Available online at http://scik.org
J. Math. Comput. Sci. 8 (2018), No. 1, 114-129
https://doi.org/10.28919/jmcs/3508
ISSN: 1927-5307

SOLUTION OF SPACE TIME FRACTIONAL GENERALIZED KdV EQUATION, KdV BURGER EQUATION AND BONA-MAHONEY-BURGERS EQUATION WITH DUAL POWER-LAW NONLINEARITY USING COMPLEX FRACTIONAL TRANSFORMATION

UTTAM GHOSH[1,*], SANTANU RAUT[2], SUSMITA SARKAR[1] AND SHANTANU DAS[3]

[1]Department of Applied Mathematics, University of Calcutta, Kolkata, India

[2]Mathabhanga College, Cooch Behar, West Bengal, India

[3]Reactor Control Systems Design Section, E & I Group, BARC, Mumbai, India

Abstract: Fractional calculus is a rising subject in the current research field. The researchers of different disciplines are using fractional calculus models to investigate different practical problems. In this paper, we found the exact solutions of space-time fractional generalized KdV equation, KdV Burger equation and Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity. The solutions are expressed in terms of hyperbolic, trigonometric and rational functions.

Keywords: generalized KdV equation; Kdv Burger equation; Bona-Mahoney-Burgers equation; complex fractional transformation; Tanh method**.**

2010 AMS Subject Classification: 35Q53.

1. Introduction

It is well known that to formulate the real world phenomenon most of the places non-linear integer order or fractional differential equations arise [1-3]. The authors are using the fractional differential models to formulate the systems which have memory [4]. To solve the non-linear classical or fractional differential equations which appear in the physical systems we have to consider linear approximation and therefore the solution loses some information. To solve those

*Corresponding author

E-mail address: uttam_math@yahoo.co.in

Received September 18, 2017

non-linear classical and fractional differential equations different research groups have developed many methods. The Adomian Decomposition Method [5-7], Homotopy Perturbation Method (HPM) [8-10], sub-equation method, Generalized Tanh method [11], Generalized exp method [12] etc. Some researchers extended the methods used to solve the integer order differential equations for solving the fractional order differential equations. The fractional sub-equation method is one of them [13-14]. The Tanh method is an important method to find exact solution of the non-linear differential equations. It was introduced by Huiblin and Kelin [15] to find the travelling wave solutions of non-linear differential equations. Wazwaz [16] used this method to find soliton solutions of the Fisher equation. Fan [17] modified the Tanh method to solve KdV-Burgers and Boussinesq equation. The Fractional sub-equation method [13-14] and Generalized Tanh-method [11] are both based on the Homogeneous balance principal. He [19-20] developed the complex fractional transformation method to convert the fractional order differential equation to integer order differential equations.

In this paper we used the complex fractional transformation and generalized Tanh method to solve three non-linear space-time fractional differential equations arises in fluid dynamics and plasma dynamics. The space-time fractional generalized KdV equation, KdV-Burger equation and Bona-Mahoney-Burgers equation with dual power-law nonlinearity first converted to integer order differential equation using the complex fractional transformation and then those equations are solved using generalized Tanh method. In this method the solutions are expressed in terms the hyperbolic, trigonometric and the rational functions.

Organization of the paper is as follows: In section-2 we gave the review of fractional calculus, complex fractional transformation and generalized Tanh method. Section-3,4 and 5 is devoted to solve the space –time fractional generalized KdV equation, KdV-Burger equation and Bona-Mahoney-Burgers equation respectively and numerical presentation of the solutions are given for all the solutions for different values of order of derivative. Finally a conclusion is drawn.

2. Review of fractional calculus, Complex transforms method and Generalized Tanh method

Here we discussed some basic definitions of the fractional derivatives complex fractional transformation and the generalized tanh method.

a) Fractional derivative

There are different definitions of fractional derivative in which Riemann–Liouville (R-L) fractional derivative is one of the widely used definition of fractional derivative. For any continuous and integrable function $f(x)$ it is defined as,

$$
\begin{aligned}
&{}_{t_0}^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-\xi)^{n-\alpha-1} f(\xi) d\xi \\
&t > t_0, \qquad n-1 < \alpha \le n
\end{aligned}
\tag{2.1}
$$

In terms of this definition, derivative of the constant K is ${}_{t_0}^{RL}D_t^{\alpha} K = \frac{K}{\Gamma(1+\alpha)}(t-t_0)^{-\alpha} \neq 0$; where as in the classical sense the fractional derivative of the constant should have been equal to zero. After that M. Caputo [1-2] generalized the fractional order derivative in the new way and overcome this difficulty. He proposed the definition in the following form:

Let α be a positive number and n be a positive integer satisfy $n-1 < \alpha \le n$ and $f^n(t)$ exists (that is $n-$th order ordinary derivative exists). Then α -th order fractional derivative is defined by,

$$
\begin{aligned}
&{}_{t_0}^{C}D_t^{\alpha} f(t) = {}_{t_0}^{RL}J_t^{n-\alpha}\left[f^{(n)}(t) \right] = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi, \\
&t > t_0, \qquad n-1 < \alpha \le n
\end{aligned}
\tag{2.2}
$$

where ${}_{t_0}^{RL}J_t^{\alpha}$ is the R-L integral operator, defined as follows

$$
{}_{t_0}^{RL}J_t^{\alpha}\left[f(t) \right] = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\xi)^{\alpha-1} f(\xi) d\xi, \qquad \alpha > 0
$$

The Caputo derivative fractional derivative a constant becomes zero and it is analogous to standard classical calculus. But it is applicable only when the function accepts differentiability. Then Jumarie [21] gave a new definition which is applicable for continuous (but not necessarily differentiable) functions.

Let $f(x)$ is defined in $0 \le x \le a$, $f(0)$ is finite. Jumarie [21] defines the fractional derivatives of a continuous function $f(x)$,

$$ {}_0^J D_x^\alpha \left[f(x) \right] = f^{(\alpha)}(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^x (x-\xi)^{-\alpha-1} f(\xi) \mathrm{d}\xi, & \alpha < 0 \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_0^x (x-\xi)^{-\alpha} \left(f(\xi) - f(0) \right) \mathrm{d}\xi, & 0 < \alpha < 1 \\ \left(f^{(\alpha-n)}(x) \right)^{(n)}, \quad n \le \alpha < n+1, & n \ge 1 \end{cases} \tag{2.3} $$

Let the general form of fractional differential equation of two independent variables x, t and one dependent variable $u(x,t)$ is,

$$ F\left(u, D_x^\alpha u, D_t^\alpha u, \ldots, D_x^{2\alpha} u, D_t^{2\alpha} u\right) = 0, \qquad 0 < \alpha < 1 \tag{2.4} $$

where $D_x^\alpha u = \frac{\partial^\alpha u(x,t)}{\partial x^\alpha}$ denotes the Jumarie fractional partial derivative of the form,

$$ \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x (x-\xi)^{-\alpha} \left(u(\xi,t) - u(0,t) \right) \mathrm{d}\xi, \qquad 0 < \alpha \le 1 \tag{2.5} $$

where the function $u(x,t)$ is continuous but not necessarily differentiable. A constant function's fractional derivative is zero with Jumarie fractional derivative.

b) The fractional Complex Transformation

The complex fractional transformation [18-20] reduces the fractional order ordinary or partial differential can be reduce to the integer order ordinary or fractional order differential equation. Let a fractional order ordinary differential equation be of the form,

$$ D^\alpha u = f(x), \qquad D^\alpha = \frac{d^\alpha}{dx^\alpha} \tag{2.6} $$

where D^α as Jumarie fractional derivative operator. We introduce the complex fractional transformation $X = \frac{(px)^\alpha}{\Gamma(1+\alpha)}$ where p is constant. Using fractional Taylor series of Jumarrie type we get $\alpha! dx \simeq x^{(\alpha)}(t)\left(dt\right)^\alpha$. Thus, we can write,

$$ \alpha! dx \simeq \left(\frac{d^\alpha x}{\left(dt\right)^\alpha} \right) \left(dt\right)^\alpha = d^\alpha x $$

The relation $d^\alpha x \simeq \alpha! dx = \left(\Gamma(\alpha+1)\right) dx$ is the heart of complex transformation which makes the fractional differential $d^\alpha f$ to the standard differential df . Using this Jumarie conversion we get

$$\frac{d^{\alpha}u}{dx^{\alpha}} = p^{\alpha}\frac{du}{dX}$$

Now, we consider a fractional partial differential equation with two independent variables in the form,

$$D_t^{\alpha}u(x,t) + D_x^{\alpha}u(x,t) = f(x,t) \tag{2.7}$$

Where $0 < \alpha \le 1$ and $D_t^{\alpha}u(x,t) = \frac{\partial^{\alpha}u}{\partial t^{\alpha}}, D_x^{\alpha}u(x,t) = \frac{\partial^{\alpha}u}{\partial x^{\alpha}}$ Then by using the complex fractional transformations, $X = \frac{(px)^{\alpha}}{\Gamma(1+\alpha)}, T = \frac{(qt)^{\alpha}}{\Gamma(1+\alpha)}$ we get the following transformed equation

$$\frac{\partial^{\alpha}u}{\partial x^{\alpha}} = p^{\alpha}\frac{\partial u}{\partial X}, \qquad \frac{\partial^{\alpha}u}{\partial t^{\alpha}} = q^{\alpha}\frac{\partial u}{\partial T}$$

Then (1.7) reduces to the integer order partial differential equation as

$$q^{\alpha}\frac{\partial u}{\partial T} + p^{\alpha}\frac{\partial u}{\partial X} = F(X,T) \tag{2.8}$$

(c) Generalized Tanh Method

In this method the exact solutions of non-linear partial differential equations are expressed in terms of hyperbolic, trigonometric and rational functions. Let us consider a non-linear partial differential equation of the form,

$$\mathrm{L}\left(u_t, u_x, u_y, u_{tt}, u_{xx}, ...\right) = 0 \tag{2.9}$$

satisfied by $u(x,y,t)$. Now we use the travelling wave transformation $\xi = lx + my + ct$, where $(l;m)$ are the wave vector and c is the velocity of propagating waves. Then the equation (2.9) reduces to the ordinary differential equation, that is

$$\mathrm{L}\left(u, u', u''........\right) = 0 \tag{2.10}$$

Fan and Hon [11] introduced a generalized method called Tanh method which is based on the a priori assumption that the travelling wave solutions can be expressed as the power series expansion of solutions of the non-linear Riccati differential and the homogeneous balance principle.

Let ϕ be the solution of non-linear Riccati differential

$$\phi'(\zeta) = \sigma + \phi^2 \tag{2.11}$$

Then Solution of the equation (2.11) can be written in the form

$$\phi(\xi)=\begin{cases}\left.\begin{array}{l}-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\xi)\\-\sqrt{-\sigma}\coth(\sqrt{-\sigma}\xi)\end{array}\right\} & \text{for } \sigma<0\\ \left.\begin{array}{l}\sqrt{\sigma}\tan(\sqrt{\sigma}\xi)\\-\sqrt{\sigma}\cot(\sqrt{\sigma}\xi)\end{array}\right\} & \text{for } \sigma>0.\\ -\dfrac{1}{\xi} & \text{for } \sigma=0\end{cases} \tag{2.12}$$

Let

$$u=s(\phi)=a_0+a_1\phi+a_2\phi^2+........+a_n\phi^n \tag{2.13}$$

be the solution of the equation (2.10) with $a_i, i=0,1,2.....n$ as constants. Then the highest power of ϕ in $u'(\phi)$ becomes $n+1$; similarly that of u'' is $n+2$. Then we balance the highest power of ϕ in the highest order derivative term and the non-linear term to determine *n*. Then put the reduce form of (2.13) in (2.10) and comparing the coefficients of ϕ the constants can be determined.

3. Solution of KdV Equation using Generalized Tanh Method and complex fractional transformation

Let us consider the well known space-time fractional generalized KdV equation

$$D_t^{\alpha}u+\mu u D_x^{\alpha}u+\beta u^2 D_x^{\alpha}u+D_x^{3\alpha}u=0 \tag{3.1}$$

where μ,β are the arbitrary constants. Now we consider a transformation that is

$$\xi=\frac{l_1x^{\alpha}}{\Gamma(1+\alpha)}+\frac{m_1t^{\alpha}}{\Gamma(1+\alpha)}=lx^{\alpha}+mt^{\alpha} \qquad l=\frac{l_1}{\Gamma(1+\alpha)}, \qquad m=\frac{m_1}{\Gamma(1+\alpha)} \tag{3.2}$$

Then the equation (3.1) reduces to the ordinary differential form,

$$mu_{\xi}+l\mu uu_{\xi}+\beta lu^2u_{\xi}+l^3u_{\xi\xi\xi}=0 \tag{3.3}$$

Integrating (3.3) once with respect to ξ, we get the following

$$mu+\frac{1}{2}l\mu u^2+\frac{1}{3}\beta lu^3+l^3u_{\xi\xi}=0 \tag{3.4}$$

The integral constant is taken as zero because the soliton solutions are localized solutions and hence u and it's derivatives will vanish when $\xi\to\pm\infty$.

The equation (3.4) can be written as

$$
\begin{aligned}
u_{\xi\xi} &= -\frac{m}{l^3}u - \frac{1}{2l^2}\mu u^2 - \frac{1}{3l^2}\beta u^3 \\
u_{\xi\xi} &= Au + Bu^2 + Cu^3
\end{aligned}
\tag{3.5}
$$

Where A, B, C are given by $A = -\frac{m}{l^3}, B = -\frac{\mu}{2l^2}, C = -\frac{\beta}{3l^2}$. Now change the independent variable $\xi \to \phi$ where ϕ satisfies the equation (2.11). Hence the equation (3.5) reduces to the form,

$$
\left(\sigma+\phi^2\right)^2 \frac{d^2u}{d\phi^2} + 2\phi\left(\sigma+\phi^2\right)\frac{du}{d\phi} = Au + Bu^2 + Cu^3 \tag{3.6}
$$

Let us consider the solution of the equation (3.6) in the form,

$$
u(\xi) = S = \sum_{i=0}^{n} a_i \phi^i \tag{3.7}
$$

Putting $u(\xi)$ from (3.7) the equation (3.6) can be expressed as,

$$
\begin{aligned}
&\left(\sigma+\phi^2\right)^2\left[2a_2 + 3a_3\phi + ...n(n-1)a_n u^{n-2}\right] \\
&\qquad + \phi\left(\sigma+\phi^2\right)\left[a_1 + 2a_2\phi + ...na_n u^{n-1}\right] = \\
&\qquad\qquad A\left[a_0 + a_1\phi + ...a_n u^n\right] \\
&\qquad\qquad\quad + B\left[a_0 + a_1\phi + ...a_n u^n\right]^2 + \left[a_0 + a_1\phi + ...a_n u^n\right]^3
\end{aligned}
\tag{3.8}
$$

Equating the order of the highest degree of ϕ from the non-linear term and the highest order derivative term from (3.8) we get $n+2=3n$, i.e. $n=1$. Thus $u(\xi)$ in (3.7) take the form $u(\xi) = a_0 + a_1\phi$, and therefore $\frac{du}{d\phi} = a_1$, $\frac{d^2u}{d\phi^2} = 0$. Putting these values in the equation (3.6), we get

$$
2\phi\left(\sigma+\phi^2\right)a_1 = A\left(a_0 + a_1\phi\right) + B\left(a_0 + a_1\phi\right)^2 + C\left(a_0 + a_1\phi\right)^3 \tag{3.9}
$$

Comparing the coefficient of ϕ^0, ϕ^1, ϕ^2 and ϕ^3 in the above equation (3.9), we get

$$
A + Ba_0 + Ca_0^2 = 0, \qquad A + 2Ba_0 + 3Ca_0^2 = 2\sigma, \qquad B + 3Ca_0 = 0, \qquad Ca_1^3 = 2a_1
$$

Solving the above equations, we get,

$$
a_0 = -\frac{\mu}{2\beta}, \qquad a_1 = \sqrt{-\frac{6l^2}{\beta}}
$$

as $a_1 \neq 0$ and l and m are connected by the following relation

$$\mu^2 l = 12\beta\left(m - 2l^3\sigma\right); \qquad \sigma = \frac{1}{2l^3}\left(m - \frac{l\mu^2}{12\beta}\right)$$

Hence the general solution of the equation (3.1) can be written as,

$$u(x,t) = \begin{cases} \left.\begin{matrix} -\dfrac{\mu}{2\beta} - \sqrt{-\dfrac{6l^2}{\beta}}\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\left(lx^\alpha + mt^\alpha\right)\right) \\ -\dfrac{\mu}{2\beta} - \sqrt{-\dfrac{6l^2}{\beta}}\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\left(lx^\alpha + mt^\alpha\right)\right) \end{matrix}\right\} & \text{for} \quad \sigma < 0 \\ -\dfrac{\mu}{2\beta} + \sqrt{-\dfrac{6l^2}{\beta}}\sqrt{\sigma}\tan\left(\sqrt{\sigma}\left(lx^\alpha + mt^\alpha\right)\right) & \text{for} \quad \sigma > 0 \\ -\dfrac{\mu}{2\beta} - \sqrt{-\dfrac{6l^2}{\beta}}\sqrt{\sigma}\cot\left(\sqrt{\sigma}\left(lx^\alpha + mt^\alpha\right)\right) & \text{for} \quad \sigma > 0 \\ -\dfrac{\mu}{2\beta} - \sqrt{-\dfrac{6l^2}{\beta}}\dfrac{1}{\left(lx^\alpha + mt^\alpha\right)} & \text{for} \quad \sigma = 0 \end{cases} \tag{3.10}$$

Hence we obtain the generalized solution of the space time fractional generalized KdV equation in terms of hyperbolic, trigonometric and the rational functions.

4. Solution of KdV Burger Equation using Generalized Tanh Method and complex fractional transformation

Let us consider the well known KdV Burger Equation

$$D_t^\alpha u + uD_x^\alpha u + \delta D_x^{3\alpha} u = \gamma D_x^{2\alpha} u \tag{4.1}$$

Now we consider the transformation defined in (3.2) .Then the equation transforms into

$$l^3\delta u_{\xi\xi\xi} = -l\,u_\xi - luu_\xi + l^2\gamma u_{\xi\xi} \tag{4.2}$$

Integrating (4.2) once both side with respect to ξ we get,

$$l^3\delta u_{\xi\xi} = -m\,u - \frac{1}{2}lu^2 + l^2\gamma u_\xi \tag{4.3}$$

Here also the integral constant is taken as zero because the soliton solutions are localized solutions and hence u and its derivatives will vanish when $\xi \to \pm\infty$. Now again change the independent variable $\xi \to \phi$ where ϕ satisfies the equation (2.11).

Hence the equation (4.3) becomes,

$$l^3\delta\left[\left(\sigma+\phi^2\right)^2\frac{d^2u}{d\phi^2}+2\phi\left(\sigma+\phi^2\right)\frac{du}{d\phi}\right]=l^2\gamma\left(\sigma+\phi^2\right)\frac{du}{d\phi}-mu-\frac{lu^2}{2} \tag{4.4}$$

Let us consider the solution of the equation (4.4) in the form as in (3.7), i.e. $u(\xi)=S=\sum_{i=0}^{n}a_i\phi^i$.

Putting this form in equation (4.4) reduces to the form as following

$$\begin{aligned}
&l^3\delta\begin{bmatrix}\left(\sigma+\phi^2\right)^2\left(2a_2+6a_3\phi+......n(n-1)a_nu^{n-2}\right)\\ +2\phi\left(\sigma+\phi^2\right)\left(a_1+2a_2\phi+......na_nu^{n-1}\right)\end{bmatrix}=\\
&-m\left[a_0+a_1\phi+......a_nu^n\right]-\frac{l}{2}\left[a_0+a_1\phi+......a_nu^n\right]^2\\
&+l^2\gamma\left(\sigma+\phi^2\right)\left[a_1+2a_2\phi+......na_nu^{n-1}\right]
\end{aligned} \tag{4.5}$$

Equating the order of the highest degree of ϕ in both sides of the above we get $n+2=2n$, i.e. $n=2$. Thus $u(\xi)$ becomes in the form $u(\xi)=a_0+a_1\phi+a_2\phi^2$ thus we have

$$\frac{du}{d\phi}=a_1+2a_2\phi;\qquad \frac{d^2u}{d\phi^2}=2a_2$$

Putting these values in the equation (4.4) we get

$$\begin{aligned}
&2l^3\delta\left[\left(\sigma+\phi^2\right)^2a_2+\phi\left(\sigma+\phi^2\right)\left(a_1+2a_2\phi\right)\right]=\\
&l^2\gamma\left(\sigma+\phi^2\right)\left(a_1+2a_2\phi\right)-m\left(a_0+a_1\phi+a_2\phi^2\right)-\frac{l}{2}\left(a_0+a_1\phi+a_2\phi^2\right)^2
\end{aligned} \tag{4.6}$$

Comparing the coefficient of $\phi^0,\phi^1,\phi^2,\phi^3$ *and* ϕ^4 in the above equation, we get following set of equations

$$\left.\begin{aligned}
&\phi^4: l^3\delta 6a_2=-\frac{la_2{}^2}{2}\\
&\phi^3: l^3\delta 2a_1=2a_2\gamma l^2-la_1a_2\\
&\phi^2: l^3\delta 8a_2\sigma=l^2\gamma a_1-\omega a_2-\frac{l}{2}\left(a_1{}^2+2a_0a_2\right)\\
&\phi: l^3\delta 2\sigma a_1=2a_2l^2\sigma\gamma-ma_1-la_0a_1\\
&\phi^0: l^3\delta 2\sigma^2a_2=a_1l^2\sigma\gamma-ma_0-\frac{la_0{}^2}{2}
\end{aligned}\right\} \tag{4.7}$$

Solving the above equations in (4.7) we get,

$$a_0=-12\delta l^2\sigma-\frac{m}{l},\qquad a_1=\frac{12l\gamma}{5},\qquad a_2=-12l^2\delta$$

And l and m are connected by the relation

$$m = 576 l^6 \sigma^2 \delta^2$$

Thus the general solution of (4.1) can be written as

$$u(x,t) = \begin{cases} \left. \begin{array}{l} a_0 - a_1\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\left(kx^\alpha + \omega t^\alpha\right)\right) + a_2\sqrt{-\sigma}\tanh^2\left(\sqrt{-\sigma}\left(kx^\alpha + \omega t^\alpha\right)\right) \\ a_0 - a_1\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\left(kx^\alpha + \omega t^\alpha\right)\right) + a_2\sqrt{-\sigma}\coth^2\left(\sqrt{-\sigma}\left(kx^\alpha + \omega t^\alpha\right)\right) \end{array} \right\} & \text{for} \quad \sigma < 0 \\ \left. \begin{array}{l} a_0 + a_1\sqrt{\sigma}\tan\left(\sqrt{\sigma}\left(kx^\alpha + \omega t^\alpha\right)\right) + a_2\sqrt{\sigma}\tan^2\left(\sqrt{\sigma}\left(kx^\alpha + \omega t^\alpha\right)\right) \\ a_0 - a_1\sqrt{\sigma}\cot\left(\sqrt{\sigma}\left(kx^\alpha + \omega t^\alpha\right)\right) + a_2\sqrt{\sigma}\cot^2\left(\sqrt{\sigma}\left(kx^\alpha + \omega t^\alpha\right)\right) \end{array} \right\} & \text{for} \quad \sigma > 0 \\ a_0 - a_1\dfrac{1}{\left(kx^\alpha + \omega t^\alpha\right)} + a_2\dfrac{1}{\left(kx^\alpha + \omega t^\alpha\right)} \quad \text{for} \quad \sigma = 0 & \end{cases} \tag{4.8}$$

Thus we obtained the exact solution of space time fractional generalized KdV Burger equation using the transformation of He [5,6] and the Generalized Tanh method.

5. Solution of Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity using Generalized Tanh Method and complex fractional transformation

Here we consider the Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity

$$D_t^\alpha u + aD_x^\alpha u + (b_2 u^{2n} + b_3 u)D_x^\alpha u + cD_{xx}^{2\alpha} u + kD_{xxt}^{3\alpha} u = 0 \tag{5.1}$$

Here a represents the strength of defection or drifting. Strengths of the two nonlinear terms are measured by $b_{2,}\ b_3$ while the exponent n stands for the power law of nonlinearity parameter. The parameters c, k are the dissipative diffraction coefficient. Putting $n = 1$ Eq. (5.1) becomes

$$D_t^\alpha u + aD_x^\alpha u + (b_2 u^2 + b_3 u)D_x^\alpha u + cD_{xx}^{2\alpha} u + kD_{xxt}^{3\alpha} u = 0 \tag{5.2}$$

Now use the transformation (3.2) in equation (5.2) and then it reduces to the following form,

$$mu_\xi + lau_\xi + l(b_2 u^2 + b_3 u)u_\xi + cl^2 u_{\xi\xi} + kl^3 m u_{\xi\xi\xi} = 0 \tag{5.3}$$

Integrating equation (5.3) with respect to ξ and using the boundary condition of solitary type solution we get

$$mu + alu + \left(\frac{1}{3}b_2u^3 + \frac{1}{2}b_3u^2\right)l + cl^2u_\xi + kl^3mu_{\xi\xi} = 0 \tag{5.4}$$

The equation (5.4) can be written as

$$\begin{aligned} &u_{\xi\xi} + \frac{m+al}{kl^3m}u + \frac{b_3}{2kl^2m}u^2 + \frac{b_2}{3kl^2m}u^3 + \frac{c}{kml}u_\xi = 0 \\ &u_{\xi\xi} + Du_\xi + Au + Bu^2 + Cu^3 = 0 \end{aligned} \tag{5.5}$$

where A, B, C, D given below (Note that term D here should be distinguished from derivative operator-that we have used)

$$A = \frac{m+al}{kl^3m}, \qquad B = \frac{b_3}{2kl^2m}, \qquad C = \frac{b_2}{3kl^2m}, \qquad D = \frac{c}{klm}$$

Now again change the independent variable $\xi \to \phi$ where ϕ satisfies the equation (2.11). Hence the equation (5.5) becomes

$$\left(\sigma+\phi^2\right)^2 \frac{d^2u}{d\phi^2} + 2\phi\left(\sigma+\phi^2\right)\frac{du}{d\phi} + D\left(\sigma+\phi^2\right)\frac{du}{d\phi} + Au + Bu^2 + Cu^3 = 0 \tag{5.6}$$

Let the solution of the equation (5.5) be expressed as $u(\xi) = S = \sum_{i=0}^{n} a_i\phi^i$ i.e. in (3.7). Then the equation (5.2) can be expressed as

$$\begin{aligned} &\left(\sigma+\phi^2\right)^2\left[2a_2 + 3a_3\phi + ...n(n-1)a_nu^{n-2}\right] \\ &\quad + 2\phi\left(\sigma+\phi^2\right)\left[a_1 + 2a_2\phi + ...na_nu^{n-1}\right] \\ &\quad + D\left(\sigma+\phi^2\right)\left[a_1 + 2a_2\phi + ...na_nu^{n-1}\right] \\ &\quad = -A\left[a_0 + a_1\phi + ...a_nu^n\right] \\ &\quad - B\left[a_0 + a_1\phi + ...a_nu^n\right]^2 - \left[a_0 + a_1\phi + ...a_nu^n\right]^3 \end{aligned} \tag{5.7}$$

Equating the order of the highest degree of ϕ in both sides of the above we get $n+2=3n$ i.e. $n=1$. Then $u(\xi)$ becomes in the form $u(\xi) = a_0 + a_1\phi$ and so we have

$$\frac{du}{d\phi} = a_1, \qquad \frac{d^2u}{d\phi^2} = 0$$

Putting these values in the equation (5.6) we get

$$(2\phi + D)\left(\sigma+\phi^2\right)a_1 = A\left(a_0 + a_1\phi\right) + B\left(a_0 + a_1\phi\right)^2 + C\left(a_0 + a_1\phi\right)^3 \tag{5.8}$$

Comparing the coefficient of ϕ^0, ϕ^1, ϕ^2 and ϕ^3 from the equation (5.8) we get,

$$\left.\begin{array}{l} Aa_0 + Ba_0^2 + Ca_0^3 + D\sigma a_1 = 0 \\ Aa_1 + 2Ba_0 a_1 + 3Ca_0^2 a_1 + 2\sigma a_1 = 0 \\ Ba_1^2 + 3Ca_0 a_1^2 + Da_1 = 0 \\ 2a_1 + Ca_1^3 = 0 \end{array}\right\} \tag{5.9}$$

Solving the equations in (5.9) we get

$$a_0 = \frac{\sqrt{-6b_2}}{18kml^2\sqrt{klm}} - \frac{b_3 l}{2b_2}, \qquad a_1 = l\sqrt{-\frac{3klm}{b_2}}$$

And l and m are connected by the following relation

$$\frac{b_3}{2klm}\left(\frac{\sqrt{-6b_2}}{18kml^2\sqrt{klm}} - \frac{b_3}{2b_2}\right)^2 + \frac{2b_2}{3klm}\left(\frac{\sqrt{-6b_2}}{18kml^2\sqrt{klm}} - \frac{b_3}{2b_2}\right)^3$$
$$+2\sigma\left(\frac{\sqrt{-6b_2}}{18kml^2\sqrt{klm}} - \frac{b_3}{2b_2}\right) - \frac{c}{km} l\sigma\sqrt{-\frac{3klm}{b_2}} = 0$$

Hence the general solution of the equation (5.2) can be written as,

$$u(x,t) = \begin{cases} \left(\frac{\sqrt{-6b_2}}{18kl^2m\sqrt{klm}} - \frac{b_3}{2b_2}\right) - l\sqrt{-\frac{3klm}{b_2}}\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\left(lx^\alpha + mt^\alpha\right)\right) & \text{for} \quad \sigma < 0 \\ \left(\frac{\sqrt{-6b_2}}{18kl^2m\sqrt{klm}} - \frac{b_3}{2b_2}\right) - l\sqrt{-\frac{3klm}{b_2}}\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\left(lx^\alpha + mt^\alpha\right)\right) & \text{for} \quad \sigma < 0 \\ \left(\frac{\sqrt{-6b_2}}{18kl^2m\sqrt{klm}} - \frac{b_3}{2b_2}\right) + l\sqrt{-\frac{3klm}{b_2}}\sqrt{\sigma}\tan\left(\sqrt{\sigma}\left(lx^\alpha + mt^\alpha\right)\right) & \text{for} \quad \sigma > 0 \\ \left(\frac{\sqrt{-6b_2}}{18kl^2m\sqrt{klm}} - \frac{b_3}{2b_2}\right) - l\sqrt{-\frac{3klm}{b_2}}\sqrt{\sigma}\cot\left(\sqrt{\sigma}\left(lx^\alpha + mt^\alpha\right)\right) & \text{for} \quad \sigma > 0 \\ \left(\frac{\sqrt{-6b_2}}{18kl^2m\sqrt{klm}} - \frac{b_3}{2b_2}\right) - l\sqrt{-\frac{3klm}{b_2}}\frac{1}{\left(lx^\alpha + mt^\alpha\right)} & \text{for} \quad \sigma = 0 \end{cases} \tag{5.10}$$

6. Numerical presentation of the solutions

In this section, we gave the graphical presentation of the solution obtained in the previous sections for different values of the order of fractional derivative (α).

Figure-1 and figure-2 respectively represents the graphical presentation of the first (in terms Tanh) and third (in terms tan) solution of (3.10) respectively for $\mu = 4, \beta = -1, l = 1, m = 1$. In figure-1 the solution for $\alpha = 1$ represents the shock solution. With the decrease of α the shock type solution disappears and range of the solution increases. Figure-2 represents the periodic solution for $\alpha = 1$ and with the decrease of α the range of the solution increases and periodic pattern reduces to a new type of pattern.

Figure-3 and 4 represents the graphical representation of the first and third solution (4.8). In figure-3 for $\alpha = 1$ the pattern of the solution is soliton type and with the decrease of α a new type of graphical presentation arises. Here also the range of solution increases with the decrease of order of derivative. In figure-4 multiple inverted soliton like solution arises for $\alpha = 1$.

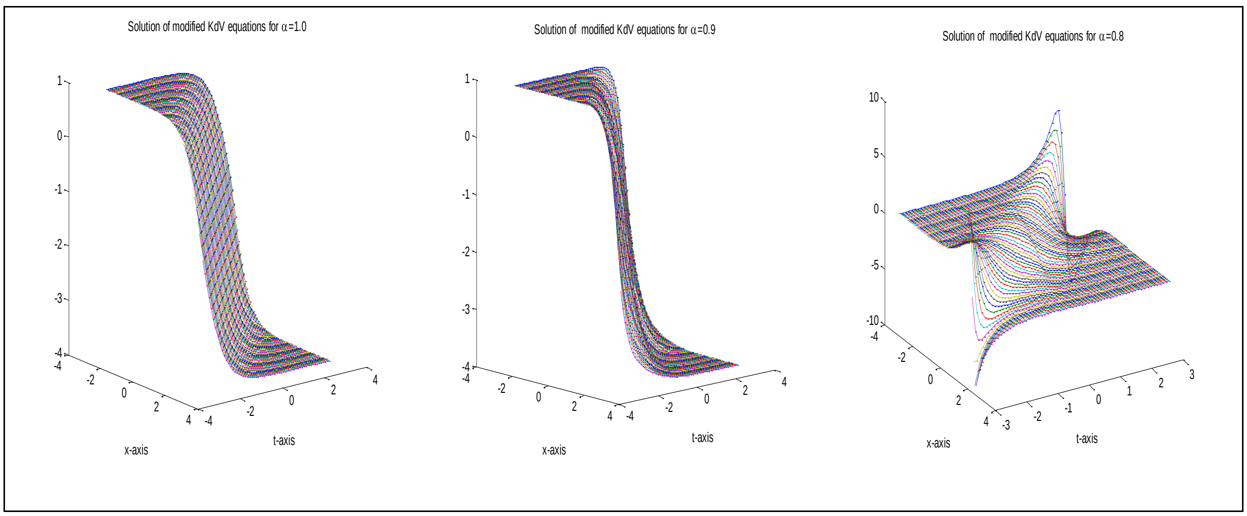

Fig-1: Graphical presentation of the tanh Solution of mKdV equation for $\alpha = 1.0, 0.9, 0.8$.

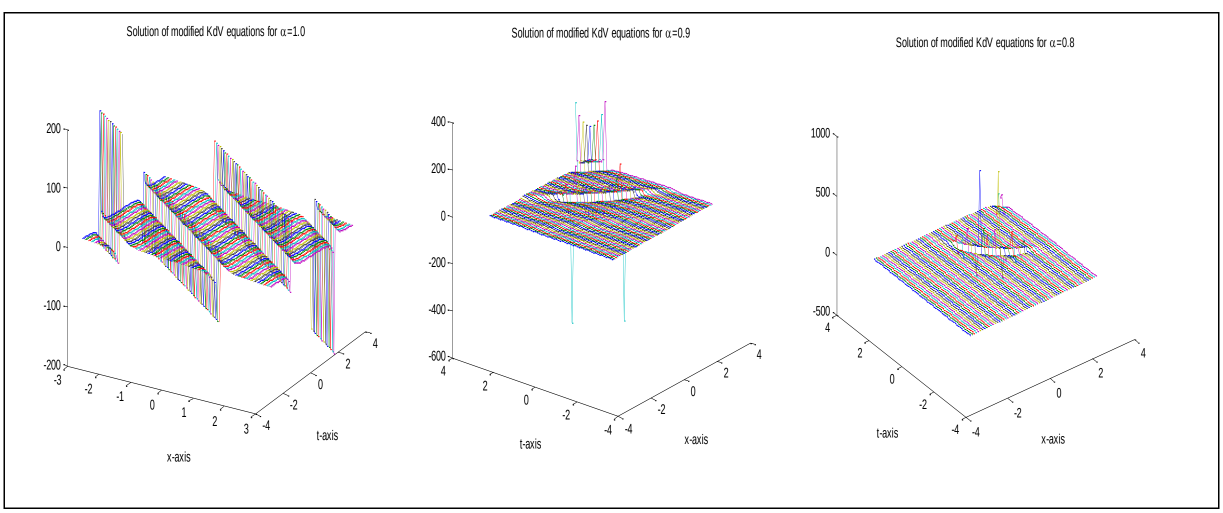

Fig-2: Fig-1: Graphical presentation of the tan Solution of mKdV for $\alpha = 1.0, 0.9, 0.8$.

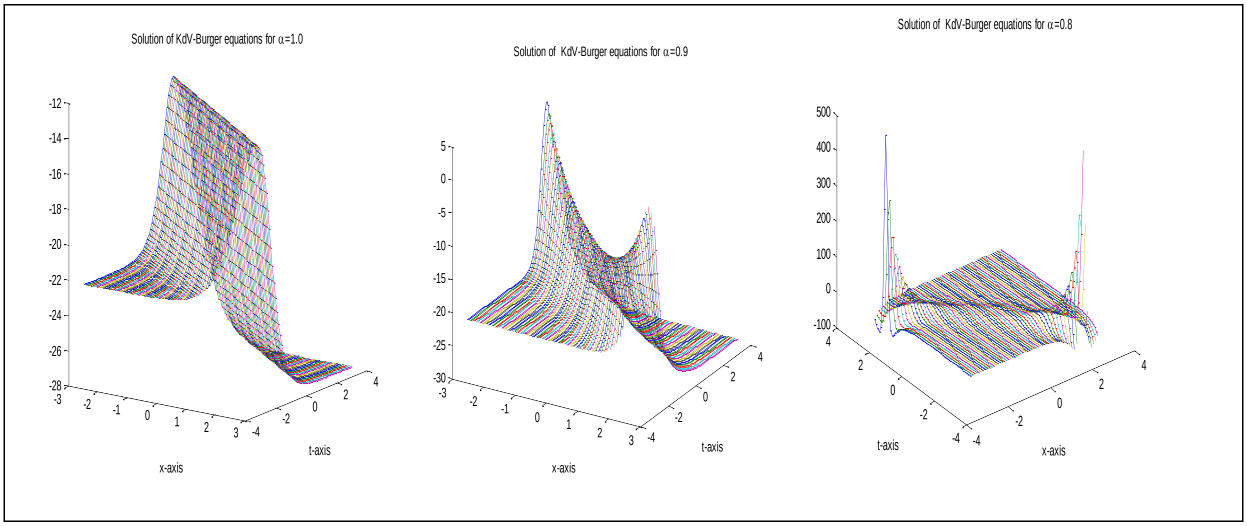

Fig-3: Graphical presentation of the tanh Solution of KdV-Burger equation for $\alpha = 1.0, 0.9, 0.8$.

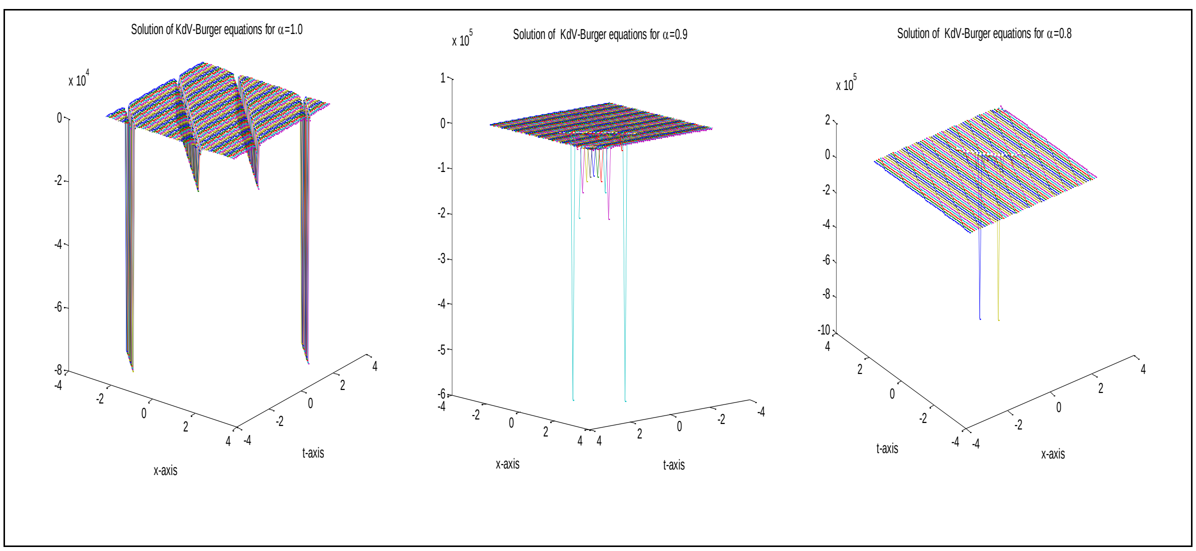

Fig-4: Graphical presentation of the tan Solution of KdV-Burger equation for $\alpha = 1.0, 0.9, 0.8$.

Figure-3 and figure-4 represents the graphical presentation of the first and third solution of (4.8) respectively for $\delta = 1, \gamma = -1, l = 1, m = 1$.

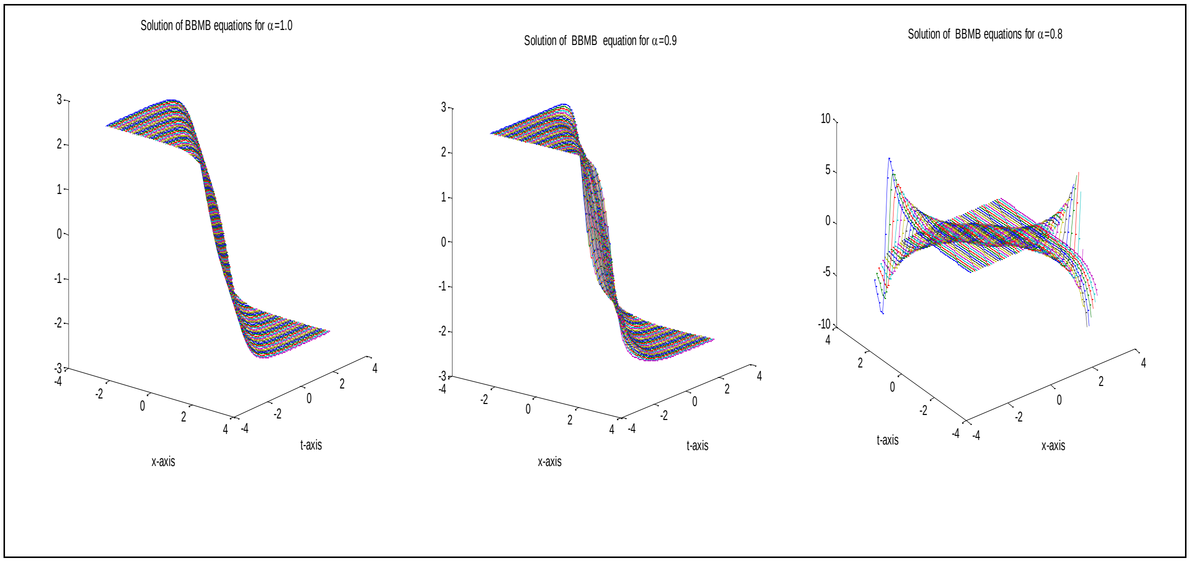

Fig-5: Graphical presentation of the tan Solution of BBMB equation for $\alpha = 1.0, 0.9, 0.8$.

7. Conclusion

In this paper we found the solutions of space-time fractional generalized KdV equation, KdV Burger equation, and Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity using complex fractional transformation and the generalized Tanh hyperbolic method. Using this method the solutions are expressed in terms the hyperbolic, trigonometric and the rational functions. The obtained solutions are new type which is also predicted from the graphical presentation of the solution. Using these type formulations other non-linear fractional differential equations can be easily solved. From the numerical presentation of the solution it is clear that with the change of order of derivative solution pattern changes.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] S. Das. Functional Fractional Calculus 2nd Edition, Springer-Verlag. (2011).

[2] I.Podlubny. Fractional Differential Equations. Mathematics in Science and Engineering, vol. 198. Academic Press, San Diego (1999).

[3] X. Li and W. Chen. Analytical study of fractional anomulus diffusion in a half-plane. J. Phys. A: Math. Theor. 43(49) (2010), 1-11.

[4] D. Qian, C. Li, R. P. Agarwal and P. J.Y Wong. Stability analysis of fractional differential system with Riemann-Liouville derivative. Math. Comput. Model. 52(2010), 862-874.

[5] A.M.A. El-Sayed, M. Gaber, The adomian decomposition method for solving partial differential equations of fractal order in finite domains, Phys. Lett. A 359(20) (2006).175-182.

[6] A.M.A. El-Sayed, S.H. Behiry, W.E. Raslan, Adomian's decomposition method for solving an intermediate fractional advectiondispersion equation, Comput. Math. Appl 59(5) (2010),1759-1765.

[7] J. Cang, Y. Tan, H. Xu and S. J. Liao, Series solutions of non-linear Riccati differential equations with fractional order, Chaos, Solitons Fractals 40 (2009), 1-9.

[8] J.H. He. Homotopy perturbation technique, Comput Methods. Appl. Mech. Eng. 178(3-4) (1999), 257-262.

[9] J.H .He. A coupling method of a homotopy technique and a perturbation technique for nonlinear problems. Int. J. Nonlinear Mech. 35(2000), 37–43.

[10] H. Jafari, S. Momani, Solving fractional diffusion and wave equations by modified homotopy perturbation method, Phys. Lett. A 370 (2007), 388-396.

[11] E. Fan, Y.C. Hon, Generalized tanh Method Extended to Special types of Non-linear equations, Z. Naturforsh 57(2002), 692-700.

[12] S. Zhang, Q.A. Zong, D. Liu, Q. Gao, A generalized exp-function method for fractional Riccati differential equations, Commun. Fract. Calc. 1 (2010), 48-51.

[13] S. Zhang and H. Q. Zhang, Fractional sub-equation method and its applications to nonlinear fractional PDEs, Phys. Lett. A. 375(2011) , 1069.

[14] U. Ghosh, S. Sarkar, S. Das. Analytical Solutions of Classical and Fractional KP-Burger Equation and Coupled KdV Equation. Comput. Methods Sci. Technol. 22(3) (2016), 143-152.

[15] L. Huibin, W. Kelin, Exact solutions for two nonlinear equations: I, J. Phys. A: Math. Gen. 23(1990), 3923-3928.

[16] A.M. Wazwaz, The tanh method for travelling wave solutions of nonlinear equations, Appl. Math. Comput. 154 (2004), 713-723.

[17] E. Fan, Extended Tanh-Function Method and Its Applications to Nonlinear Equations, Phys. Lett. A 277(4-5) (2000), 212-218.

[18] R. W. Ibrahim, Fractional complex transforms for fractional differential equations, Adv. Differ. Equ. 2012 (2012), Article ID 192.

[19] He, J.H., Elagan, S.K.: Li ZB. Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus. Phys. Lett. A 376 (2012), 257–259

[20] Li, Z.B., He, J.H.: Fractional complex transform for fractional differential equations. Math. Comput. Appl. 15 (2010), 970–973.

[21] G. Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series of non differentiable functions further results, Comput. Math. Appl. 51(9-10) (2006), 1367-1376.