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A NEW GENERALIZATION OF CONVOLVED $(p,q)-$FIBONACCI AND $(p,q)-$LUCAS POLYNOMIALS

CAN KIZILATEŞ[1,*], NAIM TUGLU[2]

[1]Bülent Ecevit University, Faculty of Art and Science, Department of Mathematics, Zonguldak-Turkey

[2]Gazi University, Faculty of Science, Department of Mathematics, 06500 Teknikokullar Ankara-Turkey

Abstract. In the present paper, we define the extended convolved $(p,q)-$Fibonacci and $(p,q)-$Lucas polynomials. Then we give some formulas of mixed-multiple sums for $(p,q)-$Fibonacci and $(p,q)-$Lucas polynomials.

Keywords: $(p,q)-$Fibonacci polynomials; $(p,q)-$Lucas polynomials; the extended convolved $(p,q)-$Fibonacci and $(p,q)-$Lucas polynomials.

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1. Introduction

The $(p,q)-$Fibonacci and $(p,q)-$Lucas polynomials are recursive sequences that generalize several polynomial and number sequences defined by recurrence relations of order two, such as Fibonacci numbers, Lucas numbers, Pell numbers, Jacobsthal numbers, Fibonacci polynomials, Lucas polynomials, Chebyshev polynomials, Morgan–Voyce polynomials, Pell polynomials, Jacobsthal polynomials, among others; see Table 1. These numbers and polynomials play a fundemental role in mathematics and have numerous important applications in combinatorics,

*Corresponding author
E-mail address: cankizilates@gmail.com
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number theory, numerical analysis, etc. Therefore, they have been studied extensively, and various generalizations of them have been introduced.

Let $p(x)$ and $q(x)$ be polynomials with real coefficients. The $(p,q)-$Fibonacci polynomials $U_n(x)$ are defined by

$$U_n(x) = p(x)U_{n-1}(x) + q(x)U_{n-2}(x),$$

with the initial values $U_0(x) = 0$ and $U_1(x) = 1$ and the $(p,q)-$Lucas polynomials $V_n(x)$ are defined by

$$V_n(x) = p(x)V_{n-1}(x) + q(x)V_{n-2}(x),$$

with the initial values $V_0(x) = 2$ and $V_1(x) = p(x)$.

The generating functions of the sequences $(U_n(x))$ and $(V_n(x))$ are

$$\sum_{n=0}^{\infty} U_{n+1}(x)t^n = \frac{1}{1-p(x)t-q(x)t^2}, \quad \sum_{n=0}^{\infty} U_n(x)t^n = \frac{t}{1-p(x)t-q(x)t^2}$$
$$\sum_{n=0}^{\infty} V_{n+1}(x)t^n = \frac{p(x)+2q(x)t}{1-p(x)t-q(x)t^2}, \quad \sum_{n=0}^{\infty} V_n(x)t^n = \frac{2-p(x)t}{1-p(x)t-q(x)t^2}$$

respectively.

For fundamental properties of these sequences, we refer to Lee and Asci [2], Wang [3], Horadam [19], Falcon [4], Falcon and Plaza [5], Nallı and Haukkanen [6].

$p(x)$	$q(x)$	$U_n(x)$	$V_n(x)$
x	1	Fibonacci polynomials	Lucas polynomials
$2x$	1	Pell polynomials	Pell Lucas polynomials
1	$2x$	Jacobsthal polynomials	Jacobsthal Lucas polynomials
$3x$	-2	Fermat polynomials	Fermat Lucas polynomials
$2x$	-1	Chebyshev polynomials of the second kind	Chebyshev polynomials of the first kind
x	$-\alpha$	Dickson polynomials of the second kind	Dickson polynomials of the first kind
$x+1$	x	Delannoy polynomials	Corona polynomials

Table 1 . Special cases of the $(p,q)-$Fibonacci and $(p,q)-$Lucas polynomials

The convolved Fibonacci numbers $F_j^{(r)}$ are defined by

$$(1-x-x^2)^{-r} = \sum_{j=0}^{\infty} F_{j+1}^{(r)} x^j, \quad r \in \mathbb{Z}^+.$$

If $r=1$, we have classical Fibonacci numbers. Note that

$$F_{m+1}^{(r)}=\sum_{j_1+j_2+\cdots+j_r=m} F_{j_1+1}\ldots F_{j_r+1}.$$

These numbers have been studied in several papers [9, 10, 11, 12, 13, 14, 15, 16]. In [10, 11], Ramirez studied the convolved $k-$Fibonacci numbers and convolved $h(x)-$Fibonacci polynomials. Moreover, in [18], Sahin and Ramirez defined the convolved generalized Lucas polynomials and gave some determinantal and permanental representations of these polynomials. Recently, in [17], Ye and Zhang, studied a common generalization of convolved generalized Fibonacci and Lucas polynomials and gave some properties of these new sequences.

In this paper, we define extented convolved $(p,q)-$Fibonacci and $(p,q)-$Lucas polynomials. Then, we give some recurrence relations for these polynomials. Finally, as applications we obtain some formulas of mixed–multiple sums for $(p,q)-$Fibonacci and $(p,q)-$Lucas polynomials.

2. The Extended Convolved (p,q)-Fibonacci and (p,q)-Lucas Polynomials

Definition 2.1. Let r and m be positive integers with $r\geq m$. Then the extended convolved $(p,q)-$Fibonacci and $(p,q)-$Lucas polynomials $K_{p,q,n}^{(r,m)}(x)$ are defined by

$$\sum_{n=0}^{\infty} K_{p,q,n}^{(r,m)}(x)t^n=\frac{(p(x)+2q(x)t)^m}{(1-p(x)t-q(x)t^2)^r}. \tag{2.1}$$

We can easily see that

$$K_{p,q,n}^{(r,0)}(x)=U_{n+1}^{(r)}(x), K_{p,q,n}^{(r,r)}(x)=V_{n+1}^{(r)}(x),$$

$$K_{p,q,n}^{(1,0)}(x)=U_{n+1}(x),\ K_{p,q,n}^{(1,1)}(x)=V_{n+1}(x),$$

$$K_{1,1,n}^{(1,0)}(x)=F_{n+1},\ K_{1,1,n}^{(1,1)}(x)=L_{n+1}.$$

Moreover, if we take $p(x)=h(x)$ and $q(x)=1$ in (2.1), we get

$$K_{h(x),1,n}^{(r,m)}(x)=T_{h,n}^{(r,m)}(x)$$

where $T_{h,n}^{(r,m)}(x)$ are extended convolved $h(x)-$Fibonacci-Lucas polynomials in [17].

Theorem 2.1. *For the extended convolved $(p,q)-$Fibonacci and $(p,q)-$Lucas polynomials, we have*

$$K_{p,q,n}^{(r,m)}(x) = p(x)K_{p,q,n-1}^{(r,m)}(x) + q(x)K_{p,q,n-2}^{(r,m)}(x) + K_{p,q,n}^{(r-1,m)}(x), \tag{2.2}$$

$$K_{p,q,n}^{(r,m+1)}(x) = p(x)K_{p,q,n}^{(r,m)}(x) + 2q(x)K_{p,q,n-1}^{(r,m)}(x), \tag{2.3}$$

$$K_{p,q,n}^{(r,m)}(x) = \frac{n+1}{r-1}K_{p,q,n+1}^{(r-1,m-1)}(x) - \frac{2(m-1)q(x)}{r-1}K_{p,q,n}^{(r-1,m-2)}(x). \tag{2.4}$$

Proof. From the identity

$$\frac{(p(x)+2q(x)t)^m}{(1-p(x)t-q(x)t^2)^r} = \frac{(p(x)+2q(x)t)^m\,(p(x)t+q(x)t^2)}{(1-p(x)t-q(x)t^2)^r} + \frac{(p(x)+2q(x)t)^m}{(1-p(x)t-q(x)t^2)^{r-1}},$$

(2.2) can be obtained. From the identity

$$\frac{(p(x)+2q(x)t)^{m+1}}{(1-p(x)t-q(x)t^2)^r} = (p(x)+2q(x)t)\frac{(p(x)+2q(x)t)^m}{(1-p(x)t-q(x)t^2)^r},$$

(2.3) can be derived. From the identity

$$\frac{d}{dt}\frac{(p(x)+2q(x)t)^m}{(1-p(x)t-q(x)t^2)^r} = r\frac{(p(x)+2q(x)t)^{m+1}}{(1-p(x)t-q(x)t^2)^{r+1}} + 2q(x)m\frac{(p(x)+2q(x)t)^{m-1}}{(1-p(x)t-q(x)t^2)^r},$$

(2.4) can be obtained. Thus the proof is completed.

Theorem 2.2. *The extended convolved $(p,q)-$Fibonacci and $(p,q)-$Lucas polynomials satisfy the following identity:*

$$K_{p,q,n}^{(r+1,m)}(x) = \frac{1}{r(4q(x)+p^2(x))}\left\{p(x)(n+1)K_{p,q,n+1}^{(r,m)}(x) + 2q(x)(2r-m+n)K_{p,q,n}^{(r,m)}(x)\right\} \tag{2.5}$$

Proof. Since

$$\begin{aligned}\frac{d}{dt}\frac{(p(x)+2q(x)t)^r}{(1-p(x)t-q(x)t^2)^r} &= \frac{d}{dt}\left(\frac{(p(x)+2q(x)t)^m}{(1-p(x)t-q(x)t^2)^r}(p(x)+2q(x)t)^{r-m}\right)\\ &= (p(x)+2q(x)t)^{r-m}\frac{d}{dt}\frac{(p(x)+2q(x)t)^m}{(1-p(x)t-q(x)t^2)^r}\\ &\quad + \frac{2q(x)(r-m)\,(p(x)+2q(x)t)^{r-1}}{(1-p(x)t-q(x)t^2)^r},\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\frac{(p(x)+2q(x)t)^r}{(1-p(x)t-q(x)t^2)^r} &= r\frac{(p(x)+2q(x)t)^{r-1}}{(1-p(x)t-q(x)t^2)^{r-1}}\frac{d}{dt}\left\{\frac{p(x)+2q(x)t}{1-p(x)t-q(x)t^2}\right\}\\ &= -2q(x)r\frac{(p(x)+2q(x)t)^{r-1}}{(1-p(x)t-q(x)t^2)^r}\\ &\quad + r(4q(x)+p^2(x))\frac{(p(x)+2q(x)t)^{r-1}}{(1-p(x)t-q(x)t^2)^{r+1}},\end{aligned}$$

thus we have

$$\begin{aligned}&\frac{2q(x)(r-m)(p(x)+2q(x)t)^{r-1}}{(1-p(x)t-q(x)t^2)^r}+(p(x)+2q(x)t)^{r-m}\frac{d}{dt}\frac{(p(x)+2q(x)t)^m}{(1-p(x)t-q(x)t^2)^r}\\ &\quad = -2q(x)r\frac{(p(x)+2q(x)t)^{r-1}}{(1-p(x)t-q(x)t^2)^r}+r(4q(x)+p^2(x))\frac{(p(x)+2q(x)t)^{r-1}}{(1-p(x)t-q(x)t^2)^{r+1}}.\end{aligned}$$

That is

$$\begin{aligned}&\frac{2q(x)(r-m)(p(x)+2q(x)t)^{m}}{(1-p(x)t-q(x)t^2)^r}+(p(x)+2q(x)t)\frac{d}{dt}\frac{(p(x)+2q(x)t)^m}{(1-p(x)t-q(x)t^2)^r}\\ &\quad = -2q(x)r\frac{(p(x)+2q(x)t)^{m}}{(1-p(x)t-q(x)t^2)^r}+r(4q(x)+p^2(x))\frac{(p(x)+2q(x)t)^{m}}{(1-p(x)t-q(x)t^2)^{r+1}}.\end{aligned}$$

By comparing the coefficients of t^n, we have

$$K_{p,q,n}^{(r+1,m)}(x)=\frac{1}{r(4q(x)+p^2(x))}\left\{\ p(x)(n+1)K_{p,q,n+1}^{(r,m)}(x)+2q(x)(2r-m+n)K_{p,q,n}^{(r,m)}(x)\ \right\}.$$

3. Some Applications of Extended Convolved $(p,q)-$Fibonacci and $(p,q)-$Lucas Polynomials

The multiple sum of Fibonacci numbers have been studied several papers in [8, 20, 21]. In this section, we consider the some mixed–multiple sums for $(p,q)-$Fibonacci and $(p,q)-$Lucas polynomials by applying the recurrence relations of extended convolved $(p,q)-$Fibonacci and $(p,q)-$Lucas polynomials. Moreover, we give some special case for the mixed–multiple sums of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas numbers.

From the generating functions of $K_{p,q,n}^{(r,m)}(x)$, we get

$$K_{p,q,n-r+m}^{(r,m)}(x)=\sum_{a_1+\cdots+a_{r-m}+b_1+\cdots+b_m=n}U_{a_1}(x)\dots U_{a_{r-m}}(x)V_{b_1+1}(x)\dots V_{b_m+1}(x). \tag{3.1}$$

Taking $r=2$, $m=0,1,2$ in (3.1), then we give the following theorem.

Theorem 3.1. *For the $(p,q)-$Fibonacci and $(p,q)-$Lucas polynomials, we have*

$$\sum_{a+b=n} U_a(x)U_b(x) = \frac{1}{(4q(x)+p^2(x))}\left\{p(x)(n-1)U_n(x)+2q(x)nU_{n-1}(x)\right\}, \tag{3.2}$$

$$\sum_{a+b=n} U_a(x)V_{b_1+1}(x) = nU_{n+1}(x), \tag{3.3}$$

$$\sum_{a+b=n} V_{a+1}(x)V_{b+1}(x) = p(x)(n+1)U_{n+2}(x)+2q(x)nU_{n+1}(x). \tag{3.4}$$

Proof. In order to proof of theorem, we will apply (2.5), (2.3) and (2.4), respectively. Thus we have

$$\begin{aligned} K^{(2,0)}_{p,q,n-2}(x) &= \frac{1}{4q(x)+p^2(x)}\left\{p(x)(n-1)K^{(1,0)}_{p,q,n-1}(x)+2nq(x)^{(1,0)}_{p,q,n-2}(x)\right\} \\ &= \frac{1}{4q(x)+p^2(x)}\left\{p(x)(n-1)U_n(x)+2nq(x)U_{n-1}(x)\right\}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} K^{(2,1)}_{p,q,n-1}(x) &= \frac{1}{4q(x)+p^2(x)}\left\{p(x)nK^{(1,1)}_{p,q,n}(x)+2q(x)nK^{(1,1)}_{p,q,n-1}(x)\right\} \\ &= \frac{1}{4q(x)+p^2(x)}\left\{np^2(x)U_{n+1}(x)+4nq(x)U_{n+1}(x)\right\} \\ &= nU_{n+1}(x), \end{aligned} \tag{3.6}$$

$$\begin{aligned} K^{(2,2)}_{p,q,n}(x) &= p(x)K^{(2,1)}_{p,q,n}(x)+2q(x)K^{(2,1)}_{p,q,n-1}(x) \\ &= p(x)(n+1)U_{n+2}(x)+2q(x)nU_{n+1}(x). \end{aligned} \tag{3.7}$$

Thus the proof is completed.

Corollary 3.1. [17, 20] *Especially, if we take $p(x)=1$ and $q(x)=1$, we have the following mixed-multiple sums for Fibonacci and Lucas numbers*

$$\sum_{a+b=n} F_aF_b = \frac{1}{5}\left\{(n-1)F_n+2nF_{n-1}\right\},$$

$$\sum_{a+b=n} F_aL_{b+1} = nF_{n+1},$$

$$\sum_{a+b=n} L_{a+1}L_{b+1} = (n+1)F_{n+2}+2nF_{n+1}.$$

Corollary 3.2. *Especially, if we take $p(x)=2$ and $q(x)=1$, we have the following mixed-multiple sums for Pell and Pell-Lucas numbers*

$$\sum_{a+b=n} P_a P_b = \frac{1}{4}\left\{(n-1)P_n + nP_{n-1}\right\},$$

$$\sum_{a+b=n} P_a Q_{b+1} = nP_{n+1},$$

$$\sum_{a+b=n} Q_{a+1} Q_{b+1} = (n+1)P_{n+2} + 2nP_{n+1}.$$

Corollary 3.3. *Especially, if we take $p(x)=1$ and $q(x)=2$, we have the following mixed-multiple sums for Jacobsthal and Jacobsthal -Lucas numbers*

$$\sum_{a+b=n} J_a J_b = \frac{1}{9}\left\{(n-1)J_n + 4nJ_{n-1}\right\},$$

$$\sum_{a+b=n} J_a j_{b+1} = nJ_{n+1},$$

$$\sum_{a+b=n} j_{a+1} j_{b+1} = (n+1)J_{n+2} + 4nJ_{n+1}.$$

Taking $r=3$, $m=0,1,2$ in (3.1), then we give the following theorem.

Theorem 3.2. *For the $(p,q)-$Fibonacci and $(p,q)-$Lucas polynomials, we have*

$$\sum_{a+b+c=n} U_a(x)U_b(x)U_c(x) = \frac{1}{2(4q(x)+p^2(x))^2}\left\{\left[p^2(x)(n-1)(n-2)+4q(x)(n^2-1)\right]U_n(x)\right.$$
$$\left. -6p(x)q(x)nU_{n-1}(x)\right\},$$

$$\sum_{a+b+c=n} U_a(x)U_b(x)V_{c+1}(x) = \frac{n-1}{2(4q(x)+p^2(x))^2}\left\{\left(p^3(x)+4p(x)q(x)\right)nU_{n+1}(x)+2p^2(x)\right.$$
$$\left.\times\left(3q(x)+n-2\right)U_n(x)+8q(x)(n+1)U_n(x)\right\}$$
$$+\frac{12np(x)q(x)(q(x)-1)U_{n-1}(x)}{2(4q(x)+p^2(x))^2},$$

$$\sum_{a+b+c=n} U_a(x)V_{b+1}(x)V_{c+1}(x) = \frac{1}{2(4q(x)+p^2(x))}\left\{\left[n(n+1)p^3(x)+2p(x)q(x)n(2n+1)\right]U_{n+1}(x)\right.$$
$$\left. +(n+1)\left[np^2(x)q(x)+(4n-4)q^2(x)\right]U_n(x)\right\},$$

$$\sum_{a+b+c=n} V_{a+1}(x)V_{b+1}(x)V_{c+1}(x) = \frac{1}{2(4q(x)+p^2(x))}\Big\{\Big[(n+1)(n+2)p^5(x)$$
$$+p^3(x)q(x)(n+1)(7n+8)+12n(n+1)p(x)q^2(x)\Big]U_{n+1}(x)$$
$$+\Big[(n+1)(n+2)p^4(x)q(x)+6(n+1)^2p^2(x)q^2(x)$$
$$+8(n^2-1)q^3(x)\Big]U_n(x)\Big\}.$$

Proof. In order to proof of theorem, we will apply (2.5), (2.3) and (2.4), respectively. Thus we have

$$K^{(3,0)}_{p,q,n-3}(x) = \frac{1}{2(4q(x)+p^2(x))}\Big\{p(x)(n-2)K^{(2,0)}_{p,q,n-2}(x)+2q(x)(n+1)K^{(2,0)}_{p,q,n-3}(x)\Big\}$$
$$= \frac{1}{2(4q(x)+p^2(x))^2}\Big\{\Big[p^2(x)(n-1)(n-2)+4q(x)(n^2-1)\Big]U_n(x)$$
$$-6p(x)q(x)nU_{n-1}(x)\Big\},$$

$$K^{(3,1)}_{p,q,n-2}(x) = p(x)K^{(3,0)}_{p,q,n-2}(x)+2q(x)K^{(2,0)}_{p,q,n-3}(x)$$
$$= \frac{n-1}{2(4q(x)+p^2(x))^2}\Big\{\big(p^3(x)+4p(x)q(x)\big)nU_{n+1}(x)+2p^2(x)\times$$
$$(3q(x)+n-2)U_n(x)+8q(x)(n+1)U_n(x)\Big\}+\frac{12np(x)q(x)(q(x)-1)U_{n-1}(x)}{2(4q(x)+p^2(x))^2},$$

$$K^{(3,2)}_{p,q,n-1}(x) = \frac{n}{2}K^{(2,1)}_{p,q,n}(x)-q(x)K^{(2,0)}_{p,q,n-1}(x)$$
$$= \frac{1}{2(4q(x)+p^2(x))}\Big\{\Big[n(n+1)p^3(x)+2p(x)q(x)n(2n+1)\Big]U_{n+1}(x)$$
$$+(n+1)\Big[np^2(x)q(x)+(4n-4)q^2(x)\Big]U_n(x)\Big\},$$

$$K^{(3,3)}_{p,q,n}(x) = p(x)K^{(3,2)}_{p,q,n}(x)+2q(x)K^{(3,2)}_{p,q,n-1}(x)$$
$$= \frac{1}{2(4q(x)+p^2(x))}\Big\{\Big[(n+1)(n+2)p^5(x)+p^3(x)q(x)(n+1)(7n+8)\Big]U_{n+1}(x)$$
$$+12n(n+1)p(x)q^2(x)U_{n+1}(x)+(n+1)(n+2)p^4(x)q(x)U_n(x)$$
$$+\Big[6(n+1)^2p^2(x)q^2(x)+8(n^2-1)q^3(x)\Big]U_n(x)\Big\}.$$

Corollary 3.3. [17, 20] *Especially, if we take* $p(x)=1$ *and* $q(x)=1$, *we have the following mixed-multiple sums for Fibonacci and Lucas numbers*

$$\sum_{a+b+c=n} F_aF_bF_c = \frac{1}{50}\left\{(5n^2-3n-2)F_n - 6F_{n-1}\right\},$$

$$\sum_{a+b+c=n} F_aF_bL_{c+1} = \frac{1}{10}(n-1)\left\{nF_{n+1}+2(n+1)F_n\right\},$$

$$\sum_{a+b+c=n} F_aL_{b+1}L_{c+1} = \frac{1}{10}\left\{(5n^2+3n)F_{n+1}+(5n^2+n-4)F_n\right\},$$

$$\sum_{a+b+c=n} L_{a+1}L_{b+1}L_{c+1} = (n+1)\left\{(2n+1)F_{n+1}+\frac{3}{2}nF_n\right\}.$$

Corollary 3.4. *Especially, if we take* $p(x)=2$ *and* $q(x)=1$, *we have the following mixed-multiple sums for Pell and Pell-Lucas numbers*

$$\sum_{a+b+c=n} P_aP_bP_c = \frac{1}{128}\left\{(8n^2-12n+4)P_n - 12nP_{n-1}\right\},$$

$$\sum_{a+b+c=n} P_aP_bQ_{c+1} = \frac{n-1}{8}\left\{nP_{n+1}+(n+1)P_n\right\},$$

$$\sum_{a+b+c=n} P_aQ_{b+1}Q_{c+1} = \frac{1}{16}\left\{(16n^2+12n)P_{n+1}+(n+1)(8n-4)P_n\right\},$$

$$\sum_{a+b+c=n} Q_{a+1}Q_{b+1}Q_{c+1} = (7n^2+15n+8)P_{n+1}+(3n^2+6n+8)P_n.$$

Corollary 3.5. *Especially, if we take* $p(x)=1$ *and* $q(x)=2$, *we have the following mixed-multiple sums for Jacobsthal and Jacobsthal -Lucas numbers*

$$\sum_{a+b+c=n} J_aJ_bJ_c = \frac{1}{162}\left\{(9n^2-3n-6)J_n - 12nJ_{n-1}\right\},$$

$$\sum_{a+b+c=n} J_aJ_bj_{c+1} = \frac{1}{54}\left\{(n-1)\left(3nJ_{n+1}+(6n+8)J_n\right)+8nJ_{n-1}\right\},$$

$$\sum_{a+b+c=n} J_aj_{b+1}j_{c+1} = \frac{1}{18}\left\{(9n^2+5n)J_{n+1}+(n+1)(18n-16)J_n\right\},$$

$$\sum_{a+b+c=n} j_{a+1}j_{b+1}j_{c+1} = \frac{1}{18}\left\{\left(63n^2+81n+18\right)J_{n+1}+(90n^2+54n-36)J_n\right\}.$$

Corollary 3.6. *Especially, if we take $p(x) = 1$ and $q(x) = 2$, we have the following mixed-multiple sums for Jacobsthal and Jacobsthal -Lucas numbers*

$$\sum_{a+b+c=n} J_a J_b J_c = \frac{1}{162}\left\{(9n^2-3n-6)J_n - 12nJ_{n-1}\right\},$$

$$\sum_{a+b+c=n} J_a J_b j_{c+1} = \frac{1}{54}\left\{(n-1)\left(3nJ_{n+1}+(6n+8)J_n\right)+8nJ_{n-1}\right\},$$

$$\sum_{a+b+c=n} J_a j_{b+1} j_{c+1} = \frac{1}{18}\left\{(9n^2+5n)J_{n+1}+(n+1)(18n-16)J_n\right\},$$

$$\sum_{a+b+c=n} j_{a+1} j_{b+1} j_{c+1} = \frac{1}{18}\left\{\left(63n^2+81n+18\right)J_{n+1}+(90n^2+54n-36)J_n\right\}.$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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