get

$$f_{k,n}=kf_{k,n-1}+(f_{k,n+1}-2kf_{k,n}-k^n)/k+(k-2/k)f_{k,n-1}+k^{n-1}.$$

The proof is completed by some simple calculations.

Theorem 5.2. **(Binet Formula)** *The characteristic polynomial equation of recurrence formula (5.3) is $x^2-3kx+2(k^2-1)=0$, whose solutions are f_1 and f_2. Any terms of the falling $k-$binomial transform of k-Jacobsthal sequence can be calculated by means of Binet formula. It is demonstrated by*

$$f_{k,n} = \frac{f_1^n - f_2^n}{f_1 - f_2}. \tag{5.6}$$

where $f_1 = \frac{3k+\sqrt{k^2+8}}{2}$, $f_2 = \frac{3k-\sqrt{k^2+8}}{2}$.

Theorem 5.3. **(Generating function)** *Let us demonstrate the generating function as* $f_k(x) = f_{k,0} + f_{k,1}x + f_{k,2}x^2 + \ldots$

$$f_k(x) = \frac{x}{1-3kx+2(k^2-1)x^2}. \tag{5.7}$$

Proof. If we multiply the equality $f_k(x)$ with $-3kx$ and $2(k^2-1)x^2$, we get

$$\begin{aligned} -3kxf_k(x) &= -3kxf_{k,0} - 3kx^2 f_{k,1} + \ldots \\ 2(k^2-1)x^2 f_k(x) &= 2(k^2-1)x^2 f_{k,0} + 2(k^2-1)x^3 f_{k,1} + \ldots \end{aligned}$$

From these equalities and the recurrence relation (5.3), we obtain

$$f_k(x) = \frac{x}{1-3kx+2(k^2-1)x^2}.$$

Triangle of the falling k− binomial transform of the k-Jacobsthal sequence

In this part we introduce a new triangle of numbers for each k by using the following rules:

- The left diagonal of the triangle consists of the elements of the k- Jacobsthal numbers.
- Any number of the left diagonal is the sum of the number to its left and k times the number diagonally above it to the left.
- On the right diagonal is the falling $k-$binomial transform of the k-Jacobsthal sequence.

For example the following triangle is for 3-Jacobsthal sequence and its falling 3- binomial transform.

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 3 & & 6 & & 9 & & \\ & 11 & & 20 & & 38 & & 65 & \\ 39 & & 72 & & 132 & & 246 & & 441 \end{array}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] A . F. Horadam, Jacobsthal Representation Numbers, Fibonacci Q. 34 (1) (1996), 40-54.

[2] S. Uygun, H. Eldoğan, The k-Jacobsthal and k-Jacobsthal Lucas sequences, General Math. Notes, 36 (1) (2016), 34-47.

[3] H. Prodinger, Some information about the binomial transform, Fibonacci Q. 32 (5) (1994), 412-415.

[4] K. W. Chen, Identities from the binomial transform, J. Num. Theory, 124 (2007), 142-150.

[5] S. Falcon, A. Plaza, On the Fibonacci k-numbers, Chaos Solitons Fractals, 32 (5) (2007), 1615-1624.

[6] S. Falcon, A. Plaza, The k-Fibonacci Sequence and Pascal 2 triangle, Chaos Solitons Fractals, 33 (1) (2007), 38-49.

[7] S. Falcon, A. Plaza, Binomial Transforms of the k-FibonacciSequence, International J. Nonlinear Sci. Numer. Simul. 10 (11-12) (2009), 1527-1538.

[8] P. Bhadouria, D. Jhala, B. Singh, Binomial Transforms of the k-Lucas Sequence, J. Math. Computer Sci. 8 (2014), 81-92.