Available online at http://scik.org
J. Math. Comput. Sci. 7 (2017), No. 6, 1006-1021
https://doi.org/10.28919/jmcs/3483
ISSN: 1927-5307

A NEW RESULT ON REVERSE ORDER LAWS FOR $\{1,2,3\}$-INVERSE OF A TWO-OPERATOR PRODUCT

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Abstract. In this note, reverse order laws for $\{1,2,3\}$-inverse of a two-operator product is mainly investigated by making full use of block-operator matrix technique. First, an example is given, which demonstrates there is a gap in the main result in [*X. J. Liu, S. X. Wu, D. S. Cvetković-Ilić. New results on reverse order law for* $\{1,2,3\}$*- and* $\{1,2,4\}$*-inverses of bounded operators. Mathematics of Computation, 2013, 82(283): 1597-1607*]. Next, The new necessary and sufficient conditions for $B\{1,2,i\}A\{1,2,i\} \subseteq (AB)\{1,2,i\}(i \in \{3,4\})$ are presented respectively, when all ranges $R(A)$, $R(B)$ and $R(AB)$ are closed. Which will fill up the gap in the above paper.

Keywords: reverse order law; generalized inverse; block-operator matrix.

2010 AMS Subject Classification: 15A09.

1. Introduction

Throughout this paper, let $\mathscr{H}$, $\mathscr{K}$ and $\mathscr{L}$ be separable Hilbert spaces and $\mathscr{B}(\mathscr{K},\mathscr{H})$ be the set of all bounded linear operators from $\mathscr{K}$ into $\mathscr{H}$ and abbreviate $\mathscr{B}(\mathscr{K},\mathscr{H})$ to $\mathscr{B}(\mathscr{H})$ if $\mathscr{K} =$

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Received August 22, 2017

$\mathscr{H}$. If $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$, write $N(A)$ and $R(A)$ for the null space and the range of A, respectively. For an operator $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$, a generalized inverse of A is an operator $G \in \mathscr{B}(\mathscr{K},\mathscr{H})$ which satisfies some of the following four equations, which is said to be the Moore-Penrose conditions:

$$(1) AGA = A, \ \ (2) GAG = G, \ \ (3) (AG)^* = AG, \ \ (4) \ (GA)^* = GA.$$

Let $A\{i, j, \cdots, l\}$ denote the set of operators $G \in \mathscr{B}(\mathscr{K},\mathscr{H})$ which satisfy equation $(i), (j), \cdots, (l)$ from among the above equations. An operator $G \in A\{i, j, \cdots, l\}$ is called an $\{i, j, \cdots, l\}$-inverse of A, and also denoted by $A^{(ij\cdots l)}$. The unique $\{1,2,3,4\}$-inverse of A is denoted by A^+, which is called the Moore-Penrose inverse of A. As is well known, A is Moore-Penrose invertible if and only if $R(A)$ is closed.

Since 1960s, considerable attention has been paid to the reverse order law for generalized inverses of multiple-matrix and multiple-operator products. It is a classical result of Greville in [9] that $(AB)^+ = B^+A^+$ if and only if $R(A^*AB) \subseteq R(B)$ and $R(BB^*A^*) \subseteq R(A^*)$ for any complex matrices A and B. This result was extended to linear bounded operators on Hilbert spaces by Bouldin [2] and Izumino [10]. In the next decades, reverse order laws for other types generalized inverses are studied, for example, $\{1,3\}$-inverse in [8], $\{1,2,3\}$-inverse in[13], [11] and [17], group inverse in [5]. And many interesting results have been obtained, see[1-18]. In particular, reverse order laws for $\{1,2,3\}$- and $\{1,2,4\}$-inverses were considered on matrix algebra by Xiong and Zheng [17] who obtained the equivalent condition for $B\{1,2,i\}A\{1,2,i\} \subseteq (AB)\{1,2,i\} (i \in \{3,4\})$. 2011, Liu and Yang [11] shown that $B\{1,2,i\}A\{1,2,i\} \subseteq (AB)\{1,2,i\} (i \in \{3,4\})$ and $B\{1,2,i\}A\{1,2,i\} = (AB)\{1,2,i\} (i \in \{3,4\})$ were equivalent when A, B are matrices. Continuing to use the same space decomposition method in [15], X. J. Liu, S. X. Wu and D. S. Cvetkovic-Ilic gave the following result in [12],

Theorem 1.1. ([12]) *Let $\mathscr{H},\mathscr{K}$ and $\mathscr{L}$ be Hilbert spaces and let $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$, $B \in \mathscr{B}(\mathscr{L},\mathscr{H})$ be such that $R(A), R(B), R(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:*

(i)$B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}$.

(ii)$R(B) = R(A^*AB) \oplus (R(B) \cap N(A)), R(AB) = R(A)$.

But, it is regretful that there is a gap in the above result.

Example 1.1. *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

By direct computation, we have

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \neq 0,$$

$$(AB)^{(123)} = \begin{pmatrix} 1 & 0 & 0 \\ x_{21} & 0 & 0 \end{pmatrix}, \quad B^{(123)}A^{(123)} = \begin{pmatrix} 1 & 0 & 0 \\ y_{21} & 0 & 0 \end{pmatrix},$$

where x_{21}, y_{21} are arbitrary. It is clearly that $B\{1,2,3\}A\{1,2,3\} = (AB)\{1,2,3\}$, but $R(A) \neq R(AB)$.

The main result in [18] could fill up the gap in Theorem 1.1. In this paper, we shall give a new result about the reverse order law for $\{1,2,3\}$- and $\{1,2,4\}$-reverses by the relationship of the range conclusion. In section 2, we shall give some preliminaries. Some necessary and sufficient conditions for an operator $G \in \mathscr{B}(\mathscr{K},\mathscr{H})$ to be in $A\{1,2,3\}$ and $A\{1,2,4\}$ are pointed. In section 3, we will derive a new sufficient and necessary conditions for $B\{1,2,i\}A\{1,2,i\} \subseteq (AB)\{1,2,i\}(i \in \{3,4\})$ respectively, when $R(A)$, $R(B)$, $R(AB)$ are closed. And also our result will fill up the gap in Theorem 1.1.

2. **Preliminaries**

In this section, we mainly introduce some notations and lemmas. Let $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$ with closed range. Then under the orthogonal decompositions $\mathscr{H} = R(A^*) \oplus N(A)$ and $\mathscr{K} = R(A) \oplus N(A^*)$ respectively, A has the matrix form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix} \to \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix}, \tag{2.1}$$

where $A_1 \in \mathscr{B}(R(A^*), R(A))$ is invertible. The Moore-Penrose inverse A^+ of A has the matrix form as follows

$$A^+ = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix} \to \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix}.$$

The $\{1,3\}$, $\{1,2,3\}$- inverses also have similarly matrix forms.

Lemma 2.1.([12]) *Let $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ with closed range and the matrix form (2.1). Then $A^{(13)}$ and $A^{(123)}$ have the matrix form*

$$A^{(13)} = \begin{pmatrix} A_1^{-1} & 0 \\ G_3 & G_4 \end{pmatrix} : \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix} \to \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix} \tag{2.2}$$

and

$$A^{(123)} = \begin{pmatrix} A_1^{-1} & 0 \\ G_3 & 0 \end{pmatrix} : \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix} \to \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix}, \tag{2.3}$$

with respect to the orthogonal decompositions $\mathscr{K} = R(A) \oplus N(A^*)$ and $\mathscr{H} = R(A^*) \oplus N(A)$ respectively, for any $G_3 \in \mathscr{B}(R(A), N(A))$ and $G_4 \in \mathscr{B}(N(A^*), N(A))$.

Lemma 2.2.([18]) *Let $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ with closed range. If A has the matrix decomposition*

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{pmatrix} \to \begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \end{pmatrix} \tag{2.4}$$

under the orthogonal decompositions $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ and $\mathscr{K} = \mathscr{K}_1 \oplus \mathscr{K}_2$ respectively, then there exist $G_1 \in \mathscr{B}(\mathscr{K}_1, \mathscr{H}_1)$ and $G_3 \in \mathscr{B}(\mathscr{K}_1, \mathscr{H}_2)$ such that the $A^{(123)}$ has the form

$$A^{(123)} = \begin{pmatrix} G_1 & 0 \\ G_3 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \end{pmatrix} \to \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{pmatrix}. \tag{2.5}$$

Lemma 2.3.([18] *Let $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ with closed range. If A has the matrix form*

$$A = \begin{pmatrix} A_1 & A_2 & 0 \\ 0 & A_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \\ \mathscr{H}_3 \end{pmatrix} \to \begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \\ \mathscr{K}_3 \end{pmatrix},$$

with respect to the orthogonal decompositions $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_3$ *and* $\mathscr{K} = \mathscr{K}_1 \oplus \mathscr{K}_2 \oplus \mathscr{K}_3$ *respectively, such that* A_1 *is invertible and* A_3 *is surjective, then there are some operators* $G_{ji} \in \mathscr{B}(\mathscr{K}_i, \mathscr{H}_j)$, $i, j = 1, 2$, *satisfy*

$$\begin{cases} R(G_{21}) \subseteq N(A_3), \\ G_{22} \in A_3\{1\}, \\ G_{12} = -A_1^{-1}A_2G_{22}, \\ G_{11} = A_1^{-1} - A_1^{-1}A_2G_{21}, \end{cases} \tag{2.6}$$

such that $A^{(123)}$ *has the matrix form*

$$A^{(123)} = \begin{pmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & 0 \\ G_{31} & G_{32} & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \\ \mathscr{K}_3 \end{pmatrix} \to \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \\ \mathscr{H}_3 \end{pmatrix}$$

for any $G_{31} \in \mathscr{B}(\mathscr{K}_1, \mathscr{H}_3)$ *and* $G_{32} \in \mathscr{B}(\mathscr{K}_2, \mathscr{H}_3)$.

In [10], the authors have given the necessary and sufficient conditions for $G \in A\{1,2,3\}$ and $G \in A\{1,2,4\}$ for any matrix A. Now, we generalize these results to an operator on an infinite dimensional Hilbert space.

Lemma 2.4. *Let* $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ *and* $G \in \mathscr{B}(\mathscr{K}, \mathscr{H})$. *If* A *has closed range, then*

(1) $G \in A\{1,2,3\}$ *if and only if* $A^*AG = A^*$ *and* $R(G^*) = R(A)$.

(2) $G \in A\{1,2,4\}$ *if and only if* $GAA^* = A^*$ *and* $R(G) = R(A^*)$.

Proof. Note that $G \in A\{1,2,4\}$ if and only if $G^* \in A^*\{1,2,3\}$. It is sufficient to show one of the two statements holds. We next show the statement (1) holds for A with closed range. Since $R(A)$ is closed, A has the matrix form as the formula (2.1). So

$$A^* = \begin{pmatrix} A_1^* & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix} \to \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix}.$$

For $G \in B(\mathscr{K}, \mathscr{H})$, if $G \in A\{1,2,3\}$, then G has the matrix form as the formula (2.3) by Lemma 2.1. Thus

$$G^* = \begin{pmatrix} (A_1^{-1})^* & G_3^* \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix} \to \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix}.$$

It follows that $R(G^*) = (A_1^{-1})^* R(A_1^*) + G_3^* N(A) = R(A)$ and

$$A^*AG = \begin{pmatrix} A_1^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_1^{-1} & 0 \\ G_3 & 0 \end{pmatrix} = \begin{pmatrix} A_1^* & 0 \\ 0 & 0 \end{pmatrix} = A^*.$$

Conversely, let $G \in \mathscr{B}(\mathscr{K},\mathscr{H})$ satisfies $A^*AG = A^*$ and $R(G^*) = R(A)$. We next show $G \in A\{1,2,3\}$.

Since $A^*AG = A^*$, we have $G^*A^*AG = (AG)^*AG = (AG)^*$. Hence $(AG)^* = (AG)^{**} = AG$ and $AGA = G^*A^*A = A^{**} = A$. The Moore-Penrose conditions (3) and (1) hold. Thus, from Lemma 2.1, G has the matrix form as the formula (2.2):

$$G = \begin{pmatrix} A_1^{-1} & 0 \\ G_3 & G_4 \end{pmatrix} : \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix} \to \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix}.$$

and then

$$G^* = \begin{pmatrix} (A_1^{-1})^* & G_3^* \\ 0 & G_4^* \end{pmatrix} : \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix} \to \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix}.$$

Because $R(G^*) = R(A)$, by a simple calculation $G_4 = 0$ and the Moore-Penrose condition (2) holds. Therefore $G \in A\{1,2,3\}$. The proof is complete.

The proof of Theorem 2.4 implies the following result.

Corollary 2.5. *Let $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$ and $G \in \mathscr{B}(\mathscr{K},\mathscr{H})$. If A has closed range, then*

(1) $G \in A\{1,3\}$ *if and only if* $A^*AG = A^*$.

(2) $G \in A\{1,4\}$ *if and only if* $GAA^* = A^*$.

3. Reverse order law for $\{1,2,3\}$- and $\{1,2,4\}$-inverses

In this section, we shall give our main result. Reverse order laws for $\{1,2,3\}$-inverse and $\{1,2,4\}$-inverse have been considered on matrix algebra in [11], [17] and on C^*-algebra in [4]. Xiong and Zheng [17] obtained the equivalent condition for $B\{1,2,i\}A\{1,2,i\} \subseteq (AB)\{1,2,i\} (i \in \{3,4\})$. And another equivalent conditions of above inclusions were given under conditions of operators A, B, AB and $A - ABB^+$ are regular in [4], which equivalent to the rang of A, B, AB and

$A-ABB^{+}$ are closed since A is regular if and only if A^{+} exists. Here, the sufficient and necessary conditions for $B\{1,2,i\}A\{1,2,i\}\subseteq(AB)\{1,2,i\}(i\in\{3,4\})$ will be presented respectively, when $R(A)$, $R(B)$ and $R(AB)$ are closed. And the range of $A-ABB^{+}$ not necessarily closed.

Theorem 3.1. *Let $A\in\mathscr{B}(\mathscr{H},\mathscr{K})$ and $B\in\mathscr{B}(\mathscr{L},\mathscr{H})$ such that all ranges $R(A)$, $R(B)$ and $R(AB)$ are closed. If $B\{1,2,3\}A\{1,2,3\}\subseteq(AB)\{1,2,3\}$, then $R(A^{*}AB)=R(B)$ or $R(A^{*})\subseteq R(B)$ holds.*

Proof. Case 1, $AB=0$. Next we prove $A=0$ or $B=0$.

Suppose that $A\neq 0$ and $B\neq 0$, then A and B have the matrix forms as follows,

$$A=\begin{pmatrix} A_{11} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}:\begin{pmatrix} R(A^{*})\\ R(B)\\ N(A)\ominus R(B)\end{pmatrix}\rightarrow\begin{pmatrix} R(A)\\ N(A^{*})\end{pmatrix}, \tag{3.1}$$

$$B=\begin{pmatrix} 0 & 0\\ B_{21} & 0\\ 0 & 0\end{pmatrix}:\begin{pmatrix} R(B^{*})\\ N(B)\end{pmatrix}\rightarrow\begin{pmatrix} R(A^{*})\\ R(B)\\ N(A)\ominus R(B)\end{pmatrix}. \tag{3.2}$$

By Lemma2.1, we have the $\{1,2,3\}$-inverses of A and B have the matrix forms,

$$A^{(123)}=\begin{pmatrix} A_{11}^{-1} & 0\\ G_{21} & 0\\ G_{31} & 0\end{pmatrix}:\begin{pmatrix} R(A)\\ N(A^{*})\end{pmatrix}\rightarrow\begin{pmatrix} R(A^{*})\\ R(B)\\ N(A)\ominus R(B)\end{pmatrix},$$

$$B^{(123)}=\begin{pmatrix} 0 & F_{12} & 0\\ 0 & B_{21}^{-1} & 0\end{pmatrix}:\begin{pmatrix} R(A^{*})\\ R(B)\\ N(A)\ominus R(B)\end{pmatrix}\rightarrow\begin{pmatrix} R(B^{*})\\ N(B)\end{pmatrix},$$

where $G_{21}\in\mathscr{B}(R(A),R(B)),G_{31}\in\mathscr{B}(R(A),N(A)\ominus R(B)),F_{21}\in\mathscr{B}(R(B),R(B^{*}))$ are arbitrary. Hence

$$B^{(123)}A^{(123)}=\begin{pmatrix} F_{12}G_{21} & 0\\ B_{21}^{-1}G_{21} & 0\end{pmatrix}:\begin{pmatrix} R(A)\\ N(A^{*})\end{pmatrix}\rightarrow\begin{pmatrix} R(B^{*})\\ N(B)\end{pmatrix}.$$

From $B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}$, it is easy to get $B_{21}^{-1}G_{21} = 0$, so $G_{21} = 0$ since $B \neq 0$. But G_{21} is arbitrary by Lemma2.1, then $A = 0$. It is a contradiction with the assumption. Hence $A = 0$ or $B = 0$ in this case. It is natural to get that the result holds.

Case 2, $AB \neq 0$.

Let $\mathscr{H} = R(B) \oplus N(B^*)$ and $\mathscr{K} = R(B^*) \oplus N(B)$ respectively, and take any $G \in A\{1,2,3\}$ and $F \in B\{1,2,3\}$. Then B and F as well as A and G are of the matrix forms as follows from Lemma 2.1, 2.2 and formulae (2.3) and (2.5).

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(B^*) \\ N(B) \end{pmatrix} \to \begin{pmatrix} R(B) \\ N(B^*) \end{pmatrix}$$

and

$$F = \begin{pmatrix} B_1^{-1} & 0 \\ F_3 & 0 \end{pmatrix} : \begin{pmatrix} R(B) \\ N(B^*) \end{pmatrix} \to \begin{pmatrix} R(B^*) \\ N(B) \end{pmatrix}, \tag{3.1}$$

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(B) \\ N(B^*) \end{pmatrix} \to \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix}$$

and

$$G = \begin{pmatrix} G_1 & 0 \\ G_3 & 0 \end{pmatrix} : \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix} \to \begin{pmatrix} R(B) \\ N(B^*) \end{pmatrix}. \tag{3.2}$$

We firstly claim that $FG \in (AB)\{1,2,3\}$ if and only if $G_1 \in A_1\{1,3\}$ and $G_1^*R(B) = R(AB)$. In fact,

$$AB = \begin{pmatrix} A_1B_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(B^*) \\ N(B) \end{pmatrix} \to \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix},$$

$$B^*A^* = \begin{pmatrix} B_1^*A_1^* & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix} \to \begin{pmatrix} R(B^*) \\ N(B) \end{pmatrix}$$

and

$$FG = \begin{pmatrix} B_1^{-1}G_1 & 0 \\ F_3G_1 & 0 \end{pmatrix} : \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix} \to \begin{pmatrix} R(B^*) \\ N(B) \end{pmatrix}.$$

Therefore,

$$B^*A^*ABFG = \begin{pmatrix} B_1^*A_1^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_1B_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B_1^{-1}G_1 & 0 \\ F_3G_1 & 0 \end{pmatrix} = \begin{pmatrix} B_1^*A_1^*A_1G_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This means that

$$B^*A^*ABFG = B^*A^* \text{ if and only if } A_1^*A_1G_1 = A_1^*.$$

It follows that

$$B^*A^*ABFG = B^*A^* \text{ if and only if } G_1 \in A_1\{1,3\} \tag{3.3}$$

from Corollary 2.5. On the other hand,

$$(FG)^* = \begin{pmatrix} G_1^*(B_1^{-1})^* & G_1^*F_3^* \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(B^*) \\ N(B) \end{pmatrix} \to \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix}.$$

Then

$$R((FG)^*) = G_1^*(B_1^{-1})^*R(B) + G_1^*F_3^*N(B) = G_1^*R(B).$$

Thus,

$$R((FG)^*) = R(AB) \text{ if and only if } G_1^*R(B) = R(AB). \tag{3.4}$$

It follows that $FG \in AB\{1,2,3\}$ if and only if

$$G_1 \in A_1\{1,3\} \text{ and } G_1^*R(B) = R(AB)$$

from Lemma 2.4 and formulae (3.3) and (3.4).

Moreover, if we set

$$\begin{cases} \mathscr{H}_1 & = R(B) \ominus (R(B) \cap N(A)) \\ \mathscr{H}_2 & = N(B^*) \ominus (N(B^*) \cap N(A)) \\ \mathscr{H}_3 & = R(B) \cap N(A) \\ \mathscr{H}_4 & = N(B^*) \cap N(A) \end{cases} \quad \text{and} \quad \begin{cases} \mathscr{K}_1 & = R(AB) \\ \mathscr{K}_2 & = R(A) \ominus R(AB) \\ \mathscr{K}_3 & = N(A^*) \end{cases} \tag{3.5}$$

respectively, then it is known that $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_3 \oplus \mathscr{H}_4$ and $\mathscr{K} = \mathscr{K}_1 \oplus \mathscr{K}_2 \oplus \mathscr{K}_3$. In particular, it is elementary that A is of the matrix form

$$A = \begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \\ \mathscr{H}_3 \\ \mathscr{H}_4 \end{pmatrix} \to \begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \\ \mathscr{K}_3 \end{pmatrix}, \tag{3.6}$$

such that A_{11} is invertible and A_{22} is surjective. Then there are some operators $G_{ji} \in \mathscr{B}(\mathscr{K}_i, \mathscr{H}_j)(i = 1,2,3, j = 1,2,3,4)$ satisfying

$$\begin{cases} R(G_{21}) \subseteq N(A_{22}) \\ G_{22} \in A_{22}\{1\}, \\ G_{12} = -A_{11}^{-1}A_{12}G_{22} \\ G_{11} = A_{11}^{-1} - A_{11}^{-1}A_{12}G_{21} \end{cases} \tag{3.7}$$

such that G has the matrix form

$$G = \begin{pmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & 0 \\ G_{31} & G_{32} & 0 \\ G_{41} & G_{42} & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \\ \mathscr{K}_3 \end{pmatrix} \to \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \\ \mathscr{H}_3 \\ \mathscr{H}_4 \end{pmatrix} \tag{3.8}$$

from Lemma 2.3. We note that all of G_{31}, G_{32}, G_{41} and G_{42} are arbitrary. From the matrix forms (3.6) and (3.8), we have

$$A_1 = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_3 \end{pmatrix} \to \begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \end{pmatrix} \tag{3.9}$$

and

$$G_1 = \begin{pmatrix} G_{11} & G_{12} \\ G_{31} & G_{32} \end{pmatrix} : \begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \end{pmatrix} \to \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_3 \end{pmatrix}. \tag{3.10}$$

If $\mathscr{K}_2 = \{0\}$, then $R(A) = R(AB)$ and $A_{22} = 0$. In this case, it is immediate that

$$A_1 = \begin{pmatrix} A_{11} & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_3 \end{pmatrix} \to \mathscr{K}_1$$

and

$$G_1 = \begin{pmatrix} G_{11} \\ G_{31} \end{pmatrix} : \begin{pmatrix} \mathscr{K}_1 \end{pmatrix} \to \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_3 \end{pmatrix}$$

from the formulae (3.9) and (3.10). Since $B\{1,2,3\}A\{1,2,3\} \subseteq AB\{1,2,3\}$, $FG \in (AB)\{1,2,3\}$. So $G_1 \in A_1\{1,3\}$ and $G_1^* R(B) = R(AB)$ from the claim above. Thus $G_{11} = A_{11}^{-1}$ and $A_{12}G_{21} = 0$ by the formula (3.7). Because of the arbitrary of G in $A\{1,2,3\}$, $A_{12} = 0$ and hence $A_{12}^* = 0$. Observing the matrix form (3.6) of A, we deduce that $R(A^*AB) = R(B) \ominus (R(B) \cap N(A))$. Therefore $R(A^*) = R(A^*AB) \subseteq R(B)$ since $R(A) = R(AB)$.

If $\mathscr{K}_2 \neq \{0\}$, then A_{22} is invertible. In fact, it is known that A_{22} is surjective from (3.6). If $N(A_{22}) \neq \{0\}$, then $A_{12} \neq 0$. Otherwise, $N(A_{22}) \subseteq N(A) \cap \mathscr{H}_2$. This is a contradiction since $N(A)$ orthogonal to $\mathscr{H}_2$. It is also known that $N(A_{12}) \cap N(A_{22}) = \{0\}$ by the definition of $\mathscr{H}_2$. On the other hand, there exists nonzero $G_{21} \in \mathscr{B}(\mathscr{K}_1, \mathscr{H}_2)$ such that $A_{22}G_{21} = 0$ by the assumption that $N(A_{22}) \neq \{0\}$. Therefore $A_{12}G_{21} \neq 0$. Combining above G_{21} with (3.7), an operator $G \in A\{1,2,3\}$ can be defined with the property $A_{12}G_{21} \neq 0$. However if $B\{1,2,3\}A\{1,2,3\} \subseteq AB\{1,2,3\}$, then for any $F \in B\{1,2,3\}$ and $G \in A\{1,2,3\}$ with the matrix forms (3.1) and (3.2), we have that $G_1 \in A_1\{1,3\}$ according to the claim above. This implies $G_{11} = A_{11}^{-1}$ and $G_{12} = 0$ in (3.9) and (3.10). It follows from (3.7) that both $A_{12}G_{22} = 0$ and $A_{12}G_{21} = 0$, a contradiction. Therefore, A_{22} is invertible and $A_{12} = 0$. Moreover,

$$A^* = \begin{pmatrix} A_{11}^* & 0 & 0 \\ 0 & A_{22}^* & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \\ \mathscr{K}_3 \end{pmatrix} \to \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \\ \mathscr{H}_3 \\ \mathscr{H}_4 \end{pmatrix}.$$

Therefore $R(A^*AB) = R(B) \ominus (R(B) \cap N(A))$. Meanwhile,

$$G_1^* = \begin{pmatrix} (A_{11}^{-1})^* & G_{31}^* \\ 0 & G_{32}^* \end{pmatrix} : \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_3 \end{pmatrix} \to \begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \end{pmatrix}.$$

Hence $\mathscr{H}_3 = 0$, that is, $R(B) \cap N(A) = \{0\}$ since $G_1^* R(B) = R(AB)$ for any $G_{32} \in \mathscr{B}(\mathscr{K}_2, \mathscr{H}_3)$. Hence $R(A^*AB) = R(B)$. The proof is complete.

Theorem 3.2. *Let $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$ and $B \in \mathscr{B}(\mathscr{K},\mathscr{H})$ such that all ranges $R(A)$, $R(B)$ and $R(AB)$ are closed. If $R(A^*AB) = R(B)$ or $R(A^*) \subseteq R(B)$, then $(AB)\{1,2,3\} \subseteq B\{1,2,3\}A\{1,2,3\}$.*

Proof if $AB = 0$, by the discussion for $AB = 0$ in the proof of Theorem 3.1, we can get the result holds. So assume that $AB \neq 0$ and denote $\mathscr{H}_i (i = 1,2,3,4), \mathscr{K}_j (j = 1,2,3)$ as in (3.5). If $R(A^*) \subseteq R(B)$, then $R(A) = R(AA^*) = R(AB)$ and $R(A^*AB) = R(A^*A) = R(A^*) = R(B) \ominus (R(B) \cap N(A))$. So $\mathscr{H}_2 = \{0\}$, $\mathscr{K}_2 = \{0\}$, $A_{12} = 0$ and $A_{22} = 0$. Then A has the matrix form as follows,

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_3 \\ \mathscr{H}_4 \end{pmatrix} \to \begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_3 \end{pmatrix}, \tag{3.11}$$

Let $\mathscr{J}_2 = B^+\mathscr{H}_3$ and $\mathscr{J}_1 = R(B^*) \ominus \mathscr{J}_2$. B has the following matrix form,

$$B = \begin{pmatrix} B_{11} & 0 & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{J}_1 \\ \mathscr{J}_2 \\ N(B) \end{pmatrix} \to \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_3 \\ \mathscr{H}_4 \end{pmatrix}, \tag{3.12}$$

which B_{11} and B_{22} invertible. According to Lemma 2.1, $\{1,2,3\}$-inverse $A^{(123)}$ and $B^{(123)}$ of A and B has the matrix forms,

$$A^{(123)} = \begin{pmatrix} A_{11}^{-1} & 0 \\ G_{31} & 0 \\ G_{41} & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_3 \end{pmatrix} \to \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_3 \\ \mathscr{H}_4 \end{pmatrix},$$

$$B^{(123)} = \begin{pmatrix} B_{11}^{-1} & 0 & 0 \\ -B_{22}^{-1}B_{21}B_{11}^{-1} & B_{22}^{-1} & 0 \\ F_{31} & F_{32} & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_3 \\ \mathscr{H}_4 \end{pmatrix} \to \begin{pmatrix} \mathscr{J}_1 \\ \mathscr{J}_2 \\ N(B) \end{pmatrix},$$

which $G_{31}, G_{41}, F_{31}, F_{32}$ are arbitrary. This follows that

$$B^{(123)}A^{(123)} = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & 0 \\ -B_{22}^{-1}B_{21}B_{11}^{-1}A_{11}^{-1} + B_{22}^{-1}G_{31} & 0 \\ F_{31}A_{11}^{-1} + F_{32}G_{31} & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{J}_1 \\ \mathscr{J}_2 \\ N(B) \end{pmatrix} \to \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_3 \\ \mathscr{H}_4 \end{pmatrix}, \tag{3.13}$$

Combining formulae (3.11) and (3.12), we have

$$AB=\begin{pmatrix} A_{11}B_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}:\begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_3 \end{pmatrix}\to\begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_3 \\ \mathscr{H}_4 \end{pmatrix},$$

Using Lemma 2.1 again, we get that

$$(AB)^{(123)}=\begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & 0 \\ M_{21} & 0 \\ M_{31} & 0 \end{pmatrix}:\begin{pmatrix} \mathscr{J}_1 \\ \mathscr{J}_2 \\ N(B) \end{pmatrix}\to\begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_3 \\ \mathscr{H}_4 \end{pmatrix}, \tag{3.14}$$

where M_{21}, M_{31} are arbitrary. It follows from formulae (3.13) and (3.14) that $B\{1,2,3\}A\{1,2,3\}\subseteq (AB)\{1,2,3\}$.

If $R(A^*AB)=R(B)$, $R(B)\subseteq R(A^*)$ and $N(A)\subseteq N(B^*)$ hold. So $\mathscr{H}_1=R(B)$ and $\mathscr{H}_3=\{0\}$. Hence A has the matrix form

$$A=\begin{pmatrix} A_{11} & A_{12} & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}:\begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \\ \mathscr{H}_4 \end{pmatrix}\to\begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \\ \mathscr{K}_3 \end{pmatrix}, \tag{3.15}$$

with respect to the orthogonal decompositions $\mathscr{H}=\mathscr{H}_1\oplus\mathscr{H}_2\oplus\mathscr{H}_4$ and $\mathscr{K}=\mathscr{K}_1\oplus\mathscr{K}_2$, respectively, such that A_{11} and A_{22} are invertible. B has the matrix form

$$B=\begin{pmatrix} B_{11} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}:\begin{pmatrix} R(B^*) \\ N(B) \end{pmatrix}\to\begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_3 \\ \mathscr{H}_4 \end{pmatrix}, \tag{3.16}$$

with respect to the orthogonal decompositions $\mathscr{K}=R(B^*)\oplus N(B)$ and $\mathscr{H}=\mathscr{H}_1\oplus\mathscr{H}_2\oplus\mathscr{H}_4$, respectively, such that B_{11} is invertible. By formulae (3.15) and (3.16), it is easy to get that

$$AB=\begin{pmatrix} A_{11}B_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}:\begin{pmatrix} R(B^*) \\ N(B) \end{pmatrix}\to\begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \\ \mathscr{K}_3 \end{pmatrix},$$

Thus

$$A^*AB = \begin{pmatrix} A_{11}^*A_{11}B_{11} & 0 \\ A_{12}^*A_{11}B_{11} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(B^*) \\ N(B) \end{pmatrix} \to \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \\ \mathscr{H}_4 \end{pmatrix},$$

We obtain $A_{12}^*A_{11}B_{11} = 0$ since $R(A^*AB) = R(B)$, and so $A_{12} = 0$. Using Lemma 2.1, $\{1,2,3\}$-inverses $A^{(123)}$ and $B^{(123)}$ of A and B have matrix forms

$$A^{(123)} = \begin{pmatrix} A_{11}^{-1} & 0 & 0 \\ 0 & A_{22}^{-1} & 0 \\ G_{41} & G_{42} & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \\ \mathscr{K}_3 \end{pmatrix} \to \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \\ \mathscr{H}_4 \end{pmatrix},$$

$$B^{(123)} = \begin{pmatrix} B_{11}^{-1} & 0 & 0 \\ F_{21} & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \\ \mathscr{H}_4 \end{pmatrix} \to \begin{pmatrix} R(B^*) \\ N(B) \end{pmatrix},$$

$$(AB)^{(123)} = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & 0 & 0 \\ M_{21} & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \\ \mathscr{K}_3 \end{pmatrix} \to \begin{pmatrix} R(B^*) \\ N(B) \end{pmatrix}, \tag{3.17}$$

respectively, which $G_{41}, G_{42}, F_{21}, M_{21}$ are arbitrary. So

$$B^{(123)}A^{(123)} = \begin{pmatrix} B_{11}^{-1}A_{11}^{-1} & 0 & 0 \\ F_{21}A_{11}^{-1} & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \\ \mathscr{K}_3 \end{pmatrix} \to \begin{pmatrix} R(B^*) \\ N(B) \end{pmatrix}. \tag{3.18}$$

Comparing the formula (3.17) with the formula (3.18), $B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}$ holds since the arbitrariness of F_{21}, M_{21}. The proof is completed.

Combining Theorem 3.1 with Theorem 3.2, we give our main results,

Corollary 3.3. *Let $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ and $B \in \mathscr{B}(\mathscr{K}, \mathscr{H})$ such that all ranges $R(A)$, $R(B)$ and $R(AB)$ are closed. Then the following statements are equivalent,*

(1) $B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}$*;*

(2) $R(A^*AB) = R(B)$ *or* $R(A^*) \subseteq R(B)$.

From the relationship of $\{1,2,3\}$-inverse and $\{1,2,4\}$-inverse, we can obtain the following result without proof.

Corollary 3.4. *Let $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$ and $B \in \mathscr{B}(\mathscr{K},\mathscr{H})$. If $R(A)$, $R(B)$, $R(AB)$ are closed, then the following statements are equivalent,*

(1) $B\{1,2,4\}A\{1,2,4\} \subseteq (AB)\{1,2,4\}$;

(2) $R(B) \subseteq R(A^*)$ or $R(BB^*A^*) = R(A^*)$.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgements

This paper is supported by the National Natural Science Foundation of China (11501345), the Natural Science Basic Research Plan of Henan Province(1523000410221) and the Hainan Province Higher Education Scientific Research Grant of China (HNKY2014-34).

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