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PROPERTIES OF A SQUARE ROOT GAMMA DISTRIBUTION

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Abstract: In this paper, the probability density function properties of a square root Gamma distribution (SRGD) were established. Firstly, a generalized expression for the k^{th} moment (k =1, 2, 3, . . .) was found. Secondly, not only that the moments and characteristic functions were established, it was also found that the moments can also be recovered from these two basic functions by using the laid down statistical rules governing these functions. Finally the measures of skewness, kurtosis and coefficient of variation were also established.

Keywords: Gamma distribution; Square root transformation; Moments

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1. Introduction

Multiplicative error models (MEMs) are particularly suitable for modeling non-negative time series (Brownlees et al., (2011), which is often the kind of data we encounter in everyday practice. Let $X_t, t \in N$ be a discrete time process defined on $[0,\infty), t \in N$ and let $f\left(X_{t-1}\right)$, the information available for forecasting X_t. For a real-valued time series data, $X_t, t \in N$ follows a MEM if it can be expressed as

$$X_t = f\left(X_{t-1}\right)e_t \tag{1}$$

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where $f\left(X_{t-1}\right)$, is a quantity that evolves deterministically according to the parameters M_t, the trend-cycle component and S_t, the seasonal component. Model (1) is a multiplicative time series model, where

$$f\left(X_{t-1}\right)=M_t S_t \tag{2}$$

e_t in (1) is a random variable with probability density function (pdf) defined over a $[0,+\infty]$ support, with unit mean and unknown constant variance.

$$e_t \sim V^{+}\left(1,\sigma^2\right) \tag{3}$$

In principle, the conditional distribution of the error term e_t can be specified by means of any pdf having the characteristics in (3). Examples are Gamma, Log-Normal, Weibull, Inverted- Gamma and mixtures of them (Brownlees et al., (2011)). Engle and Gallo (2006) favor a Gamma (ϕ,ϕ)(which implies $\sigma^2 = 1/\phi$); Bauwens and Giot (2000), in ACD framework considered a Weilbull $\Gamma\left((1+\phi)^{-1},\phi\right)$ (in this case, $\sigma^2 = \Gamma(1+2\phi)/\Gamma\left((1+\phi)^2-1\right)$). De Luca and Gallo (2010) investigated (possibly time – varying mixtures, while Lanne (2006) adopts mixtures and a conditional expectation specification with time varying parameters.

Many time series Analyst assume normality and it is well known that variance stabilization implies normalization of the series. The popular and common are the power transformation such as $Log_e X_t, 1/X_t, 1/\sqrt{X_t}, X_t^2$, and $1/X_t^2$. For further details on transformation see [Bartlett (1947); Box and Cox, (1964); Akpanta and Iwueze (2009)] .

Studies on the effects of transformation on the error component of the multiplicative time

Series model are not new in the statistical literature. The overall aim of such studies is to establish the conditions for successful transformation. A successful transformation is achieved when the desirable properties of a data set remains unchanged after transformation. These basic properties or assumptions of interest are; (i) Normality (ii)

Unit mean and (iii) constant variance.

Suppose e_t in (1) is Gamma $\left(1,\sigma^2\right)$, Ohakwe et al.,(2012) had studied the implication on e_t of applying a square root transformation on model (1) and discovered that the square root transformed error component was found to be normal with unit mean and variance, approximately 4 times that of the original error before transformation except when the shape parameter is equal to one. Furthermore, it was also found that applying a square root transformation on Gamma $\left(1,\sigma^2\right)$ distribution yields a different kind of distribution given by

$$f\left(y\right)=\frac{2\alpha^{\alpha}}{\Gamma\left(\alpha\right)}y^{2\alpha-1}e^{-y^2\alpha},\ y>0 \quad . \tag{4}$$

that belongs to the Generalized Gamma Distribution (GGD) given in Walck (2000) as

$$f\left(x;a,b,c\right)=\frac{ac\left(ax\right)^{bc-1}e^{-\left(ax\right)^{c}}}{\Gamma\left(b\right)} \tag{5}$$

Equation (4) can be expressed as (5) with $a=\sqrt{\alpha}$; $b=\alpha$ and $c=2$. The expression given in (4) was found to be a pdf and its first and second moments were obtained in Ohakwe et al., (2012). However attempts were not made to establish the other important statistical measures such as the moment generating function (mgf), Characteristics function (cf), skewness, kurtosis and coefficient of variation and these form the basis of this paper.

Thus, the aim of this paper is not only to establish the existence of the above mentioned statistical measures but to obtain a generalized expression for the uncorrected moments and also demonstrate that the moments can be obtained from the mgf and cf as contained in the statistical literature. This paper is organized into six Sections. The introduction, the expression for k^{th} moment (k =1, 2, 3, ….) and the moment generating function are contained in Sections one and two respectively. The characteristic function is established in Section three while the measures of skewness and kurtosis are contained in Section four. Summary of results and conclusion are contained in Section five while the references are contained in Section six.

2. General Expression for the Kth Moment (E (Y^k)) and the Moment Generating Function of the Square root Gamma Probability Function

This Section would be split into two parts. The first part, Section 2.1 would contain the derivation for the kth moment (E (Y^k)), k = 1, 2, 3, . . .) of the distribution under study while the second part, Section 2.2 would contain the derivation of its moment generating function (mgf = $M_X(t)$)

2.1: General Expression for the Kth Moment (E (Y^k))

In this Section we would obtain an expression for the Kth moment (E (Y^k)), k = 1, 2, 3, . . .) of

the Square root Gamma probability density function.

By definition

$$E\left(Y^k\right) = \int_0^\infty y^k f(y)dy \tag{6}$$

Substituting (4) into (6), we have that

$$E\left(Y^k\right) = \frac{2\alpha^\alpha}{\Gamma(\alpha)}\int_0^\infty y^k y^{2\alpha-1}e^{-y^2\alpha}dy = \frac{2\alpha^\alpha}{\Gamma(\alpha)}\int_0^\infty y^{(2\alpha+k)-1}e^{-y^2\alpha}dy \tag{7}$$

If we let $p = y^2\alpha$ in (7), the following results are true

$$y = \alpha^{-\frac{1}{2}} p^{\frac{1}{2}} \ and\ dy = \frac{\alpha^{-\frac{1}{2}} p^{-\frac{1}{2}}}{2} dp \tag{8}$$

hence

$$E\left(Y^k\right) = \frac{2\alpha^\alpha}{\Gamma(\alpha)}\int_0^\infty \left(\alpha^{-\frac{1}{2}} p^{\frac{1}{2}}\right)^{2\alpha+k-1} e^{-p}\,\frac{\alpha^{-\frac{1}{2}}p^{-\frac{1}{2}}}{2}dp$$

$$= \frac{\alpha^{-\frac{k}{2}}}{\Gamma(\alpha)}\int_0^\infty p^{\left(\alpha+\frac{k}{2}\right)-1} e^{-p}dp = \frac{\Gamma\left(\alpha+\frac{k}{2}\right)}{\alpha^{\frac{k}{2}}\Gamma(\alpha)}$$

thus

$$E\left(Y^{k}\right)=\frac{\Gamma\left(\alpha+\frac{k}{2}\right)}{\alpha^{\frac{k}{2}}\Gamma(\alpha)} \tag{9}$$

Substituting k = 1 and 2, we have the same results obtained in Ohakwe et al., (2012) for the first and second moments. That is

$$E(Y)=\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\sqrt{\alpha}\,\Gamma(\alpha)} \quad . \tag{10}$$

and

$$E\left(Y^{2}\right)=\frac{\Gamma(\alpha+1)}{\alpha\Gamma(\alpha)}=\frac{\alpha\Gamma(\alpha)}{\alpha\Gamma(\alpha)}=1 \tag{11}$$

2.2: Moment Generating Function $M_{x}(t)$

The mgf denoted by $M_{y}(t)$ is defined as

$$M_{y}(t)=E\left(e^{ty}\right)=\int_{0}^{\infty}e^{ty}f(y)\,dy=\int_{0}^{\infty}f(y)\left[1+ty+\frac{t^{2}y^{2}}{2!}+\frac{t^{3}y^{3}}{3!}+\frac{t^{4}y^{4}}{4!}+\ldots\right]dy$$

$$=\int_{0}^{\infty}f(y)dy+t\int_{0}^{\infty}y\,f(y)dy+\frac{t^{2}}{2!}\int_{0}^{\infty}y^{2}f(y)dy+\frac{t^{3}}{3!}\int_{0}^{\infty}y^{3}f(y)dy+\frac{t^{4}}{4!}\int_{0}^{\infty}y^{4}f(y)dy+\ldots$$

$$1+tE(Y)+\frac{t^{2}}{2!}E\left(Y^{2}\right)+\frac{t^{3}}{3!}E\left(Y^{3}\right)+\frac{t^{4}}{4!}E\left(Y^{4}\right)+\ldots=\sum_{k=0}^{\infty}\frac{t^{k}}{k!}E\left(Y^{k}\right) \tag{12}$$

or

$$M_{y}(t)=\sum_{k=0}^{\infty}\frac{t^{k}\,\Gamma\left(\alpha+\frac{k}{\alpha}\right)}{k!\,\alpha^{\frac{k}{2}}\Gamma(\alpha)} \tag{13}$$

There is no question that the mgf is a continuous function of t, therefore having obtained the moment generating function of (4), the next task is to show that evaluating the r[th] derivative (r = 1, 2, 3, . . .) of the mgf at t = 0 gives E (X [r]), that is

$$\left[M^{r}_{y}(t)\right]_{t=0} = \left[\frac{d^{r} M_{y}(t)}{d t^{r}}\right]_{t=0} = E\left(X^{r}\right) \quad (14)$$

We proceed as follows;

When r = 1, we obtain from (12) that

$$M^{1}_{y}(t) = E(Y) + \sum_{k=2}^{\infty} \frac{k t^{k-1}}{k!} E\left(Y^{k}\right) \quad (15)$$

When r = 2, we have

$$M^{2}_{y}(t) = E\left(Y^{2}\right) + \sum_{k=3}^{\infty} \frac{k(k-1)t^{k-2}}{k!} E\left(Y^{k}\right) \quad (16)$$

When r = 3, we have

$$M^{3}_{y}(t) = E\left(Y^{3}\right) + \sum_{k=4}^{\infty} \frac{k(k-1)(k-2)t^{k-3}}{k!} E\left(Y^{k}\right) \quad (17)$$

Generally the r[th] order derivative of $M_{y}(t)$ denoted by $M^{r}_{y}(t)$ is given by

$$M^{r}_{y}(t) = E\left(Y^{r}\right) + \sum_{k>r}^{\infty} \frac{k(k-1)(k-2)\ldots(k-r).1t^{k-r}}{k!} E\left(Y^{k}\right)$$

$$= \frac{\Gamma\left(\alpha+\frac{r}{2}\right)}{\alpha^{\frac{r}{2}}\Gamma(\alpha)} + \sum_{k>r}^{\infty} \frac{k(k-1)(k-2)\ldots(k-r).1t^{k-r}}{k!} \frac{\Gamma\left(\alpha+\frac{k}{2}\right)}{\alpha^{\frac{k}{2}}\Gamma(\alpha)} \quad (18)$$

Evaluating $M^{r}_{y}(t)$ at t = 0, gives $E\left(X^{r}\right)$ since all other terms are zero, thus

$$M^{r}_{y}(t=0) = \frac{\Gamma\left(\alpha+\frac{r}{2}\right)}{\alpha^{\frac{r}{2}}\Gamma(\alpha)}, r = 1, 2, 3, \ldots \quad (19)$$

3.0 Characteristic function $\left(\phi_{y}(t)\right)$

Like the mgf, the characteristics function (cf) denoted by $\left(\phi_{y}(t)\right) = E\left(e^{ity}\right)$ is the expectation of another type of function of the random variable Y and is differentiable and continuous in t. It generates moments in a manner almost similar to the mgf. It is the moment of a complex function of the random variable Y and exists for all real values of t unlike the mgf which exists only if the moment of the distribution exists. A simple relationship between $M_y(t)$ and $(\phi_y(t))$ as given in Uche (2003) is

$$\phi_y(t) = M_y(it) \tag{20}$$

The cf is also a generator of integral moments. The procedure for obtaining these moments is similar to that of the mgf.

Now, for the distribution under study

$$\phi^1{}_y(t) = \frac{d\phi_y(t)}{dt} = \int_0^\infty iye^{ity} f(y)dy$$

hence

$$\phi^1{}_y(t=0) = \frac{d\phi_y(t=0)}{dt} = \int_0^\infty iy \ f(y)dy = iE(y)$$

In the same way

$$\phi^r{}_y(t) = \frac{d^r\phi_y(t)}{dt^r} = \int_0^\infty i^r y^r e^{ity} f(y)dy$$

thus the r^th derivative of the cf is given by

$$\phi^r{}_y(t=0) = \frac{d^r\phi_y(t=0)}{dt^r} = \int_0^\infty i^r y^r \ f(y)dy = i^r E(y) \tag{21}$$

therefore

$$E(X^r) = \frac{1}{i^r}\frac{d^r\phi_y(t=0)}{dt^r} \tag{22}$$

Like the mgf given in (12), the cf is given as

$$\phi_y(t) = \sum_{k=0}^{\infty}\frac{i^k t^k}{k!}E(Y^k) = \sum_{k=0}^{\infty}\frac{i^k t^k \Gamma\left(\alpha+\frac{k}{\alpha}\right)}{k!\alpha^{\frac{k}{2}}\Gamma(\alpha)} \tag{23}$$

While the r^th derivative of the cf is given by

$$\phi^r{}_y(t) = i^r E(Y^r) + \sum_{k>r}^{\infty}\frac{k(k-1)(k-2)\ldots(k-r).1.i^k t^{k-r}}{k!}E(Y^k)$$

$$= i^r \frac{\Gamma\left(\alpha+\frac{r}{2}\right)}{\alpha^{\frac{r}{2}}\Gamma(\alpha)} + \sum_{k>r}^{\infty} \frac{k(k-1)(k-2)...(k-r).1.i^k t^{k-r}}{k!} \frac{\Gamma\left(\alpha+\frac{k}{2}\right)}{\alpha^{\frac{k}{2}}\Gamma(\alpha)} \qquad (24)$$

Evaluating (24) at t = 0, yields

$$\phi^r{}_y(t) = i^r E\left(Y^r\right) = i^r \frac{\Gamma\left(\alpha+\frac{r}{2}\right)}{\alpha^{\frac{r}{2}}\Gamma(\alpha)}, r=1,2,3,\ldots \qquad (25)$$

hence

$$\frac{1}{i^r}\phi^r{}_y(t=0) = E\left(X^r\right) = \frac{\Gamma\left(\alpha+\frac{r}{2}\right)}{\alpha^{\frac{r}{2}}\Gamma(\alpha)}, r=1,2,3,\ldots \qquad (26)$$

4.0 Measures of Skewness, Kurtosis and Coefficient of Variation

In this Section, we would obtain the expressions for three important statistical measures namely, Skewness, Kurtosis and Coefficient of Variation denoted by γ_1, γ_2 and cv respectively. By definition

$$\gamma_1 = E\left(Y-E(Y)\right)^3 = E\left(Y^3\right) - 3E(Y)E\left(Y^2\right) + 2\left[E(Y)\right]^3 \qquad (27)$$

$$\gamma_1 = E\left(Y-E(Y)\right)^4 = E\left(Y^4\right) - 4E(Y)E\left(Y^3\right) + 6E\left(Y^2\right)\left[E(Y)\right]^2 - 3\left[E(Y)\right]^4 \qquad (28)$$

and

$$cv = \frac{\sqrt{E\left(Y^2\right)-(E(Y))^2}}{E(Y)} \qquad (29)$$

Obviously from (27), (28) and (29), the following results are true

$$\gamma_1 = \frac{\Gamma\left(\alpha+\frac{3}{2}\right)}{\alpha^{\frac{3}{2}}\Gamma(\alpha)} - 3\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\alpha^{\frac{1}{2}}\Gamma(\alpha)} + 2\left[\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\alpha^{\frac{1}{2}}\Gamma(\alpha)}\right]^3 \qquad (30)$$

$$\gamma_2 = \frac{\alpha+1}{\alpha} - 4\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\alpha+\frac{3}{2}\right)}{\alpha^2\left[\Gamma(\alpha)\right]^2} + 6\left[\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\alpha^{\frac{1}{2}}\Gamma(\alpha)}\right]^2 - 3\left[\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\alpha^{\frac{1}{2}}\Gamma(\alpha)}\right]^4 \qquad (31)$$

and

$$cv = \frac{\sqrt{\alpha\left[\Gamma(\alpha)\right]^2 - \left[\Gamma\left(\alpha+\frac{1}{\alpha}\right)\right]^2}}{\Gamma\left(\alpha+\frac{1}{2}\right)} \tag{32}$$

5.0 Concluding Remarks

The fundamental finding of this study is that the square root Gamma distribution (SRGD) is found to be a proper probability density function in that the existence of the basic properties required of a probability density function were ascertained. Firstly, a generalized expression for the k^{th} moment (k =1, 2, 3, . . .) was found. Secondly, not only that the moments and characteristic functions were established, it was also found that the moments can also be recovered from these two basic functions by using the laid down statistical rules governing these functions. Finally the measures of skewness, kurtosis and coefficient of variation were also established. Finally, the results of this study will broaden the field of distribution theory that the square root of a Gamma distribution yields a different kind of distribution that belongs to the generalized gamma family herein called the square root gamma distribution (SRGD).

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