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SOME SEPARATION AXIOMS IN FUZZY SOFT BITOPOLOGICAL SPACES

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Abstract. It is known that separation axioms are playing a vital role in study of topological spaces. In this paper, Some Separation axioms have been studied in context of fuzzy soft bitopological spaces. we introduce and study the notions of pairwise fuzzy soft T_i-spaces; $(i = 0,1,2)$. This study focuses on question: If a fuzzy soft bitopological space (X,E,τ_1,τ_2) is a pairwise fuzzy soft T_i-space; $(i = 0,1,2)$, what can be said about the following situations:

(1) both (X,E,τ_1) and (X,E,τ_2) are fuzzy soft T_i-spaces; $(i = 0,1,2)$,

(2) (X,E,τ_{12}) is a supra fuzzy soft T_i-space; $(i = 0,1,2)$.

(3) fuzzy soft subspaces $(X,E,\tau_{1_Y},\tau_{2_Y})$ are fuzzy soft T_i-spaces for $\phi \neq Y \subset X$; $(i = 0,1,2)$. Finally, characterizations theorem is proved for pairwise fuzzy soft Hausdorff space.

Keywords: soft set; fuzzy set; fuzzy soft set; fuzzy soft point; fuzzy soft topological space; fuzzy soft bitopological space; supra fuzzy soft topological space; $\tau_i(i = 1,2)$-fuzzy soft open (closed) set.

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1. Introduction

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In the year 1965, Zadeh [25] introduced the concept of fuzzy set theory and its applications can be found in many branches of mathematical and engineering sciences including management science, control engineering, computer science and artificial intelligence (see [3], [4]).

In the year 1999, Russian researcher Molodtsov [15] initiated the concept of soft sets as a new mathematical tool to deal with uncertainties while modeling problems in engineering physics, computer science, economics, social sciences and medical sciences. In 2003, Maji et. al [14] studied the theory of soft sets initiated by Molodtsov. They defined equality of two soft sets, subset and super set of a soft set, complement of a soft set, null soft set and absolute soft set with examples. Soft binary operations like AND, OR and also the operations of union and intersection were also defined. In 2005, D. Chen [2] presented a new definition of soft set parametrization reduction and a comparison of it with attribute reduction in rough set theory.

In 1963, J. C. Kelly [8] first initiated the concept of bitopological spaces and other authors have contributed to development and construction some properties of such spaces (see, [21], [19]).

In 2014, B. M. Ittanagi [5] introduced and studied the concept of soft bitopological spaces and other authors have contributed to development and construction some properties of such spaces (see [6], [7], [18], [9] , [11], [10], [17], [20], [24]).

In 2015, Mukherjee and Park [16] were first introduced the notion of fuzzy soft bitopological space and they studied some of its basic properties. Also, my work in [23] was an extension and continuation of studying in this trend by introducing and characterizing a new type of fuzzy soft sets in fuzzy soft bitopological spaces. It is known that separation axioms are playing a vital role in study of topological spaces. In this paper, Some Separation axioms have been studied in context of fuzzy soft bitopological spaces. we introduce and study the notions of pairwise fuzzy soft T_i-spaces; $(i = 0, 1, 2)$. This study focuses on question: If a fuzzy soft bitopological space (X, E, τ_1, τ_2) is a pairwise fuzzy soft T_i-space; $(i = 0, 1, 2)$, what can be said about the following situations:

(1) both (X, E, τ_1) and (X, E, τ_2) are fuzzy soft T_i-spaces; $(i = 0, 1, 2)$,

(2) (X, E, τ_{12}) is a supra fuzzy soft T_i-space; $(i = 0, 1, 2)$.

(3) fuzzy soft subspaces $(X,E,\tau_{1_Y},\tau_{2_Y})$ are fuzzy soft T_i-spaces for $\phi \neq Y \subset X$; $(i=0,1,2)$. Finally, characterizations theorem is proved for pairwise fuzzy soft Hausdorff space.

2. Preliminaries

In this section we have presented the basic definitions and results of fuzzy soft set and fuzzy soft bitopological space which will be a central role in our paper.

Throughout our discussion, X refers to an initial universe, E the set of all parameters for X and $P(X)$ denotes the power set of X.

Definition 2.1. [25] A fuzzy set A in a non-empty set X is characterized by a membership function $\mu_A : X \to [0,1] = I$ whose value $\mu_A(x)$ represents the "degree of membership" of x in A for $x \in X$. Let I^X denotes the family of all fuzzy sets on X.

A member A in I^X is contained in a member B of I^X denoted $A \leq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ for every $x \in X$ (see [25]).

Let $A,B \in I^X$, we have the following fuzzy sets (see [25]).

(1) $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ for all $x \in X$. (Equality),

(2) $C = A \wedge B \in I^X$ by $\mu_C(x) = min\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$. (Intersction),

(3) $D = A \vee B \in I^X$ by $\mu_C(x) = max\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$.(Union),

(4) $E = A^c \in I^X$ by $\mu_E(x) = 1 - \mu_A(x)$ for all $x \in X$. (Complement).

Definition 2.2. [25] An empty fuzzy set on X denoted by 0_X is a function which maps each $x \in X$ to 0. That is, $0_X(x) = 0$ for all $x \in X$.

A universal fuzzy set denoted by 1_X is a function which maps each $x \in X$ to 1. That is, $1_X(x) = 1$ for all $x \in X$.

Definition 2.3. [15] Let $A \subseteq E$. A pair (F,A) is called a soft set over X if F is a mapping $F : A \to P(X)$.

Definition 2.4. [13] Let $A \subseteq E$. A pair (f,A), denoted by f_A, is called a fuzzy soft set over X, where f is a mapping given by $f : A \to I^X$ defined by $f_A(e) = \mu^e_{f_A}$ where

$$\mu^e_{f_A} = \begin{cases} \tilde{0}, & \text{if } e \notin A; \\ otherwise, & \text{if } e \in A. \end{cases}$$

$\widetilde{(X,E)}$ denotes the family of all fuzzy soft sets over X.

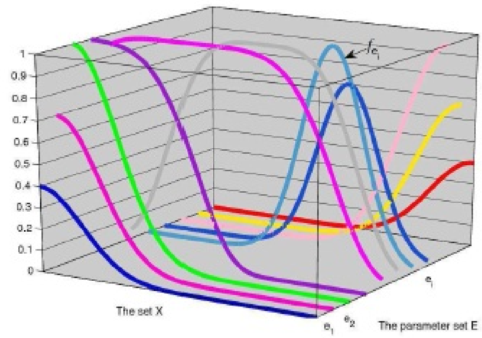

FIGURE 1. A fuzzy soft set f_A

Definition 2.5. [14] A fuzzy soft set $F_A \tilde{\in} \widetilde{(X,E)}$ is said to be:

(a) NULL fuzzy soft set, denoted by $\tilde{0}_E$, if for all $e \in A, f_A(e) = 0_X$.

(b) absolute fuzzy soft set, denoted by $\tilde{1}_E$, if for all $e \in E, f_A(e) = 1_X$.

Definition 2.6. [22] The complement of a fuzzy soft set (f,A), denoted by $(f,A)^c$, is defined by $(f,A)^c = (f^c,A), f_A^c : E \to I^X$ is a mapping given by $\mu^e_{f_A^c} = 1_X - \mu^e_{f_A}$, where $1_X(x) = 1$, for all $x \in X$. Clearly $(f_A^c)^c = f_A$.

Definition 2.7. [22] Let $f_A, g_B \tilde{\in} \widetilde{(X,E)}$. f_A is fuzzy soft subset of g_B, denoted by $f_A \tilde{\subseteq} g_B$, if $A \subseteq B$ and $\mu^e_{f_A} \leq \mu^e_{g_B}$ for all $e \in A$, i.e. $\mu^e_{f_A}(x) \leq \mu^e_{g_B}(x)$ for all $x \in X$ and for all $e \in A$.

Definition 2.8. [22] Let $f_A, g_B \tilde{\in} \widetilde{(X,E)}$.The union of f_A and g_B is also a fuzzy soft set h_C, where $C = A \cup B$ and for all $e \in C, h_C(e) = \mu^e_{h_c} = \mu^e_{f_A} \vee \mu^e_{g_B}$. Here we write $h_C = f_A \tilde{\cup} g_B$.

Definition 2.9. [22] Let $f_A, g_B \tilde{\in} \widetilde{(X,E)}$.The intersection of f_A and g_B is also a fuzzy soft set d_C, where $C = A \cap B$ and for all $e \in C, d_C(e) = \mu^e_{d_c} = \mu^e_{f_A} \wedge \mu^e_{g_B}$. Here we write $d_C = f_A \tilde{\cap} g_B$.

Definition 2.10. [12] The fuzzy soft set $f_A \tilde{\in} \widetilde{(X,E)}$ is called fuzzy soft point if there exist $x \in X$ and $e \in E$ such that $\mu^e_{f_A}(x) = \alpha(0 < \alpha \leq 1)$ and $\mu^e_{f_A}(y) = 0$ for each $y \in X - \{x\}$, and this fuzzy soft point is denoted by x^e_α or f_e.

Definition 2.11. [12] The fuzzy soft point f_e is said to be belonging to the fuzzy soft set (g,A), denoted by $f_e \tilde{\in} (g,A)$, if for the element $e \in A$, $\alpha \leq \mu^e_{g_A}(x), (0 < \alpha \leq 1)$.

Definition 2.12. [1] Let f_A be fuzzy soft set over X. The two fuzzy soft points $f_{e_1}, f_{e_2} \tilde{\in} f_A$ are said to be equal if $\mu_{f_{e_1}}(x) = \mu_{f_{e_2}}(x)$ for all $x \in X$. Thus $f_{e_1} \neq f_{e_2}$ (i.e. f_{e_1}, f_{e_2} are two distinct fuzzy soft points)if and only $\mu_{f_{e_1}}(x) \neq \mu_{f_{e_2}}(x)$for all$x \in X$.

Definition 2.13. [22] A fuzzy soft topology τ over (X,E) is a family of fuzzy soft sets over (X,E) satisfying the following properties

(i) $\tilde{0}_E, \tilde{1}_E \in \tau$

(ii) if $f_A, g_B \in \tau$, then $f_A \tilde{\cap} g_B \in \tau$,

(iii) if $f_{A_\alpha} \in \tau$ for all $\alpha \in \Delta$ an index set, then $\tilde{\bigcup}_{\alpha\in\Delta} f_{A_\alpha} \in \tau$.

Definition 2.14. [16] If τ is a fuzzy soft topology on (X,E) the triple (X,E,τ) is said to be a fuzzy soft topological space. Also each member of τ is called a fuzzy soft open set in (X,E,τ).

The complement of a fuzzy soft open set is a fuzzy soft closed set.

Definition 2.15. [12] Let (X,E,τ) be a fuzzy soft topological space and let $Y \subseteq X$. Let h_E^Y be a fuzzy soft set over (Y,E) such that $h_E^Y : E \to I^Y$ such that $h_E^Y(e) = \mu^e_{h_E^Y}$ for all $e \in E$,

$$\mu^e_{h_E^Y}(x) = \begin{cases} 1, & x \in Y; \\ 0, & x \notin Y. \end{cases}$$

Let $\tau_Y = \{h_E^Y \tilde{\cap} g_B : g_B \in \tau\}$, then the fuzzy soft topology τ_Y on (Y,E) is called fuzzy soft subspace topology for (Y,E) and (Y,E,τ_Y) is called fuzzy soft subspace of (X,E,τ).

Definition 2.16. [16] Let (X,E,τ_1) and (X,E,τ_2) be the two different fuzzy soft topologies on (X,E). Then (X,E,τ_1,τ_2) is called a fuzzy soft bitopological space on which no seperation axioms are assumed unless explicitly stated.

The members of $\tau_i(i=1,2)$ are called $\tau_i(i=1,2)$-fuzzy soft open sets and the complement of $\tau_i(i=1,2)$- fuzzy soft open sets are called $\tau_i(i=1,2)$-fuzzy soft closed sets.

Definition 2.17. [16] Let (X,E,τ_1,τ_2) be a fuzzy soft bitopological space and $f_E \tilde{\in} \widetilde{(X,E)}$. Then the $\tau_i(i=1,2)$-fuzzy soft closure of f_E, denoted by $\tau_i cl(f_E)$, is the intersection of all $\tau_i(i=1,2)$-fuzzy soft closed supersets of f_E.

Clearly, $\tau_i cl(f_E)$ is the smallest $\tau_i(i=1,2)$-fuzzy soft closed set over (X,E) which contains f_E.

Definition 2.18. [16] A fuzzy soft set $f_E \tilde{\in} \widetilde{(X,E)}$ is called $\tau_1\tau_2$-fuzzy soft open set if $f_E = g_E \tilde{\cup} h_E$ such that $g_E \tilde{\in} \tau_1$ and $h_E \tilde{\in} \tau_2$.

The complement of $\tau_1\tau_2$-fuzzy soft open set is called $\tau_1\tau_2$-fuzzy soft closed set.

The family of all $\tau_1\tau_2$-fuzzy soft open (closed) sets in (X,E,τ_1,τ_2) is denoted by $\tau_1\tau_2 FSO(X,\tau_1,\tau_2)_E$ $(\tau_1\tau_2 FSC(X,\tau_1,\tau_2)_E)$, respectively.

Definition 2.19. [23] Let (X,E,τ_1,τ_2) be a soft bitopological space. Then, the family of all $\tau_1\tau_2$-fuzzy soft open sets is a supra fuzzy soft topology on (X,E). This supra fuzzy soft topology, will denoted by τ_{12}, i.e., $\tau_{12} = \tau_1\tau_2 FSO(X,\tau_1,\tau_2)_E = \{g_E = g_{1_E}\tilde{\cup}g_{2_E} : g_{i_E} \in \tau_i, i = 1,2\}$ and the triple (X,E,τ_{12}) is the supra fuzzy soft topological space associated to the fuzzy soft bitopological space (X,E,τ_1,τ_2).

3. Pairwise Fuzzy soft T_i-Spaces ; $(i = 0,1,2)$

In this section, we introduce and study the notions of Pairwise Fuzzy soft T_i-Spaces ; $(i = 0,1,2)$.

Definition 3.1. A fuzzy soft bitopological space (X,E,τ_1,τ_2) is said to be a pairwise fuzzy soft T_0-Space if for each pair of distinct fuzzy soft points f_e, g_e in $\widetilde{(X,E)}$ there exists a τ_1-fuzzy soft open set u_E such that $f_e\tilde{\in}u_E$ and $g_e\tilde{\notin}u_E$ or τ_2-fuzzy soft open set v_E such that $f_e\tilde{\notin}v_E$ and $g_e\tilde{\in}v_E$.

Example 3.2. Let X be an initial universe set and E be the non-empty set of parameters. Consider

$\tau_1 = \{\tilde{0}_E, \tilde{1}_E\}$ Fuzzy soft indiscrete topology

$\tau_2 = \{f_E | f_E$ is a fuzzy soft set over $(X,E)\}$ Fuzzy soft discrete topology.

Then (X,E,τ_1,τ_2) is a fuzzy soft bitopological space and is a pairwise fuzzy soft T_0-Space.

Proposition 3.3. Let (X,E,τ_1,τ_2) be a fuzzy soft bitopological space. If (X,E,τ_1) or (X,E,τ_2) is a fuzzy soft T_0-Space then (X,E,τ_1,τ_2) is a pairwise fuzzy soft T_0-Space.

Proof. Let $f_e, g_e\tilde{\in}\widetilde{(X,E)}$ such that $f_e \neq g_e$. Suppose that (X,E,τ_1) or (X,E,τ_2) is a fuzzy soft T_0-Space. Then there exist some $u_E \in \tau_1$ such that $f_e\tilde{\in}u_E$ and $g_e\tilde{\notin}u_E$ or some $v_E \in \tau_2$ such that $g_e\tilde{\in}v_E$ and $f_e\tilde{\notin}v_E$. In either case we obtain the requirement and so (X,E,τ_1,τ_2) is a pairwise fuzzy soft T_0-Space. This completes the proof.

Remark 3.4. The converse of Proposition 3.3 is not true in general.

Example 3.5. Let $X = \{x_1,x_2,x_3,x_4\}$, $E = \{e_1,e_2\}$ and

$\tau_1 = \{\tilde{0}_E, \tilde{1}_E, f_E\}$, and

$\tau_2 = \{\tilde{0}_E, \tilde{1}_E, g_{1_E}, g_{2_E}, g_{3_E}\}$,

where f_E, g_{1_E}, g_{2_E} and g_{3_E} are fuzzy soft sets over (X,E) defined as follows:

$f_E = \{f(e_1) = \{x_1/0.5, x_2/0.0, x_3/0.7, x_4/0.0\}, f(e_2) = \{x_1/0.0, x_2/0.0, x_3/0.3, x_4/0.0\}$ and

$g_{1_E} = \{g_1(e_1) = \{x_1/0.0, x_2/0.0, x_3/0.3, x_4/0.5\}, g_1(e_2) = \{x_1/0.3, x_2/0.0, x_3/0.0, x_4/0.6\}\}$,

$g_{2_E} = \{g_2(e_1) = \{x_1/0.0, x_2/0.4, x_3/0.0, x_4/0.5\}, g_2(e_2) = \{x_1/0.0, x_2/0.1, x_3/0.0, x_4/0.0\}\}$,

$g_{3_E} = \{g_3(e_1) = \{x_1/0.0, x_2/0.4, x_3/0.3, x_4/0.5\}, g_3(e_2) = \{x_1/0.3, x_2/0.1, x_3/0.0, x_4/0.6\}\}$,

Then τ_1 and τ_2 are two fuzzy soft topologies over (X,E). Therefore (X,E,τ_1,τ_2) is a fuzzy soft bitopological space.

Now, for all $s = 1,2$;

$f(e_s), g_k(e_s) (k = 1,2,3) \tilde{\in} \widetilde{(X,E)}$ and $f_E \in \tau_1$ such that $f(e_s) \tilde{\in} f_E, g_k(e_s) \tilde{\notin} f_E$,

$g_1(e_s), g_2(e_s) \tilde{\in} \widetilde{(X,E)}$ and $g_{1_E} \in \tau_2$ such that $g_1(e_s) \tilde{\in} g_{1_E}, g_2(e_s) \tilde{\notin} g_{1_E}$,

$g_1(e_s), g_3(e_s) \tilde{\in} \widetilde{(X,E)}$ and $g_{1_E} \in \tau_2$ such that $g_1(e_s) \tilde{\in} g_{1_E}, g_3(e_s) \tilde{\notin} g_{1_E}$,

$g_2(e_s), g_3(e_s) \tilde{\in} \widetilde{(X,E)}$ and $g_{2_E} \in \tau_2$ such that $g_2(e_s) \tilde{\in} g_{2_E}, g_3(e_s) \tilde{\notin} g_{2_E}$.

Thus (X,E,τ_1,τ_2) is a pairwise fuzzy soft T_0-Space.

We observe that $f(e_1), f(e_2) \tilde{\in} \widetilde{(X,E)}$ and there does not exist any $f_E \tilde{\in} \tau_1$ such that $f(e_1) \tilde{\in} f_E$, $f(e_2) \tilde{\notin} f_E$ or $f(e_2) \tilde{\in} f_E$, $f(e_1) \tilde{\notin} f_E$. Therefore (X,E,τ_1) is not a fuzzy soft T_0-Space.

Similarly $g_1(e_1), g_1(e_2) \tilde{\in} \widetilde{(X,E)}$ and there does not exist any $g_E \tilde{\in} \tau_2$ such that $g_1(e_1) \tilde{\in} g_E$, $g_1(e_2) \tilde{\notin} f_E$ or $g_1(e_2) \tilde{\in} g_E$, $g_1(e_1) \tilde{\notin} f_E$, so (X,E,τ_2) is not a fuzzy soft T_0-Space.

Proposition 3.6. Let (X,E,τ_1,τ_2) be a fuzzy soft bitopological space. If (X,E,τ_1,τ_2) is a pairwise fuzzy soft T_0-Space, then (X,E,τ_{12}) is a supra fuzzy soft T_0-Space.

Proof. Let $f_e, g_e \tilde{\in} \widetilde{(X,E)}$ such that $f_e \neq g_e$. Then there exist some $u_E \in \tau_1$ such that $f_e \tilde{\in} u_E$ and $g_e \tilde{\notin} u_E$ or some $v_E \in \tau_2$ such that $g_e \tilde{\in} v_E$ and $f_e \tilde{\notin} v_E$. In either case $u_E, v_E \in \tau_{12}$. Hence (X,E,τ_{12}) is a supra fuzzy soft T_0-Space.

Remark 3.7. The converse of Proposition 3.6 is not true. This is shown by the following example:

Example 3.8. Let $X = \{x_1,x_2,x_3,x_4\}$, $E = \{e_1,e_2\}$ and

$\tau_1 = \{\tilde{0}_E, \tilde{1}_E, f_{1_E}, f_{2_E}\}$, and

$\tau_2 = \{\tilde{0}_E, \tilde{1}_E, g_E\}$,

where f_{1_E}, f_{2_E} and g_E are fuzzy soft sets over (X,E) defined as follows:

$f_{1_E} = \{f_1(e_1) = \{x_1/0.3, x_2/0.0, x_3/0.0, x_4/0.6\}, f_1(e_2) = \{x_1/0.0, x_2/0.0, x_3/0.0, x_4/0.7\}\}$,

$f_{2_E} = \{f_2(e_1) = \{x_1/0.0, x_2/0.0, x_3/0.0, x_4/0.6\}, f_2(e_2) = \{x_1/0.0, x_2/0.0, x_3/0.0, x_4/0.0 = 0_X\}\}$,

and

$g_E = \{g(e_1) = \{x_1/0.0, x_2/0.6, x_3/0.0, x_4/0.9\}, g(e_2) = \{x_1/0.1, x_2/0.4, x_3/0.0, x_4/0.0\}$.

Then τ_1 and τ_2 are two fuzzy soft topologies over $(X,E)\dot{T}$herefore (X,E,τ_1,τ_2) is a fuzzy soft bitopological space.

Now $\tau_{12} = \{\tilde{0}_E, \tilde{1}_E, f_{1_E}, f_{2_E}, \tilde{1}_E, g_E, h_E\}$ where

$h_E = f_{1_E} \tilde{\cup} g_E = \{h(e_1) = \{x_1/0.3, x_2/0.6, x_3/0.0, x_4/0.9\}, h(e_2) = \{x_1/0.1, x_2/0.4, x_3/0.0, x_4/0.7\}$.

So (X,E,τ_{12}) is a supra fuzzy soft Space.

For $f_1(e_1), g(e_1) \tilde{\in} \widetilde{(X,E)}$, we can not fined any fuzzy soft sets $f_E \in \tau_1$ or $g_E \in \tau_2$ such that $f_1(e_1) \tilde{\in} f_E, g(e_1) \tilde{\notin} f_E$ or $g(e_1) \tilde{\in} g_E, f_1(e_1) \tilde{\notin} f_E$.

Thus (X,E,τ_1,τ_2) is not pairwise fuzzy soft T_0-Space.

Now, for all $s = 1,2$;

$f_1(e_s), f_2(e_s) \tilde{\in} \widetilde{(X,E)}$ and $f_{1_E} \in \tau_1$ such that $f_1(e_s) \tilde{\in} f_{1_E}, f_2(e_s) \tilde{\notin} f_{1_E}$,

$f_1(e_s), g(e_s) \tilde{\in} \widetilde{(X,E)}$ and $g_E \in \tau_2$ such that $g(e_s) \tilde{\in} g_E, f_1(e_s) \tilde{\notin} g_E$,

$f_2(e_s), g(e_s) \tilde{\in} \widetilde{(X,E)}$ and $g_E \in \tau_2$ such that $g(e_s) \tilde{\in} g_E, f_2(e_s) \tilde{\notin} g_E$,

$g(e_s), h(e_s) \tilde{\in} \widetilde{(X,E)}$ and $h_E \in \tau_{12}$ such that $h(e_s) \tilde{\in} h_E, g(e_s) \tilde{\notin} h_E$.

Thus (X,E,τ_{12}) is a supra fuzzy soft T_0-Space.

Definition 3.9. Let (X,E,τ_1,τ_2) be a fuzzy soft bitopological space and let $Y \subseteq X$, then $(Y,E,\tau_{1_Y},\tau_{2_Y})$ is also a fuzzy soft bitopological space where $\tau_{i_Y} = \{h_E^Y \tilde{\cap} g_B : g_B \in \tau_i\}$, $i = 1,2$. This fuzzy soft bitopological space is called fuzzy soft bitopological subspace of (X,E,τ_1,τ_2).

Proposition 3.10. Let (X,E,τ_1,τ_2) be a fuzzy soft bitopological space and Y be a non-empty subset of X If (X,E,τ_1,τ_2) is a pairwise fuzzy soft T_0-Space, then $(Y,E,\tau_{1_Y},\tau_{2_Y})$ is also a pairwise fuzzy soft T_0-Space.

Proof. Let $f_e, g_e \tilde{\in} \widetilde{(X,E)}$ such that $f_e \neq g_e$. Then there exist some fuzzy soft set $u_E \in \tau_1$ or $v_E \in \tau_2$ such that $f_e \tilde{\in} u_E$ and $g_e \tilde{\notin} u_E$ or $g_e \tilde{\in} v_E$ and $f_e \tilde{\notin} v_E$. Suppose that there exist some some fuzzy soft set $u_E \in \tau_1$ such that $f_e \tilde{\in} u_E$ and $g_e \tilde{\notin} u_E$.

Now $f_e \tilde{\in} \widetilde{(Y,E)}$ implies that $f_e \tilde{\in} h_E^Y$. So $f_e \tilde{\in} h_E^Y$ and $f_e \tilde{\in} u_E$. Hence $f_e \tilde{\in} h_E^Y \tilde{\cap} u_E$.

Consider $g_e \tilde{\notin} u_E$, this means that $g_e \tilde{\notin} \{u(\acute{e})\}$ for some $\acute{e} \in E$. Then $g_e \tilde{\notin} h_E^Y \tilde{\cap} u_E$

Similarly it can also be proved that $g_e \tilde{\in} v_E$ and $f_e \tilde{\notin} v_E$ implies that $g_e \tilde{\in} h_E^Y \tilde{\cap} v_E$ and $f_e \tilde{\notin} h_E^Y \tilde{\cap} v_E$. Thus $(Y, E, \tau_{1_Y}, \tau_{2_Y})$ is a pairwise fuzzy soft T_0-Space.

Definition 3.11. A fuzzy soft bitopological space (X, E, τ_1, τ_2) is said to be a pairwise fuzzy soft T_1-Space if for every pair of distinct fuzzy soft points f_e, g_e in $(\tilde{X, E})$ there is a τ_1-fuzzy soft open set u_E such that $f_e \tilde{\in} u_E$ and $g_e \tilde{\notin} u_E$ and τ_2-fuzzy soft open set v_E such that $f_e \tilde{\notin} v_E$ and $g_e \tilde{\in} v_E$.

Example 3.12. Let $X = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$ and

$\tau_1 = \{\tilde{0}_E, \tilde{1}_E, f_{1_E}, f_{2_E}, f_{3_E}, f_{4_E}, f_{5_E}, f_{6_E}, f_{7_E}\}$, and

$\tau_2 = \{\tilde{0}_E, \tilde{1}_E, g_{1_E}, g_{2_E}, g_{3_E}, g_{4_E}, g_{5_E}, g_{6_E}\}$,

where $f_{1_E}, f_{2_E}, f_{3_E}, f_{4_E}, f_{5_E}, f_{6_E}, f_{7_E}, g_{1_E}, g_{2_E}, g_{3_E}, g_{4_E}, g_{5_E}$ and g_{6_E} are fuzzy soft sets over (X, E) defined as follows:

$f_{1_E} = \{f_1(e_1) = \{x_1/0.2, x_2/0.0, x_3/0.0, x_4/0.0\}, f_1(e_2) = \{x_1/0.3, x_2/0.0, x_3/0.6, x_4/0.0\}\}$,

$f_{2_E} = \{f_2(e_1) = \{x_1/0.0, x_2/0.0, x_3/0.5, x_4/0.0\}, f_2(e_2) = \{x_1/0.0, x_2/0.0, x_3/0.6, x_4/0.0\}\}$,

$f_{3_E} = \{f_3(e_1) = \{x_1/0.2, x_2/0.0, x_3/0.5, x_4/0.0\}, f_3(e_2) = \{x_1/0.3, x_2/0.0, x_3/0.6, x_4/0.0\}\}$,

$f_{4_E} = \{f_4(e_1) = \{x_1/0.0, x_2/0.0, x_3/0.0, x_4/0.0\} = 0_X, f_4(e_2) = \{x_1/0.0, x_2/0.0, x_3/0.6,$
$x_4/0.0\}\}$,

$f_{5_E} = \{f_5(e_1) = \{x_1/0.0, x_2/3.0, x_3/0.5, x_4/0.0\}, f_5(e_2) = \{x_1/0.0, x_2/0.7, x_3/0.0, x_4/0.0\}\}$,

$f_{6_E} = \{f_6(e_1) = \{x_1/0.0, x_2/0.0, x_3/0.5, x_4/0.0\}, f_6(e_2) = \{x_1/0.0, x_2/0.0, x_3/0.0, x_4/0.0\}$
$= 0_X\}$,

$f_{7_E} = \{f_7(e_1) = \{x_1/0.0, x_2/0.3, x_3/0.5, x_4/0.0\}, f_7(e_2) = \{x_1/0.0, x_2/0.7, x_3/0.6, x_4/0.0\}\}$,

and

$g_{1_E} = \{g_1(e_1) = \{x_1/0.0, x_2/0.8, x_3/0.0, x_4/0.0\}, g_1(e_2) = \{x_1/0.0, x_2/0.4, x_3/0.0, x_4/0.0\}\}$,

$g_{2_E} = \{g_2(e_1) = \{x_1/0.0, x_2/0.0, x_3/0.1, x_4/0.0\}, g_2(e_2) = \{x_1/0.0, x_2/0.0, x_3/0.5, x_4/0.0\}\}$,

$g_{3_E} = \{g_3(e_1) = \{x_1/0.0, x_2/0.8, x_3/0.1, x_4/0.0\}, g_3(e_2) = \{x_1/0.0, x_2/0.4, x_3/0.5, x_4/0.0\}\}$,

$g_{4_E} = \{g_4(e_1) = \{x_1/0.2, x_2/0.8, x_3/0.0, x_4/0.0\}, g_4(e_2) = \{x_1/0.9, x_2/0.4, x_3/0.0, x_4/0.0\}\}$,

$g_{5_E} = \{g_5(e_1) = \{x_1/0.2, x_2/0.0, x_3/0.1, x_4/0.0\}, g_5(e_2) = \{x_1/0.9, x_2/0.0, x_3/0.5, x_4/0.0\}\}$,

$g_{6_E} = \{g_6(e_1) = \{x_1/0.2, x_2/0.0, x_3/0.0, x_4/0.0\}, g_6(e_2) = \{x_1/0.9, x_2/0.0, x_3/0.0, x_4/0.0\}\}$,

Then τ_1 and τ_2 are two fuzzy soft topologies over (X, E). Therefore (X, E, τ_1, τ_2) is a fuzzy soft bitopological space. One can easily see that (X, E, τ_1, τ_2) is a pairwise fuzzy soft T_1-Space.

Proposition 3.13. Let (X,E,τ_1,τ_2) be a fuzzy soft bitopological space. Then (X,E,τ_1,τ_2) is a pairwise fuzzy soft T_1-Space if and only if (X,E,τ_1) and (X,E,τ_2) are fuzzy soft T_1-Spaces.

Proof. Let $f_e, g_e \tilde{\in} \widetilde{(X,E)}$ such that $f_e \neq g_e$. Suppose that (X,E,τ_1) and (X,E,τ_2) are fuzzy soft T_0-Spaces. Then there exist some $u_E \in \tau_1$ and $v_E \in \tau_2$ such that $f_e \tilde{\in} u_E$ and $g_e \tilde{\notin} u_E$ and $g_e \tilde{\in} v_E$ and $f_e \tilde{\notin} v_E$. In either case we obtain the requirement and so (X,E,τ_1,τ_2) is a pairwise fuzzy soft T_1-Space.

Conversely we assume that (X,E,τ_1,τ_2) is a pairwise fuzzy soft T_1-Space. Then there exist some $u_{1_E} \in \tau_1$ and $v_{1_E} \in \tau_2$ such that $f_e \tilde{\in} u_{1_E}$ and $g_e \tilde{\notin} u_{1_E}$ and $g_e \tilde{\in} v_{1_E}$ and $f_e \tilde{\notin} v_{1_E}$. Also there exist some $u_{2_E} \in \tau_1$ and $v_{2_E} \in \tau_2$ such that $f_e \tilde{\in} u_{2_E}$ and $g_e \tilde{\notin} u_{2_E}$ and $g_e \tilde{\in} v_{2_E}$ and $f_e \tilde{\notin} v_{2_E}$. Thus (X,E,τ_1) and (X,E,τ_2) are fuzzy soft T_1-Spaces.

Proposition 3.14. Let (X,E,τ_1,τ_2) be a fuzzy soft bitopological space. If (X,E,τ_1,τ_2) is a pairwise fuzzy soft T_1-Space, then (X,E,τ_{12}) is a supra fuzzy soft T_1-Space.

Proof. Let $f_e, g_e \tilde{\in} \widetilde{(X,E)}$ such that $f_e \neq g_e$. Then there exist some $u_E \in \tau_1$ such that $f_e \tilde{\in} u_E$ and $g_e \tilde{\notin} u_E$ and $v_E \in \tau_2$ such that $g_e \tilde{\in} v_E$ and $f_e \tilde{\notin} v_E$. So $u_E, v_E \in \tau_{12}$. Hence (X,E,τ_{12}) is a supra fuzzy soft T_1-Space.

Remark 3.15. The converse of Proposition 3.14 is not true. This is shown by the following example:

Example 3.16. Let $X = \{x_1,x_2\}$, $E = \{e_1,e_2\}$ and

$\tau_1 = \{\tilde{0}_E, \tilde{1}_E, f_E\}$, and

$\tau_2 = \{\tilde{0}_E, \tilde{1}_E, g_E\}$,

where f_E and g_E are fuzzy soft sets over (X,E) defined as follows:

$f_E = \{f(e_1) = \{x_1/0.3, x_2/0.0\}, f(e_2) = \{x_1/0.1, x_2/0.9\} = 1_X\}$,

and

$g_E = \{g(e_1) = \{x_1/0.3, x_2/0.7\}, g(e_2) = \{x_1/0.0, x_2/0.9\}$.

Then τ_1 and τ_2 are two fuzzy soft topologies over (X,E). Therefore (X,E,τ_1,τ_2) is a fuzzy soft bitopological space. Both of (X,E,τ_1) and $X,E,\tau_2)$ are not fuzzy soft T_1-Spaces and so (X,E,τ_1,τ_2) is not a pairwise fuzzy soft T_1-Space by Proposition 3.3.

Now $\tau_{12} = \{\tilde{0}_E, \tilde{1}_E, f_E, g_E, h_E\}$ where

$h_E = f_E \tilde{\cup} g_E = \{h(e_1) = \{x_1/0.3, x_2/0.7\}, h(e_2) = \{x_1/0.1, x_2/0.9\}\}$

So (X, E, τ_{12}) is a supra fuzzy soft topological Space.

For all $s = 1, 2$; $f_1(e_s), g(e_s) \tilde{\in} \widetilde{(X,E)}$, we can fined fuzzy soft sets $f_E \in \tau_1$ and $g_E \in \tau_2$ such that $f_1(e_1) \tilde{\in} f_E, g(e_1) \tilde{\notin} f_E$ and $g(e_1) \tilde{\in} g_E, f_1(e_1) \tilde{\notin} f_E$.

Thus (X, E, τ_{12}) is a supra fuzzy soft T_1-Space.

Proposition 3.17. Let (X, E, τ_1, τ_2) be a fuzzy soft bitopological space and Y be a non-empty subset of X If (X, E, τ_1, τ_2) is a pairwise fuzzy soft T_1-Space, then $(Y, E, \tau_{1_Y}, \tau_{2_Y})$ is also a pairwise fuzzy soft T_1-Space. .

Proof. Let $f_e, g_e \tilde{\in} \widetilde{(X,E)}$ such that $f_e \neq g_e$. Then there exist fuzzy soft set $u_E \in \tau_1$ and $v_E \in \tau_2$ such that $f_e \tilde{\in} u_E$, $g_e \tilde{\notin} u_E$ and $g_e \tilde{\in} v_E$, $f_e \tilde{\notin} v_E$.

Now $f_e \tilde{\in} \widetilde{(Y,E)}$ implies that $f_e \tilde{\in} h_E^Y$. So $f_e \tilde{\in} h_E^Y$ and $f_e \tilde{\in} u_E$. Hence $f_e \tilde{\in} h_E^Y \tilde{\cap} u_E$ where $u_E \in \tau_1$. Consider $g_e \tilde{\notin} u_E$, this means that $g_e \tilde{\notin} \{u(\acute{e})\}$ for some $\acute{e} \in E$. Then $g_e \tilde{\notin} h_E^Y \tilde{\cap} u_E$

Similarly it can be proved that $g_e \tilde{\in} v_E$ and $f_e \tilde{\notin} v_E$ then $g_e \tilde{\in} h_E^Y \tilde{\cap} v_E$ and $f_e \tilde{\notin} h_E^Y \tilde{\cap} v_E$.

Thus $(Y, E, \tau_{1_Y}, \tau_{2_Y})$ is a pairwise fuzzy soft T_1-Space.

Proposition 3.18. Every pairwise fuzzy soft T_1-Space is also a pairwise fuzzy soft T_0-Space. .

Proof. Straightforward.

Example 3.19. Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and

$\tau_1 = \{\tilde{0}_E, \tilde{1}_E, f_E\}$, and

$\tau_2 = \{\tilde{0}_E, \tilde{1}_E, g_E\}$,

where f_E and g_E are fuzzy soft sets over (X, E) defined as follows:

$f_E = \{f(e_1) = \{x_1/0.3, x_2/0.0\}, f(e_2) = \{x_1/0.1, x_2/0.9\} = 1_X\}$,

and

$g_E = \{g(e_1) = \{x_1/0.3, x_2/0.7\}, g(e_2) = \{x_1/0.0, x_2/0.9\}$.

It was showed in example 3.16 that (X, E, τ_1, τ_2) is not a pairwise fuzzy soft T_1-Space, but it is evident that (X, E, τ_1, τ_2) is a pairwise fuzzy soft T_0-Space.

Definition 3.20. A fuzzy soft bitopological space (X, E, τ_1, τ_2) is said to be a pairwise fuzzy soft T_2-Space or pairwise fuzzy soft Hausdorff Space if for every pair of distinct fuzzy soft points f_e, g_e in $\widetilde{(X,E)}$, there is a τ_1-fuzzy soft open set u_E and τ_2-fuzzy soft open set v_E such that $f_e \tilde{\in} u_E$, $g_e \tilde{\in} v_E$ and $u_E \tilde{\cap} v_E$.

Remark 3.21. Let (X,E,τ_1,τ_2) be a pairwise fuzzy soft T_2-Space then (X,E,τ_1) and (X,E,τ_2) need not be fuzzy soft T_2-Spaces.

Example 3.22. Let X be an infinite set and E be the set of parameters. We define

$\tau_1 = \{f_E \mid f_E$ is a fuzzy soft set over $(X,E)\}$ fuzzy soft discrete topology over (X,E).

$\tau_2 = \{0_E \tilde{\cup} f_E \mid f_E$ is a fuzzy soft set over (X,E) and $f^c(\acute{e})$ is a finite for all $\acute{e} \in E\}$

Obviously τ_1 is a fuzzy soft topology over (X,E). We verify for τ_2 as:

(1) $\tilde{0}_E \in \tau_2$ and $\tilde{1}_E^c = \tilde{0}_E \Rightarrow \tilde{1}_E \in \tau_2$.

(2) Let $\{f_{i_E} \mid i \in I\}$ be a collection of fuzzy soft sets in τ_2. For any $\acute{e} \in E$, $\{f_i^c(\acute{e})\}$ is finite for all $i \in I$ so that $\widetilde{\bigcap}_{i\in I}\{f_i^c(\acute{e})\} = \{(\widetilde{\bigcup}_{i\in I} f_i)^c(\acute{e})\}$ is also finite. This means that $\widetilde{\bigcup}_{i\in I} f_{i_E} \in \tau_2$.

(3) Let $f_E, g_E \in \tau_2$. Since $\{f^c(\acute{e})\}$ and $\{g^c(\acute{e})\}$ are finite fuzzy soft sets for all $\acute{e} \in E$ so as their union $\{f^c(\acute{e})\}\tilde{\cup}\{g^c(\acute{e})\}$. Thus $\{(f\tilde{\cap}g)^c(\acute{e})\}$ is finite for all $\acute{e} \in E$ which shows that $f_E \tilde{\cap} g_E \in \tau_2$.

Then τ_1 and τ_2 are fuzzy soft topologies on (X, E).

For any $f_e, g_e \tilde{\in} \widetilde{(X,E)}$ where $f_e \neq g_e$, $\{f_e\} \in \tau_1$ and $\{f_e\}^c \in \tau_2$ such that $f_e \tilde{\in} \{f_e\}$, $g_e \tilde{\in} \{f_e\}^c$ and $\{f_e\} \widetilde{\bigcap} \{f_e\}^c = \tilde{0}_E$.

Thus (X,E,τ_1,τ_2) be a pairwise fuzzy soft T_2-Space.

Now, we suppose that there are fuzzy soft sets $g_{1_E}, g_{2_E} \in \tau_2$ such that $f_e \tilde{\in} g_{1_E}, g_e \tilde{\in} g_{2_E}$ and $g_{1_E} \tilde{\cap} g_{2_E} = \tilde{0}_E$.

But then, we must have $g_{1_E} \tilde{\subseteq} g_{2_E}^c \Rightarrow \{g_1(\acute{e})\} \tilde{\subseteq} \{g_2(\acute{e})\}^c$ for all $\acute{e} \in E$, which is not possible because $\{g_1(\acute{e})\}$ is infinite and $\{g_2(\acute{e})\}^c$ is finite. Therefore (X,E,τ_2) is not a fuzzy soft T_2-Space.

Remark 3.23. Let (X,E,τ_1) and (X,E,τ_2) be a fuzzy soft T_2-Spaces then (X,E,τ_1,τ_2) need not be a pairwise fuzzy soft T_2-Spaces.

Example 3.24. Let X be an infinite set and E be the set of parameters. We define

$\tau(f_e)_1 = \{u_E \mid f_e \tilde{\in} u_E^c$ is a fuzzy soft set over $(X,E)\} \widetilde{\bigcup} \{u_E \mid u_E$ is a fuzzy soft set over (X,E) and $\{f^c(\acute{e})\}$ is finite for all $\acute{e} \in E\}\}$.

$\tau(g_e)_2 = \{v_E \mid g_e \tilde{\in} v_E^c$ is a fuzzy soft set over $(X,E)\} \widetilde{\bigcup} \{v_E \mid v_E$ is a fuzzy soft set over (X,E) and $\{f^c(\acute{e})\}$ is finite for all $\acute{e} \in E\}\}$.

verify for $\tau(f_e)_1$ as:

(1) $f_e \neq \tilde{0}_E \Rightarrow \tilde{0}_E \in \tau(f_e)_1$ and $\tilde{1}_E^c = \tilde{0}_E \Rightarrow \tilde{1}_E \in \tau(f_e)_1$.

(2) Let $\{u_{i_E} \mid i \in I\}$ be a collection of fuzzy soft sets in $\tau(f_e)_1$. We have following three cases:

(i) If $f_e \tilde{\in} u^c_{i_E}$ for all $i \in I$ then $f_e \tilde{\in} \widetilde{\bigcap}_{i\in I} u^c_{i_E}$ so, in this case, $\widetilde{\bigcup}_{i\in I} u_{i_E} \in \tau(f_e)_1$

(ii) If u_{i_E} is such that $\{u^c_i(\acute{e})\}$ is finite for all $\acute{e} \in E$ so $\{u^c_i(\acute{e})\}$ is finite for all $i \in I$ implies that $\bigcap_{i\in I}\{u^c_i(\acute{e})\} = \{(\bigcup_{i\in I} u_i)^c(\acute{e})\}$ is also finite this means that $\bigcup_{i\in I} u_{i_E} \in \tau(f_e)_1$

(iii) If there exist some $i,k \in I$ such that $f_e \tilde{\in} u^c_{i_E}$ and $\{u^c_k(\acute{e})\}$ is finite for all $\acute{e} \in E$. It means that $\widetilde{\bigcap}_{i\in I}\{u^c_i(\acute{e})\}(\tilde{\subset}\{u^c_k(\acute{e})\})$ is also finite for all $\acute{e} \in E$ and by definition $\bigcup_{i\in I} u_E \in \tau(f_e)_1$.

(3) Let $u_{1_E}, u_{2_E} \in \tau(f_e)_1$. Again we have following three cases:

(i) If $f_e \tilde{\in} u^c_{1_E}$ and $f_e \tilde{\in} u^c_{2_E}$ then $f_e \tilde{\in} u^c_{1_E} \tilde{\cup} u^c_{2_E}$ and therefore $u_{1_E} \tilde{\cap} u_{2_E} \in \tau(f_e)_1$

(ii) If $\{u^c_1(\acute{e})\}$ and $\{u^c_2(\acute{e})\}$ are finite for all $\acute{e} \in E$ then their union $\{u^c_1(\acute{e})\}\widetilde{\bigcup}\{u^c_2(\acute{e})\}$ is also finite. Thus $\{(u_1 \tilde{\cap} u_2)^c(\acute{e})\}$ is finite for all $\acute{e} \in E$ which shows that $u_{1_E} \tilde{\cap} u_{2_E} \in \tau(f_e)_1$.

(iii) If $f_e \tilde{\in} u^c_{1_E}$ and $\{u^c_2(\acute{e})\}$ is finite for all $\acute{e} \in E$ then $f_e \tilde{\in} \{u^c_1(\acute{e})\}\widetilde{\bigcup}\{u^c_2(\acute{e})\} = \{(u^c_1 \tilde{\cup} u^c_2)(\acute{e})\}$ and so $f_e \tilde{\in} (u_{1_E} \tilde{\cap} u_{2_E})^c$. Thus $u_{1_E} \tilde{\cap} u_{2_E} \in \tau(f_e)_1$.

Hence $\tau(f_e)_1$ is a fuzzy soft topology on (X,E).

For any $f_e, h_e \tilde{\in} \widetilde{(X,E)}$ where $f_e \neq h_e$, $f_e \tilde{\in} \{h^c_e\} \Rightarrow \{h_e\} \in \tau(f_e)_1$ and $\{h^c_e\} \in \tau(f_e)_1$ such that $h_e \tilde{\in} \{h_e\}$, $f_e \tilde{\in} \{h^c_e\}$ and $\{h_e\}\widetilde{\bigcap}\{h^c_e\} = \tilde{0}_E$.

Thus $(X,E,\tau(f_e)_1)$ is a fuzzy soft T_2-Space.

Similarly $(X,E,\tau(g_e)_2)$ is a fuzzy soft T_2-Space.

Now, $(X,E,\tau(f_e)_1,\tau(g_e)_2)$ is a fuzzy soft bitopological space. For $f_e, g_e \tilde{\in} \widetilde{(X,E)}$ where $f_e \neq g_e$, we can not find any fuzzy soft sets $u_E \in \tau(f_e)_1$ and $v_E \in \tau(g_e)_2$ such that $f_e \tilde{\in} u_E, g_e \tilde{\in} v_E$ and $u_E \tilde{\cap} v_E = \tilde{0}_E$ because $g_e \tilde{\in} v_E$ and $u_E \tilde{\cap} v_E = \tilde{0}_E$ implies that we must have $u_E \tilde{\subseteq} v_E$ which means that $\{v^c(\acute{e})\}$ is finite for all $\acute{e} \in E$ and $\{u(\acute{e})\} \tilde{\subseteq} \{v^c(\acute{e})\}$ for all $\acute{e} \in E$, and this is not possible for $\{u(\acute{e})\}$ is infinite and $\{v^c(\acute{e})\}$ is finite.

Therefore $(X,E,\tau(f_e)_1,\tau(g_e)_2)$ is not a pairwise fuzzy soft T_2-Space.

Proposition 3.25. Let (X,E,τ_1,τ_2) be a fuzzy soft bitopological space. If (X,E,τ_1,τ_2) is a pairwise fuzzy soft T_2-Space, then (X,E,τ_{12}) is a supra fuzzy soft T_2-Space.

Proof. Let $f_e, g_e \tilde{\in} \widetilde{(X,E)}$ such that $f_e \neq g_e$. Then there exist $u_E \in \tau_1$ and $v_E \in \tau_2$ such that $f_e \tilde{\in} u_E$ and $g_e \tilde{\in} v_E$ and $u_E \tilde{\cap} v_E = \tilde{0}_E$.

In either case $u_E, v_E \in \tau_{12}$. Hence (X,E,τ_{12}) is a supra fuzzy soft T_2-Space.

Remark 3.26. Let (X,E,τ_{12}) be a supra fuzzy soft T_2-Space then (X,E,τ_1,τ_2) need not be a pairwise fuzzy soft T_2-Space.

Example 3.27. Let $X=\{x_1,x_2,x_3\}$, $E=\{e_1,e_2\}$ and

$\tau_1=\{\tilde{0}_E,\tilde{1}_E,f_{1_E},f_{2_E},f_{3_E},f_{4_E}\}$, and

$\tau_2=\{\tilde{0}_E,\tilde{1}_E,g_{1_E},g_{2_E},g_{3_E}\}$,

where $f_{1_E},f_{2_E},f_{3_E},f_{4_E},g_{1_E},g_{2_E}$ and g_{3_E} are fuzzy soft sets over (X,E) defined as follows:

$f_{1_E}=\{f_1(e_1)=\{x_1/0.2,x_2/0.0,x_3/0.0\},f_1(e_2)=\{x_1/0.7,x_2/0.0,x_3/0.0\}\}$,

$f_{2_E}=\{f_2(e_1)=\{x_1/0.0,x_2/0.4,x_3/0.0\},f_2(e_2)=\{x_1/0.7,x_2/0.3,x_3/0.0\}\}$,

$f_{3_E}=\{f_3(e_1)=\{x_1/0.0,x_2/0.0,x_3/0.0\}=0_X,f_3(e_2)=\{x_1/0.7,x_2/0.0,x_3/0.0\}\}$,

$f_{4_E}=\{f_4(e_1)=\{x_1/0.2,x_2/0.4,x_3/0.0\},f_4(e_2)=\{x_1/0.7,x_2/0.3,x_3/0.0\}\}$,

and

$g_{1_E}=\{g_1(e_1)=\{x_1/0.0,x_2/0.0,x_3/0.1\},g_1(e_2)=\{x_1/0.0,x_2/0.0,x_3/0.2\}\}$,

$g_{2_E}=\{g_2(e_1)=\{x_1/0.0,x_2/0.4,x_3/0.0\},g_2(e_2)=\{x_1/0.0,x_2/0.3,x_3/0.0\}\}$,

$g_{3_E}=\{g_3(e_1)=\{x_1/0.0,x_2/0.4,x_3/0.1\},g_3(e_2)=\{x_1/0.0,x_2/0.3,x_3/0.2\}\}$.

Then τ_1 and τ_2 are two fuzzy soft topologies over (X,E). Therefore (X,E,τ_1,τ_2) is a fuzzy soft bitopological space. One can easily see that (X,E,τ_1,τ_2) is not a pairwise fuzzy soft T_2-Space because $f_4(e_1),g_3(e_1)\tilde{\in}\widetilde{(X,E)}$, and we can not find any fuzzy soft sets $u_E\in\tau_1$, $v_E\in\tau_2$ such that $f_4(e_1)\tilde{\in}u_E,g_3(e_1)\tilde{\in}v_E$ such that $u_E\tilde{\cap}v_E=\tilde{0}_E$.

Now, we have

$\tau_{12}=\{\tilde{0}_E,\tilde{1}_E,f_{1_E},f_{2_E},f_{3_E},f_{4_E},g_{1_E},g_{2_E},g_{3_E},h_{1_E},h_{2_E},h_{3_E},h_{4_E}\}$ where

$h_{1_E}=f_{1_E}\tilde{\cup}g_{1_E}=\{h_1(e_1)=\{x_1/0.2,x_2/0.0,x_3/0.1\},h_1(e_2)=\{x_1/0.7,x_2/0.0,x_3/0.2\}\}$,

$h_{2_E}=f_{2_E}\tilde{\cup}g_{1_E}=\{h_2(e_1)=\{x_1/0.0,x_2/0.4,x_3/0.1\},h_2(e_2)=\{x_1/0.7,x_2/0.3,x_3/0.2\}\}$,

$h_{3_E}=f_{3_E}\tilde{\cup}g_{1_E}=\{h_3(e_1)=\{x_1/0.0,x_2/0.0,x_3/0.1\},h_3(e_2)=\{x_1/0.7,x_2/0.0,x_3/0.2\}\}$,

$h_{4_E}=f_{4_E}\tilde{\cup}g_{1_E}=\{h_4(e_1)=\{x_1/0.2,x_2/0.4,x_3/0.1\},h_4(e_2)=\{x_1/0.7,x_2/0.3,x_9/0.2\}\}$,

$f_{1_E}\tilde{\cup}g_{2_E}=f_{4_E}\tilde{\cup}g_{2_E}=f_{4_E}$, $f_{2_E}\tilde{\cup}g_{2_E}=f_{3_E}\tilde{\cup}g_{2_E}=f_{2_E}$

$f_{1_E}\tilde{\cup}g_{3_E}=f_{4_E}\tilde{\cup}g_{3_E}=h_{4_E}$, $f_{2_E}\tilde{\cup}g_{3_E}=f_{3_E}\tilde{\cup}g_{3_E}=h_{2_E}$

So (X,E,τ_{12}) is a supra fuzzy soft topological Space.

It is obvious that for each distinct fuzzy soft points $f_e, g_e \tilde{\in} \widetilde{(X,E)}$, $f_e \neq g_e$ there exist fuzzy soft sets u_E, v_E of τ_{12} such that $f_e \tilde{\in} u_E$, $g_e \tilde{\in} v_E$ and $u_E \tilde{\cap} v_E = \tilde{0}_E$.

Thus (X, E, τ_{12}) is a fuzzy soft T_2-Space.

Proposition 3.28. Let (X, E, τ_1, τ_2) be a fuzzy soft bitopological space and Y be a non-empty subset of X If (X, E, τ_1, τ_2) is a pairwise fuzzy soft T_2-Space, then $(Y, E, \tau_{1_Y}, \tau_{2_Y})$ is also a pairwise fuzzy soft T_2-Space.

Proof. Let $f_e, g_e \tilde{\in} \widetilde{(X,E)}$ such that $f_e \neq g_e$. Then there exist fuzzy soft sets $u_E \in \tau_1$ and $v_E \in \tau_2$ such that $f_e \tilde{\in} u_E$, $g_e \tilde{\in} v_E$ and $u_E \tilde{\cap} v_E = \tilde{0}_E$.

For each $\acute{e} \in E$, $f_e \tilde{\in} \{u(\acute{e})\}$, $g_e \tilde{\in} \{v(\acute{e})\}$ and $\{u(\acute{e})\} \widetilde{\bigcap} \{v(\acute{e})\} = \tilde{0}_E$ for all $\acute{e} \in E$. This implies that $f_e \tilde{\in} h_E^Y \tilde{\cap} \{u(\acute{e})\}$, $g_e \tilde{\in} h_E^Y \tilde{\cap} \{v(\acute{e})\}$ and

$(h_E^Y \tilde{\cap} \{u(\acute{e})\}) \tilde{\cap} (h_E^Y \tilde{\cap} \{v(\acute{e})\}) = h_E^Y \tilde{\cap} (\{u(\acute{e})\} \widetilde{\bigcap} \{v(\acute{e})\}) = h_E^Y \tilde{\cap} \tilde{0}_E = \tilde{0}_E$.

Hence $f_e \tilde{\in} h_E^Y \tilde{\cap} u_E \in \tau_{1_Y}, g_e \tilde{\in} h_E^Y \tilde{\cap} v_E \in \tau_{2_Y}$

and

$(h_E^Y \tilde{\cap} u_E) \tilde{\cap} (h_E^Y \tilde{\cap} v_E) = h_E^Y \tilde{\cap} (u_E \tilde{\cap} v_E) = h_E^Y \tilde{\cap} \tilde{0}_E = \tilde{0}_E$.

Thus $(Y, E, \tau_{1_Y}, \tau_{2_Y})$ is a pairwise fuzzy soft T_2-Space.

Proposition 3.29. Every pairwise fuzzy soft T_2-Space is also a pairwise fuzzy soft T_1-Space.

Proof. If (X, E, τ_1, τ_2) is a pairwise fuzzy soft T_2-Space and $f_e, g_e \tilde{\in} \widetilde{(X,E)}$ such that $f_e \neq g_e$ then there exist fuzzy soft sets $u_E \in \tau_1$ and $v_E \in \tau_2$ such that $f_e \tilde{\in} u_E$, $g_e \tilde{\in} v_E$ and $u_E \tilde{\cap} v_E = \tilde{0}_E$.

As $u_E \tilde{\cap} v_E = \tilde{0}_E$, so $g_e \tilde{\notin} u_E$, $f_e \tilde{\notin} v_E$

Hence (X, E, τ_1, τ_2) is also a pairwise fuzzy soft T_1-Space.

Remark 3.30. The converse of Proposition 3.29 is not true i.e. a pairwise fuzzy soft T_1-Space need not be a pairwise fuzzy soft T_2-Space.

Example 3.31. The fuzzy soft bitopological space (X, E, τ_1, τ_2) in Example 3.12 is a pairwise fuzzy soft T_1-space which is not a pairwise fuzzy soft Hausdorff space.

Theorem 3.32. Let (X, E, τ_1, τ_2) be a fuzzy soft bitopological space. Then the following are equivalent:

(1) (X, E, τ_1, τ_2) be a pairwise fuzzy soft Hausdorff space.

(2) Let $f_e \tilde{\in} \widetilde{(X,E)}$, for each fuzzy soft point g_e distinct from f_e, there is a fuzzy soft set $u_E \in \tau_1$ such that $f_e \tilde{\in} u_E$ and $g_e \tilde{\in} \tilde{1}_E - \tau_2 cl(u_E)$.

Proof. (1) $\Rightarrow$ (2): Suppose that (X,E,τ_1,τ_2) be a pairwise fuzzy soft Hausdorff space and $f_e\tilde{\in}\widetilde{(X,E)}$. For any $g_e\tilde{\in}\widetilde{(X,E)}$ such that $f_e \neq g_e$, pairwise fuzzy soft Hausdorffness implies that there is a τ_1-fuzzy soft open set u_E and τ_2-fuzzy soft open set v_E such that $f_e\tilde{\in}u_E$, $g_e\tilde{\in}v_E$ and $u_E\tilde{\cap}v_E$.

So that $u_E\tilde{\subseteq}v_E^c$. Since $\tau_2cl(u_E)$ is the smallest fuzzy soft closed set in τ_2 that contains u_E and v_E^c is a fuzzy soft closed set in τ_2 so $\tau_2cl(u_E)\tilde{\subseteq}v_E^c \Rightarrow v_E\tilde{\subseteq}(\tau_2cl(u_E))^c$.

Thus $g_e\tilde{\in}v_E\tilde{\subseteq}(\tau_2cl(u_E))^c$ or $g_e\tilde{\in}\tilde{1}_E-\tau_2cl(u_E)$.

(2) $\Rightarrow$ (1): Let $f_e,g_e\tilde{\in}\widetilde{(X,E)}$ such that $f_e \neq g_e$. By (2) there is a τ_1-fuzzy soft open set u_E such that $f_e\tilde{\in}u_E$ and $g_e\tilde{\in}\tilde{1}_E-\tau_2cl(u_E)$. As $\tau_2cl(u_E)$ is a τ_2-fuzzy soft closed set so $v_E\tilde{=}\tilde{1}_E-\tau_2cl(u_E)\in\tau_2$. Now $f_e\tilde{\in}u_E$, $g_e\tilde{\in}v_E$ and
$u_E\tilde{\cap}v_E = u_E\tilde{\cap}(\tilde{1}_E-\tau_2cl(u_E))\tilde{\subseteq}u_E\tilde{\cap}(\tilde{1}_E-u_E)$ (since $u_E\tilde{\subseteq}\tau_2cl(u_E)) = \tilde{0}_E$.

Thus $u_E\tilde{\cap}v_E = \tilde{0}_E$, and hence (X,E,τ_1,τ_2) be a pairwise fuzzy soft Hausdorff space.

Corollary 3.33. Let (X,E,τ_1,τ_2) be a pairwise fuzzy soft Hausdorff space. Then for each $f_e\tilde{\in}\widetilde{(X,E)}$,

$$\{f_e\} = \widetilde{\bigcap}\{\tau_2cl(u_E) : f_e\tilde{\in}u_E\in\tau_1\}.$$

Proof. Let $f_e\tilde{\in}\widetilde{(X,E)}$, the existence of a fuzzy soft open set $f_e\tilde{\in}u_E\in\tau_1$ is guaranteed by pairwise fuzzy soft Hausdorffness. If $g_e\tilde{\in}\widetilde{(X,E)}$ such that $f_e \neq g_e$ then, by Theorem 3.32, there exists a fuzzy soft set $u_E\in\tau_1$ such that $f_e\tilde{\in}u_E$ and $g_e\tilde{\in}\tilde{1}_E-\tau_2cl(u_E) \Rightarrow g_e\tilde{\notin}\tau_2cl(u(\acute{e})) \Rightarrow g_e\tilde{\notin}\widetilde{\bigcap}_{f_e\tilde{\in}u_E\in\tau_1}(\tau_2cl(u(\acute{e})))$ for all $\acute{e}\in E$. Therefore

$$\widetilde{\bigcap}\{\tau_2cl(u_E) : f_e\tilde{\in}u_E\in\tau_1\}\tilde{\subseteq}\{f_e\}.$$

Converse inclusion is obvious as $f_e\tilde{\in}u_E\subseteq\tau_2cl(u_E)$.

Corollary 3.34. Let (X,E,τ_1,τ_2) be a pairwise fuzzy soft Hausdorff space. Then for each $f_e\tilde{\in}\widetilde{(X,E)}, \{f_e\}^c\in\tau_i$, for $i=1,2$.

Proof. By Corollary 3.33

$$\{f_e\}^c = \widetilde{\bigcup}\{(\tau_2cl(u_E))^c : f_e\tilde{\in}u_E\in\tau_1\}.$$

Since $\tau_2cl(u_E)$ is a τ_2-fuzzy soft closed set so $(\tau_2cl(u_E))^c\in\tau_2$ and by the axiom of a fuzzy soft topological space $\widetilde{\bigcup}\{(\tau_2cl(u_E))^c : f_e\tilde{\in}u_E\in\tau_1\}\in\tau_2$. Thus $\{f_e\}^c\in\tau_2$.

A similar argument holds to show $\{f_e\}^c\in\tau_1$.

4. Conclusion

In this paper, we presented and studied some classes of fuzzy soft bitopological spaces, namely pairwise fuzzy soft T_i-Spaces $(; i = 0, 1, 2)$. Characterizations of these spaces are obtained. Moreover, we studied the implications of these types of fuzzy soft separation axioms in fuzzy soft case. Also, we showed that these fuzzy soft separation axioms have hereditary properties. This is a beginning of some new generalized structure and the concept of separation axioms may be studied further for regular and normal fuzzy soft bitopological spaces that is our goal in the future work. Also, we will try to introduce and study some other properties and applications in fuzzy soft bitopological spaces based on these types of separation axioms.

Conflict of Interests

The author declare that there is no conflict of interests.

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