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FUZZY LINEAR CONJUNCTIVE GRAMMAR

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Abstract. Inspired by, Fuzzy Conjunctive Grammar (FCG) [1]. We study the Fuzzy Linear Conjunctive Grammar (FLCG).

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1. Introduction

Conjuctive grammar (CG), intoduced by Alexander Okhotin in [7]. Conjuctive grammar is a context-free grammar augmented with an explicit set-theoretic intersection operation. In particular, in [7] Okhotin defined a sub-family of conjunctive grammar called linear conjunctive grammar (LCG). LCG is an interesting sub-family of CG as they have especially efficient parsing algorithem, see[11], making them very appealing from a computational standpoint. In addition, many of the interesting language generated by conjunctive grammar can in fact be

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generated by linear conjunctive grammar.

The theory of fuzzy set was introduced by L.A.Zadeh in 1965 [9]. The notion of a different type of fuzzy grammars (Type 0, Type 1, Type 2, Type 3) defined by E.T.Lee and L.A.Zadeh in 1969 [2] is a natural generalization of the definition of formal grammars [8]. Fuzzy grammars on Boolean lattices (B-fuzzy grammar), N-fold fuzzy gramma and L-fuzzy grammar are newly defined by M.Mizumoto, J.Toyoda and K.Tanaka in 1975, 1973 and 1975 [4,5,3]. The mathematical formulation of a fuzzy automata was first proposed by W.G.Wee in 1967 [10]. J.N.Mordension and D.S.Malik gave a detailed account of fuzzy automata and languages in their book 2002 [6]. Recently, R.Pathrakumar and M.Rajasekar said fuzzy conjunctive grammar in 2017 [1].

This paper is organized as follows. In section 2 we introduced some notations and definition about FCG and FLCG. It also presents some example for the explaining the concepts. In section 3 we established the main result of this paper, the family of fuzzy linear conjunctive grammar closure under complement. Also, we study the family of FLCL closure under union, intersection and quotient.

2. Preliminaries

We first recall the different notions and introduced some definitions used in this article.

An alphabet Σ is a finite set of symbols. A word or string over Σ is a finite sequence of symbols from Σ. The empty word is denoted by ε. Let $|w|$ denote the length of the word w; so $|\varepsilon| = 0$, and for all $w \in \Sigma^+$: if $w = ax$ with $a \in \Sigma$ and $x \in \Sigma^*$, then $|w| = 1 + |x|$.

In formally, a fuzzy grammar may be viewed as a set of rules for generating the elements of a fuzzy set. More concretely, a fuzzy grammar, or simply a grammar, is a quadruple $G = (V, \Sigma, P, S)$ in which V is a finite set of non-terminal symbol, Σ is a finite set of terminal symbols disjoint from V, $S \in V$ is the designated start symbols, P is a set of fuzzy productions. More specifically, the elements of P are expressions of the form $\alpha \xrightarrow{r} \beta$, where $\alpha, \beta \in (V \cup \Sigma)^*$ and $r \in (0,1]$.

Definition 2.1. A Grammar G is defined as the quadruple $G = (V, \Sigma, P, S)$,where

- $V = \{A, B, \ldots, Z\}$ is a finite set of non-terminal symbol (Variables)

• $\Sigma = \{a, b, \ldots, z\}$ is a finite set of terminal symbols disjoint from V

• $S \in V$ is the designated start symbols

• P is a set of productions.

Definition 2.2. A fuzzy language, $\mu(L)$, is a fuzzy set in Σ^*. Thus, $\mu(L)$ is a set of ordered pairs $\mu(L) = \{(w, r)\}$, where, $w \in \Sigma^*$, $r \in [0, 1]$.

Definition 2.3. Let $\mu(L_1)$ and $\mu(L_2)$ be two fuzzy languages in Σ^*. The union , intersection and concatenation of $\mu(L_1)$ and $\mu(L_2)$ is defined as follows

• $\mu(L_1) \cup \mu(L_2) = max\{\mu(L_1), \mu(L_2)\}$ or simply defined by $\mu(L_1) \cup \mu(L_2) = \mu(L_1) \vee \mu(L_2)$

• $\mu(L_1) \cap \mu(L_2) = min\{\mu(L_1), \mu(L_2)\}$ or simply defined by $\mu(L_1) \cap \mu(L_2) = \mu(L_1) \wedge \mu(L_2)$

• $\mu(L_1.L_2) = min\{\mu(L_1), \mu(L_2)\}$ or simply defined by $\mu(L_1.L_2) = \mu(L_1) \wedge \mu(L_2)$.

Note 2.4. Let $G = (V, \Sigma, P, S)$ be a fuzzy grammar,then

• A fuzzy language $\mu(L)$ is generated by the fuzzy grammar $\mu(L, G)$

• If $\alpha_1, \ldots, \alpha_1$ are string in $(V \cup \Sigma)^*$ and

$$\alpha_1 \stackrel{r_1}{\Longrightarrow} \alpha_2 \stackrel{r_2}{\Longrightarrow}, \ldots, \alpha_{m-1} \stackrel{r_m}{\Longrightarrow} \alpha_m$$

where, $r_1, \ldots, r_m \in [0, 1]$, then α_1 is said to derive α_m in grammar G, or ,equivalentelly, α_m is derivable from α_1 in grammar G. This is expressed by $\alpha_1 \stackrel{r_m}{\Longrightarrow}_G$. The expression

$$\alpha_1 \stackrel{r_2}{\Longrightarrow} \alpha_2 \stackrel{r_2}{\Longrightarrow}, \ldots, \alpha_{m-1} \stackrel{r_m}{\Longrightarrow} \alpha_m$$

will be referred to as a derivation chain from α_1 to α_m.

Definition 2.5. A fuzzy context-free grammar (FCFG) is a quadruple $G = (V, \Sigma, P, S)$, where

• V is a finite set of non-terminal symbol (Variables)

• Σ is a finite set of terminal symbols disjoint from V

• $S \in V$ is the designated start symbols

• P is a set of fuzzy production of the form $A \stackrel{r}{\rightarrow} \alpha$, where $A \in V$, $\alpha \in (V \cup \Sigma)^*$ and $r \in (0, 1]$.

Definition 2.6. A fuzzy conjunctive grammar (FCG) is a quadruple $G = (V, \Sigma, P, S)$, where

• V is a finite set of non-terminal symbol (Variables)

• Σ is a finite set of terminal symbols disjoint from V

• $S \in V$ is the designated start symbols

• P is a finite set of fuzzy rules of the form

$A \xrightarrow{r}_G (\alpha_1/r_1\&\ldots\&\alpha_n/r_n)$, where $A \in V$, $\alpha_i \in (V\cup\Sigma)^*$ and $r = min\{r_1,\ldots,r_n\}$ for $i = 1,2,\ldots,n$. If $n = 1$ we write it as $A \xrightarrow{r} \alpha_1$ and call it an ordinary fuzzy rule. Otherwise, the rule is called proper fuzzy conjunctive.

Definition 2.7. Any element $\mathscr{B},\mathscr{C},\mathscr{D},\varepsilon$ is a conjunctive formulas over $V\cup\Sigma\cup\{(,),\&\}$ are defined by the following recursion.

- The empty string ε is a conjunctive formula.
- Every symbol in $V\cup\Sigma$ is a conjunctive formula.
- If $\mathscr{B}$ and $\mathscr{C}$ are conjunctive formulas, then $\mathscr{B}\mathscr{C}$ is a conjunctive formula.
- If $\mathscr{B}_1,\ldots,\mathscr{B}_n$ are conjunctive formulas, then $(\mathscr{B}_1\&\ldots\&\mathscr{B}_n)$ is a conjunctive formula.

Definition 2.8. Let $G = (V,\Sigma,P,S)$ be a fuzzy grammar, $\mathscr{B}$ be a formula. The fuzzy language generated by $\mathscr{B}$ is a set of all string over Σ derivable from $\mathscr{B}$:

$$\mu(L,\mathscr{B}) = max\ min\{(w,r)/w \in \Sigma^*, \mathscr{B} \overset{r}{\Longrightarrow}{}^*_G\, w\}.$$

The fuzzy language gerated by the fuzzy grammar is a set of all string over Σ derivable from its start symbol: $\mu(L,G) = \mu(L_G,S)$. The maximum is taken over all derivation chains from S to w.

A fuzzy language $\mu(L)$ is called fuzzy conjunctive, if it is generated by some fuzzy conjunctive grammar.

Definition 2.9. A fuzzy conjunctive grammar $G = (V,\Sigma,P,S)$ is said to be fuzzy linear, if each rule in P is of the form

$A \xrightarrow{r_l} u_1B_1v_1/r_{l_1}\&\ldots\&u_mB_mv_m/r_{l_m}$, $(u_i,v_i \in \Sigma^*, B_i \in V), r_l = min\{r_{l_1},\ldots,r_{l_m}\}$

$A \xrightarrow{r_w} w$, $(w \in \Sigma^*)$.

Definition 2.10. A fuzzy conjunctive grammar $G = (V,\Sigma,P,S)$ is said to be fuzzy linear normal, if each rule in P is of the form

$A \xrightarrow{r_{BC}} bB_1/r_{bB_1}\&\ldots\&bB_m/r_{bB_m}\&C_1c/r_{C_1c}\&\ldots\&C_nc/r_{C_nc}$, $(m+n \geq 1 : B_i,C_j \in V; b,c \in \Sigma)$,

$r_{BC} = min\{r_{bB_1},\ldots,r_{bB_m},r_{C_1c},\ldots,r_{C_nc}\}$

$A \xrightarrow{r_a} a$, $(a \in \Sigma)$

$A \xrightarrow{r_\varepsilon} \varepsilon$.

Examples 1.11. A fuzzy linear conjunctive grammar $G = (V, \Sigma, P, S)$ for the languiage $\mu(L) = \{(wcw, r) : w \in \{a,b\}^*\}$, where

$V = \{S, A, B, C, D, E, F, I, J, K, M, N, R\}$

$\Sigma = \{a, b, c\}$ and

P consists of the following derivation rules;

$$S \xrightarrow{0.001} Da/0.001\&aA/0.03\&aK/0.4$$

$$S \xrightarrow{0.002} Db/0.002\&aA/0.003\&aK/0.4$$

$$S \xrightarrow{0.003} Ea/0.003\&bB/0.04\&bK/0.7$$

$$S \xrightarrow{0.01} Eb/0.01\&bB/0.04\&bK/0.7$$

$$S \xrightarrow{0.02} c$$

$$C \xrightarrow{0.001} Da$$

$$C \xrightarrow{0.002} Db$$

$$C \xrightarrow{0.003} Ea$$

$$C \xrightarrow{0.01} Eb$$

$$C \xrightarrow{0.02} c$$

$$D \xrightarrow{1} aC$$

$$E \xrightarrow{1} bC$$

$$K \xrightarrow{0.03} aA/0.03\&aK/0.4$$

$$K \xrightarrow{0.04} bB/0.04\&bK/0.7$$

$$K \xrightarrow{0.05} cR$$

$$K \xrightarrow{0.05} c$$

$$A \xrightarrow{0.06} aI$$
$$A \xrightarrow{0.007} aJ$$
$$A \xrightarrow{0.008} bI$$
$$A \xrightarrow{0.009} bJ$$
$$A \xrightarrow{0.1} Fa$$
$$I \xrightarrow{1} Aa$$
$$J \xrightarrow{1} Ab$$
$$B \xrightarrow{0.2} aM$$
$$B \xrightarrow{0.3} aN$$
$$B \xrightarrow{0.4} bM$$
$$B \xrightarrow{0.5} bN$$
$$B \xrightarrow{0.6} Fb$$
$$M \xrightarrow{1} Ba$$
$$N \xrightarrow{1} Bb$$
$$R \xrightarrow{0.7} aR$$
$$R \xrightarrow{0.8} bR$$
$$R \xrightarrow{0.7} a$$
$$R \xrightarrow{0.8} b$$
$$F \xrightarrow{1} cR$$
$$F \xrightarrow{0.9} c$$

For example, the word **abcab** can be derived as follows

$$\begin{aligned}
S &\overset{0.002}{\Longrightarrow^*} Db/0.002\&aA/0.003\&aK/0.4\\
&\overset{0.04}{\Longrightarrow^*} aCb/1\&abJ/0.009\&a(bB/0.04\&bK/0.7)\\
&\overset{0.04}{\Longrightarrow^*} aCb/1\&abJ/0.009\&abB/0.04\&abK/0.7\\
&\overset{0.003}{\Longrightarrow^*} aEab/0.003\&abAb/1\&abFb/0.6\&abcR/0.05\\
&\overset{0.1}{\Longrightarrow^*} abCab/1\&abFab/0.1\&abcRb/1\&abcaR/0.7\\
&\overset{0.02}{\Longrightarrow^*} abcab/0.02\&abcab/0.9\&abcab/0.7\&abcab/0.8\\
S &\overset{r=0.002}{\Longrightarrow^*} abcab\&abcab\&abcab\&abcab
\end{aligned}$$

where, $r = min\{0.002, 0.04, 0.04, 0.003, 0.1, 0.02\} = 0.002$.
Therefore, $(abcab, 0.002) \in \mu(L, G)$.

3. Some Closure Properties of Fuzzy Linear Conjunctive Grammar

In this section we give constructive proofs that the family of fuzzy linear conjunctive language is closed under union, intersection, complement, and quotient.

Theorem 3.1. The family of fuzzy linear conjunctive languages is closed under union.

Proof. Let $\mu(L_1, G_1)$ and $\mu(L_2, G_2)$ be two fuzzy linear conjunctive languages generated by the fuzzy linear conjunctive grammmars $G_1 = (V_1, \Sigma_1, P_1, S_1)$ and $G_1 = (V_1, \Sigma_1, P_1, S_1)$ respectively. We can assume that V_1 and V_2 as well as P_1 and P_2 are disjoint. The language $\mu(L, G)$ generated by the fuzzy linear conjunctive grammar $G = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, P, S)$, where $S \notin V_1 \cup V_2$ and $P = P_1 \cup P_2 \cup \{S \xrightarrow{1} S_1, S \xrightarrow{1} S_2\}$. Now, we prove $\mu(L, G) = \mu(L_1, G_1) \vee \mu(L_2, G_2)$. Let $(w, r) \in \mu(L, G)$, then it is either,$S \overset{1}{\Longrightarrow}^* S_1 \overset{r}{\Longrightarrow}^* w$, $S \overset{1}{\Longrightarrow}^* S_2 \overset{r_1}{\Longrightarrow}^* w$ or $S \overset{1}{\Longrightarrow}^* S_2 \overset{r}{\Longrightarrow}^* w$, $S \overset{1}{\Longrightarrow}^* S_1 \overset{r_2}{\Longrightarrow}^* w$. Both the case $(w, r) \in \mu(L, G_1) \vee \mu(L, G_2)$. Hence

$$\mu(L, G) \leq \mu(L_1, G_1) \vee \mu(L_2, G_2) \tag{1}$$

Conversely, let $(w, r) \in \mu(L_1, G_1) \vee \mu(L_2, G_2)$, then (w, r) is in either $\mu(L_1, G_1)$ or $\mu(L_1, G_1)$. Suppose $(w, r) \in \mu(L_1, G_1)$, then $S_1 \overset{r}{\Longrightarrow}^* w$ is the derivation, if $(w, r) \in \mu(L_2, G_2)$, then $S_2 \overset{r}{\Longrightarrow}^*$

w is the derivation.

By construction of G, $S \overset{1}{\Longrightarrow}^* S_1 \overset{r}{\Longrightarrow}^* w$ or $S \overset{1}{\Longrightarrow}^* S_2 \overset{r}{\Longrightarrow}^* w$, then $(w,r) \in \mu(L,G)$. Hence

$$\mu(L_1,G_1) \vee \mu(L_2,G_2) \leq \mu(L,G) \tag{2}$$

From (1) and (2) we get

$$\mu(L,G) = \mu(L_1,G_1) \vee \mu(L_2,G_2)$$

This completes the proof.

Theorem 3.2. The family of fuzzy linear conjunctive languages is closed under intersection.

Proof. Let $\mu(L_1,G_1)$ and $\mu(L_2,G_2)$ be two fuzzy linear conjunctive languages generated by the fuzzy linear conjunctive grammmars $G_1 = (V_1,\Sigma_1,P_1,S_1)$ and $G_1 = (V_1,\Sigma_1,P_1,S_1)$ respectively. We can assume that V_1 and V_2 as well as P_1 and P_2 are disjoint. The languages $\mu(L,G)$ generated by the fuzzy linear conjunctive grammar $G = (V_1 \cup V_2 \cup \{S\},\Sigma_1 \cup \Sigma_2,P,S)$,
where $S \notin V_1 \cup V_2$ and $P = P_1 \cup P_2 \cup \{S \overset{r_S}{\to} S_1/r_{S_1}\&S/r_{S_2}\}$, $r_S = min\{r_{S_1},r_{S_2}\}$. Now, we prove $\mu(L,G) = \mu(L_1,G_1) \wedge \mu(L_2,G_2)$. Let $(w,r) \in \mu(L,G)$, then $S \overset{r_S}{\Longrightarrow}^* S_1\&S_2 \overset{r}{\Longrightarrow}^* w/r_1\&w/r_2 \overset{r}{\Longrightarrow}^*$ w is the derivation, where $r = min\{r_1,r_2\}$. Since $(w,r_1) \in \mu(L_1,G_1)$ and $(w,r_2) \in \mu(L_2,G_2)$, then $(w,r) \in \mu(L_1,G_1) \wedge \mu(L_2,G_2)$. Hance

$$\mu(L,G) \leq \mu(L_1,G_1) \wedge \mu(L_2,G_2) \tag{3}$$

Conversely, let $(w,r_1) \in \mu(L_1,G_1)$, then $S_1 \overset{r_1}{\Longrightarrow} w$ is the derivation and let $(w,r_2) \in \mu(L_2,G_2)$, then $S_2 \overset{r_{S_2}}{\Longrightarrow} w$ is the derivation. By construction of G, $S \overset{r_S}{\Longrightarrow}^* S_1\&S_2 \overset{r}{\Longrightarrow}^* w/r_1\&w/r_2 \overset{r}{\Longrightarrow}^* w$, then $(w,r) \in \mu(L,G)$, where $r = min\{r_1,r_2\}$. Hance

$$\mu(L_1,G_1) \wedge \mu(L_2,G_2) \leq \mu(L,G) \tag{4}$$

From (3) and (4) we get

$$\mu(L,G) = \mu(L_1,G_1) \wedge \mu(L_2,G_2).$$

This completes the proof.

Theorem 3.3. The family of fuzzy linear conjunctive languages is closed under the complement.

Proof. Let grammar $G=(V,\Sigma,P,S)$ be the fuzzy linear conjunctive normal form. We construct the following fuzzy linear conjunctive grammar G'.
Let $G'=(V_X\cup V_Y\cup V_Z\cup V_M\cup V_N\cup V_W,\Sigma,P',S')$, where

$$\begin{aligned}
V_X &= \{X_{\neg A}/A\in V\}\\
V_Y &= \{Y_{\neg A\xrightarrow{r}\alpha_1\&\ldots\&\alpha_m}/A\xrightarrow{r}\alpha_1\&\ldots\&\alpha_m\}\\
V_Z &= \{Z_{\neg a}/a\in\Sigma\}\cup\{Z_{\neg\varepsilon}\}\\
V_M &= \{M_{\neg a\Sigma^+}/a\in\Sigma\}\\
V_N &= \{N_{\neg\Sigma^+a}/a\in\Sigma\}\\
V_W &= \{W\}\\
S' &= X_{\neg S}
\end{aligned}$$

For each rule $A\xrightarrow{r}\alpha_1\&\ldots\&\alpha_m\in P$, the non terminal $Y_{\neg A\xrightarrow{r}\alpha_1\&\ldots\&\alpha_m}$ generate those and only those string that are not generated by this rule in the original grammar with membership values 1. For each terminal symbol $a\in\Sigma$, the nonterminal $Z_{\neg a}$ generate all string but the string a with membership 1, while $Z_{\neg\varepsilon}$ generate Σ^+ with membership 1, the non terminal $M_{\neg a\Sigma^+}$ generate all strings except those in $a.\Sigma^+$ with membership 1. Similarly, $N_{\neg\Sigma^+a}$ generate all strings except those that at the same time end with a and are atleast two symbols long with membership 1. Finally, the non terminal W generates Σ^* with membership 1.
Let us construct rules for P'
(i). For each production
$A\xrightarrow{r_{BC}}bB_1/r_{B_1}\&\ldots\&bB_m/r_{B_m}\&C_1c/r_{C_1}\&\ldots\&C_nc/r_{C_n}$ in P $(m+n\geq 1)$,
where, $r_{BC}=min\{r_{B_1},\ldots,r_{B_n},r_{C_1},\ldots,r_{C_n}\}$, there is a rules
$X_{\neg A}\xrightarrow{1-r_{BC}}bX_{\neg B_1}/1-r_{B_1}\&\ldots\&bX_{\neg B_m}/1-r_{B_m}\&X_{\neg C_1}c/1-r_{C_1}\&\ldots\&X_{\neg C_n}c/1-r_{C_n}$ in P'
where, $1-r_{BC}=max\{1-r_{B_1},\ldots,1-r_{B_m},1-r_{C_1},\ldots,1-r_{C_n}\}$ and

$$Y_{\neg A\xrightarrow{r_{BC}}bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc}\xrightarrow{1}M_{\neg b\Sigma^+}\ in\ P'\ (if\ m>0) \tag{5}$$

$$Y_{\neg A\xrightarrow{r_{BC}}bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc}\xrightarrow{1}N_{\neg\Sigma^+c}\ in\ P'\ (if\ n>0) \tag{6}$$

$$Y_{\neg A\xrightarrow{r_{BC}}bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc}\xrightarrow{1}bX_{\neg B_i}\ in\ P'\ (\forall\ i\in 1,\ldots,m) \tag{7}$$

$$Y_{\neg A \overset{r_{BC}}{\to} bB_1\&...\&bB_m\&C_1c\&...\&C_nc} \overset{1}{\to} X_{\neg C_j c} \; in \; P' \; (\forall \; j \in 1,\ldots,m) \tag{8}$$

(ii). For each production $A \overset{r_{\alpha_1}}{\to} \alpha_1, \ldots, A \overset{r_{\alpha_m}}{\to} \alpha_m$ in P

there is a rule $X_{\neg A} \overset{1-r_\alpha}{\to} X_{\neg \alpha_1}/1 - r_{\alpha_1}\& \ldots \& X_{\neg \alpha_m}/1 - r_{\alpha_m}$ in P'

where, $1 - r_\alpha = max\{1 - r_{\alpha_1}, \ldots, 1 - r_{\alpha_m}\}$ and

$$X_{\neg A} \overset{1}{\to} Y_{\neg A \overset{r_{\alpha_1}}{\to} \alpha_1} \& \ldots \& Y_{\neg A \overset{r_{\alpha_m}}{\to} \alpha_m} \; in \; P' \tag{9}$$

(iii). For each production $A \overset{r_a}{\to} a$ in P

there is a rules $X_{\neg A} \overset{1-r_a}{\to} a$ and

$$Y_{\neg A \overset{r_a}{\to} a} \overset{1}{\to} Z_{\neg a} \; in \; P' \tag{10}$$

(iv). For each production $S \overset{r_\varepsilon}{\to} \varepsilon$ in P

there is a rules $X_{\neg S} \overset{1-r_\varepsilon}{\to} X_{\neg \varepsilon}$ and

$$Y_{\neg S \overset{r_\varepsilon}{\to} \varepsilon} \overset{1}{\to} Z_{\neg \varepsilon} \; in \; P' \tag{11}$$

(v). For each non terminal $A \in V$ of the grammar G, if there are no rules for A in P, then P' contains the following single rule for $X_{\neg A}$;

$$X_{\neg A} \overset{1}{\to} W \tag{12}$$

Now, we will show that for every non terminal $A \in V$, then $\mu(L_{G'}, X_{\neg A}) = 1 - \mu(L_G, A)$. We prove the above result in two cases. First we consider the case $\mu(L_G, A)$ contains element which has non-zero membership and second the case having elements which has zero membership.

Case 1 [Non-zero membership]: Since grammar $G = (V, \Sigma, P, S)$ is the fuzzy linear conjunctive normal form. Then the derivation rule as

$$
\begin{aligned}
A &\overset{r_{BC}}{\Longrightarrow}^{*} bB_1/r_{B_1}\&\ldots\&bB_m/r_{B_m}\&C_1c/r_{C_1}\&\ldots\&C_nc/r_{C_n}\\
&\overset{r_{\beta\gamma}}{\Longrightarrow}^{*} b\beta_1/r_{\beta_1}\&\ldots\&b\beta_m/r_{\beta_m}\&\gamma_1c/r_{\gamma_1}\&\ldots\&\gamma_nc/r_{\gamma_n}\\
&\quad .\\
&\quad .\\
&\quad .\\
&\overset{r_w}{\Longrightarrow}^{*} w/r_{w_1}\&\ldots\&w/r_{w_n}\\
A &\overset{r}{\Longrightarrow}^{*} w
\end{aligned}
$$

Where, $r = min\{r_{BC}, r_{\beta\gamma}, \ldots, r_w\}$, $b, c \in \Sigma^*$, $B_1, \ldots, B_m, C_1, \ldots, C_n \in V$ and $\beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n \in (V \cup \Sigma)^*$. Hence, $(w, r) \in \mu(L_G, A)$.

By the construction of (i)

$$
\begin{aligned}
X_{\neg A} &\overset{1-r_{BC}}{\Longrightarrow}^{*} bX_{\neg B_1}/1-r_{B_1}\&\ldots\&bX_{\neg B_m}/1-r_{B_m}\&X_{\neg C_1}c/1-r_{C_1}\&\ldots\&X_{\neg C_n}c/1-r_{C_n}\\
&\overset{1-r_{\beta\gamma}}{\Longrightarrow}^{*} bX_{\neg\beta_1}/1-r_{\beta_1}\&\ldots\&bX_{\neg\beta_m}/1-r_{\beta_m}\&X_{\neg\gamma_1}c/1-r_{\gamma_1}\&\ldots\&X_{\neg\gamma_n}c/1-r_{\gamma_n}\\
&\quad .\\
&\quad .\\
&\quad .\\
&\overset{1-r_w}{\Longrightarrow}^{*} w/1-r_{w_1}\&\ldots\&w/1-r_{w_n}\\
X_{\neg A} &\overset{1-r}{\Longrightarrow}^{*} w
\end{aligned}
$$

Where, $1-r = max\{1-r_{BC}, 1-r_{\beta\gamma}, \ldots, 1-r_w\}$, $b, c \in \Sigma^*$, $X_{\neg B_1}, \ldots, X_{\neg B_m}, X_{\neg C_1}, \ldots, X_{\neg C_n} \in V_X$ and $X_{\neg\beta_1}, \ldots, X_{\neg\beta_m}, X_{\neg\gamma_1}, \ldots, X_{\neg\gamma_n} \in (V_X \cup \Sigma)^*$. Hence, $(w, 1-r) \in \mu(L_{G'}, A)$.

Case 2 [For zero membership]: In this case our aim is to prove that, if $(w, 0) \in \mu(L_G, A)$, then $(w, 1) \in \mu(L_{G'}, X_{\neg A})$ by using induction hypothesis on length n of string w; ie.$(w, 0) \in \mu(L_G, A)$, it implies that $(w, 0) \notin \mu(L_{G'}, A)$, which means that $(w, 1) \in \mu(L_{G'}, A)$.

Basis $n = 1$**:** Let $w = a$ (or) $w = \varepsilon$ and let $(w, 0) \in \mu(L_G, A)$, then it's corresponding production is

$A \xrightarrow{0} w \notin P$. This means that all rules for A are either of the form

$$A \xrightarrow{0} bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc \in P \tag{13}$$

(or)

$$A \xrightarrow{0} u(u \in \{\varepsilon\} \cup \Sigma) \tag{14}$$

For each rule of the form (13) the corresponding nonterminal $Y_{\neg A \xrightarrow{0} bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc}$ has atleast one rules (5) and (6) and consequently generates $(w,1)$. The same holds in respect to each rule of the form (14), if $u = a \in \Sigma$, then the nonterminal $Y_{\neg A \xrightarrow{r_a} a}$ has rule (10), which can generate any string except a, and consequently the string $(w,1)$. Similarly,if $u = \varepsilon$ and $A = S$, then the nonterminal $Y_{\neg S \xrightarrow{r_\varepsilon} \varepsilon}$ has rule (11) that generate anything except ε and thus $(w,1)$.

As we have shown, for any rule $A \xrightarrow{0} w \notin P$ for $A \in V$, the corresponding nonterminal $Y_{\neg A \xrightarrow{0} \alpha}$ generate $(w,1)$. Now, if there is atleast one rule for A in P, then every conjunct of rule (9) for the nonterminal $X_{\neg A}$ generate $(w,1)$ and thus $(w,1) \in \mu(L_{G'}, X_{\neg A})$; if there are no rules for A in P,then $(w,1)$ can be derived from $X_{\neg A}$ using rule (12).

Induction step $n \geq 2$**:** Let $n \geq 2$ and $A \overset{0(n)}{\Longrightarrow} w$. Then $A \overset{0(n-2)}{\Longrightarrow} bxc \overset{0(2)}{\Longrightarrow} w$, where $b, c \in \sum$ and $x \in \Sigma^+$, the string $(w,0) \in \mu(L_G, A)$ if and only if there is some rule

$$A \xrightarrow{0} bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc \in P \ (m+n \geq 1) \tag{15}$$

such that there exist derivations

$$B_i \overset{0}{\Longrightarrow}_G \cdots \overset{0}{\Longrightarrow}_G xc \ (\forall 1 \leq i \leq m) \tag{16}$$

$$C_j \overset{0}{\Longrightarrow}_G \cdots \overset{0}{\Longrightarrow}_G bx \ (\forall 1 \leq j \leq n) \tag{17}$$

By induction hypothesis, (16) holds if and only if for some rule (15)

$$(xc,0) \notin \mu(L_{G'}, X_{\neg B_i}) \ or \ (xc,1) \in \mu(L_{G'}, X_{\neg B_i}) \tag{18}$$

$$(bx,0) \notin \mu(L_{G'}, X_{\neg C_j}) \ or \ (bx,1) \in \mu(L_{G'}, X_{\neg C_j}) \tag{19}$$

If we assume the rule $(18,19)$ then none of the rules $(5,6,7,8)$ is not derive $(w = bxc, 0)$. If $(18,19)$ is untrue then $(xc,1) \in \mu(L_{G'}, X_{\neg B_i})$ for some i and $(bx,1) \in \mu(L_{G'}, X_{\neg C_j})$ for some j

and hence by one of the rules

$$Y_{\neg A \overset{0}{\to} bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc} \overset{1}{\to} bX_{\neg B_i}$$
$$Y_{\neg A \overset{0}{\to} bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc} \overset{1}{\to} X_{\neg C_j c}$$

the non terminal $Y_{\neg A \overset{0}{\to} bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc}$ derives $(w,1)$. Therefore $(w,0) \in \mu(L_G,A)$ if and only if $(w,0) \notin \mu(L'_G,X_{\neg A})$, that is $(w,1) \in \mu(L_G,X_{\neg A})$.

This completes the proof.

Definition 3.4. Right and left fuzzy quotient is defined as follows

$$\begin{aligned}
\mu(L_1/L_2) = \mu(L_1.L_2^{-1}) &= \{(u,r_u)/(uv,r_{uv}) \in \mu(L_1) \; and \; (v,r_v) \in \mu(L_2)\} \\
r_u &> r_{uv}, if \; r_{uv} = r_v \\
r_u &= r_{uv}, if \; r_{uv} < r_v \\
& and \\
\mu(L_2/L_1) = \mu(L_1^{-1}.L_2) &= \{(v,r_v)/(uv,r_{uv}) \in \mu(L_1) \; and \; (u,r_u) \in \mu(L_2)\} \\
r_v &> r_{uv}, if \; r_{uv} = r_u \\
r_v &= r_{uv}, if \; r_{uv} < r_u
\end{aligned}$$

where r_u, r_v and $r_{uv} \in (0,1]$.

Theorem 3.5. Let $\mu(L)$ be any fuzzy linear conjunctive languages over Σ and for any terminal $d \in \Sigma$. Then prove that, the languages $\mu(L/d)$ and $\mu(d/L)$ are fuzzy linear conjunctive.

Proof. The argument is again a direct construction. Let $G = (V,\Sigma,P,S)$ be an arbitrary fuzzy conjunctive grammar in the fuzzy linear normal form, let $d \in \Sigma$. We construct a fuzzy grammar for the fuzzy languages $\mu(L/d)$ (fuzzy right quotient) and $\mu(d/L)$ (fuzzy left quotient). Let $V' = \{A'/A \in V\}$ be a copy of V. Define a new fuzzy grammar $G' = (V \cup V',\Sigma,P \cup P',S')$, where P' consists of the following rules:

i. For each production $A \overset{r_{BC}}{\to} bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc \in P$ $(m+n \geq 1)$, such that $c = d$, and then there is a rule

$$A' \overset{r_{B'C}}{\to} bB'_1\&\ldots\&bB'_m\&C_1\&\ldots\&C_n \in P' \tag{20}$$

Where $r_{BC} \leq r_{B'C}$.

ii. For each production $A \overset{r_a}{\to} a$ in P, such that $a = d$, and then there is a rule $A' \overset{1}{\to} \varepsilon$ in P'.

Claim 1: For each nonterminal $A \in V$, $\mu(L_G, A) = \mu(L_{G'}, A)$. Every derivation valied in G is valied in G' as well, and thus $\mu(L_G, A) \leq \mu(L_{G'}, A)$. On the other hand,the bodies of rules for nonterminal form V in the grammar G' do not contain any nonterminal not in V', and therfore each valid derivation from $A \in V$ in G' is also valid in G and thus $\mu(L_{G'}, A) \leq \mu(L_G, A)$.

Claim 2: For each nonterminal $A \in V$, $\mu(L_{G'}, A') \leq \mu\{(L_G, A)/d\}$. Let $w \in \Sigma^*$ and $(w, r_{B'C}) \in \mu(L_{G'}, A')$ we shall prove that $(w, r_{B'C}) \in \mu\{(L_G, A)/d\}$, it is enough to prove that $(wd, r_{BC}) \in \mu(L_G, A)$. Now, we using induction on length l of the derivation w.

Basis $l = 2$**:** $A' \overset{r}{\Longrightarrow}'_G (w) \overset{r_w}{\Longrightarrow}'_G w$. This implies that $w = \varepsilon$ and $A' \overset{1}{\to} \varepsilon \in P'$. By the construction of P', $A \overset{r_a}{\to} d \in P$ and therefore $(wd = \varepsilon d = d, r_a) \in \mu\{(L_G, A)/d\}$, $r_a \leq 1$.

Induction step $l \geq 2$**:** Let $(w, r_{B'C})$ be derivable from A'. Then the derivation begins with an application of a rule of type (20) and thus is of the form

$$A' \overset{r_{B'C}}{\Longrightarrow}_{G'} bB'_1\&\ldots\&bB'_m\&C_1\&\ldots\&C_n \overset{r_{C'}}{\Longrightarrow}_{G'} \cdots \overset{r_w}{\Longrightarrow}_{G'} w\&\ldots\&w \overset{r_w}{\Longrightarrow}_{G'} w \in P' \tag{21}$$

Where $r_{B'C} = min\{r'_C, \ldots, r_w\}$. Let $(w = au, r_{B'C})$, it follows from (21) that $a = b$, $(u, r_{B'_i})$ is derivable from each B'_i in less than l steps and $(w, r_{C_j}) \in \mu(L'_G, C_j)$ for all j.

iii. By the induction hypothesis, $(ud, r_{B_i}) \in \mu(L_G, B_i)$ and $(wd = bud, r_{bB_i}) \in \mu(L_G, bB_i)$ for all i.

iv. By claim 1, $(w, r_{C_j}) \in \mu(L_G, C_j)$ this implies that $(wd = wc, r_{Cc_j}) \in \mu(L_G, C_j c)$ for all j.

v. By the construction of P', then rule (20) implies that $A \overset{r_{BC}}{\to} bB_1\&\ldots\&bB_m\&C_1c\&\ldots\&C_nc \in P$, where $r_{BC} = min\{r_{bB_i}, r_{Cc_j}\}$. Together, these (iii, iv, v) three results allow to derive the string $(wd, r_{BC}) \in \mu(L_G, A)$ and therefore $(w, r_{B'C}) \in \mu\{(L_G, A)/d\}$, where $r_{BC} \leq r_{B'C}$. Hence,

$$\mu(L_{G'}, A') \leq \mu(L_G, A)/d \tag{22}$$

Claim 3: For each nonterminal $A \in V$, then we prove that $\mu\{(L_G, A)/d\} \leq \mu(L_{G'}, A')$. Let $(w, r_{B'C}) \in \mu\{(L_G, A)/d\}$, then $(wd, r_{BC}) \in \mu(L_G, A)$, we shall prove that $(w, r_{B'C}) \in \mu(L_{G'}, A')$;

the argument is an induction on the length of w.

Basis $|w| = 0$**:** Let $w = \varepsilon$ and thus $(wd = \varepsilon d = d, r_a) \in \mu(L_G, A)$, then $A \xrightarrow{r_a} d \in P$. Consequently $A' \xrightarrow{1} \varepsilon \in P'$, and $(w = \varepsilon, 1) \in \mu(L_{G'}, A')$, where $r_a \leq 1$.

Induction step $|w| \geq 1$**:** Let $(wd, r_{BC}) \in \mu(L_G, A)$, then there exists a derivation

$$A \xrightarrow{r_{BC}}_G bB_1\& \ldots \& bB_m \& C_1 c \& \ldots \& C_n c \in P, \ such\ that\ d = c \tag{23}$$

Now, $w = bu$, $d = c$ and $(ud, r_{B_i}) \in \mu(L_G, B_i)$ for all i and $(w, r_{C_j}) \in \mu(L_G, C_j)$ for all j.

vi. By the induction hypothesis, $(u, r_{B'_i}) \in \mu(L_{G'}, B'_i)$ and $(w = bu, r_{bB'_i}) \in \mu(L_G, bB'_i)$, for all i.

vii. By claim 1, $(w, r_{C_j}) \in \mu(L_{G'}, C_j)$.

viii.Since rule (23) is in P, then the rule of the form (20) must be in P'.

Now, $A' \xRightarrow{r_{B'C}}_G bB'_1 \& \ldots \& bB'_m \& C_1 \& \ldots \& C_n \xRightarrow{r_{C'}}_G \cdots \xRightarrow{r_w}_G w \& \ldots \& w \xRightarrow{r_w}_G w$. Hence $(w, r_{B'C}) \in \mu(L_{G'}, A')$.

$$\mu\{(L_G, A)/d\} \leq \mu(L_{G'}, A') \tag{24}$$

From (22) and (24), $\mu(L_{G'}, A') = \mu\{(L_G, A)/d\}$.

This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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