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# SLANT LIGHTLIKE SUBMERSIONS FROM AN INDEFINITE NEARLY KÄHLER MANIFOLD INTO A LIGHTLIKE MANIFOLD

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**Abstract.** As a generalization of almost Hermitian submersions, we introduce slant lightlike submersion from an indefinite nearly Kähler manifold into a lightlike manifold. We establish the existence theorems for these submersions and investigate the geometry of foliations which are arisen from the definition of lightlike submersion. We also find necessary and sufficient condition for the leaves of the distributions to be totally geodesic foliations in indefinite nearly Kähler manifold.

**Keywords:** nearly Kähler manifold; lightlike manifold; slant submersions.

**2010 AMS Subject Classification:** 32Q60, 53C12, 53C22.

## 1. Introduction

Let $(M, g_M)$ and $(N, g_N)$ be two Riemannian manifolds. The idea of Riemannian submersion between two manifolds were introduced by O'Neill [5] and Gray [4]. Later, such submersions

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were considered between manifolds with differentiable structures. As an analogue of holomorphic submanifolds, Watson defined almost Hermitian submersions between almost Hermitian manifolds [13]. O' Neill introduced the semi-Riemannian submersions [6].

On the other hand, it is known that if $M$ and $N$ are Riemannian manifolds, then the fibres are always Riemannian manifolds. However, if $M$ and $N$ are semi-Riemannian manifolds, then the fibres may not be semi-Riemannian manifolds. Therefore, in [9], Sahin introduced a screen lightlike submersion from a lightlike manifold into a semi-Riemannian manifold. Later, Sahin and Gunduzulp in [10], introduced a lightlike submersion from a semi-Riemannian manifold into a lightlike manifold.

As a generalization of almost Hermitian submersions, Sahin introduced slant submersions from almost Hermitian manifolds into Riemannian manifolds. We have studied some research papers related to it. Some of them are: Slant lightlike submanifolds of an indefinite Cosymplectic manifold [15], slant lightlike submanifold of indefinite Kenmotsu manifolds [16], screen pseudo slant lightlike submanifolds of indefinite Sasakian manifold [17], radical transversal screen-semi-slant lightlike submanifolds of indefinite Sasakian manifolds [18] etc.

The geometry of lightlike submanifolds has extensive uses in mathematical physics and in particular in the theory of general relativity [2]. It is also well known that semi-Riemannian submersions are of interest in physics, owing to their application in the Yang-Mills theory, Kaluza-Klein theory and supergravity and superstring theories [1,3,7,8]. Moreover, we obtained the nonexistence of totally contact umbilical proper slant lightlike submanifolds of indefinite Sasakian manifold [11]. Thus all these facts and results of above papers motivated us to work on the theory of lightlike submersions with slant lightlike submersions. From these facts we get the concept of slant lightlike submersion from an indefinite nearly Kähler manifold to lightlike manifold.

In the present paper we introduce slant lightlike submersion from an indefinite nearly Kähler manifold into lightlike manifold.

The paper is organized as follows: In section 2, we collect some basic information and notions needed for this paper. In section 3, we give definition of slant Riemannian submersions and investigate the geometry of leaves of distributions. We obtain necessary and sufficient conditions for such slant lightlike submersions to be totally geodesic.

## 2. Preliminaries

Let $(M,g)$ be a real n-dimensional smooth manifold where $g$ is a symmetric tensor field of type $(0,2)$. The radical space $Rad\ T_pM$ of $T_pM$ is defined by

$$Rad\ T_pM = \{\xi \in T_pM : g(\xi,X) = 0\ ,\ \forall X \in T_pM\}.$$

The dimension of $Rad\ T_pM$ is called the nullity degree of $g$. If the mapping

$$Rad\ TM : p \in M \rightarrow Rad\ T_pM,$$

defines a smooth distribution on $M$ of rank $r > 0$, then $Rad\ TM$ is known as the radical distribution of $M$ and the manifold $M$ is known as $r$-lightlike manifold if $0 < r \leq n$, see [12].

Let $(M_1,g_1)$ be a semi-Riemannian manifold and let $(M_2,g_2)$ be an $r$-lightlike manifold. Consider a smooth submersion $f : M_1 \rightarrow M_2$, then $f^{-1}\ (p)$ is a submanifold of $M_1$ of dim $M_1$ $-$dim $M_2$ for $p \in M_2$. The kernel of $f_*$ at the point $p$ is given by

$$(\ker f_*) = \{X \in T_p(M_1) : f_*(X) = 0\},$$

and $(ker f_*)^{\perp}$is given by

$$(\ker f_*)^{\perp} = \{Y \in T_p(M_1) : g_1(Y,X) = 0, for\ all\ X \in (\ker f_*)\},$$

Since $T_p(M_1)$ is a semi-Riemannian vector space, $(\ker f_*)^{\perp}$ may be not complementary to $(\ker f_*)$. Hence, we assume that

$$\Delta = (\ker f_*) \cap (\ker f_*)^{\perp} \neq \{0\}.$$

Thus, we get the following four cases of submersions:

**Case** 1: $0 < dim\Delta < min\{\dim(\ker f_*), \dim(\ker f_*)^{\perp}\}$. Then $\Delta$ is called as radical subspace of $T_p(M_1)$.

Since $kerf_*$ is a real lightlike vector space and $S(\ker f_*)$ is the complementary non degenerate subspace of $\Delta$ in $S(Kerf_*)$ and we obtain

$$(\ker f_*) = \Delta \perp S(\ker f_*).$$

Similarly, we have

$$(\ker f_*)^{\perp} = \Delta \perp S(\ker f_*)^{\perp},$$

where $S(\ker f_*)^{\perp}$ is a complementary non degenerate subspace of $\Delta$ in $(\ker f_*)^{\perp}$.

Since $S(\ker f_*)^{\perp}$ is non-degenerate in $T_p(M_1)$, we get

$$T_p(M_1) = S(\ker f_*) \perp (S(\ker f_*))^{\perp},$$

where $(S(kerf_*))^{\perp}$ is the complementary subspace of $S(\ker f_*)$ in $T_p(M_1)$. Since $S(\ker f_*)$ and $(S(\ker f_*))^{\perp}$ are non-degenerate in $T_p(M_1)$, we get

$$(S(\ker f_*))^{\perp} = S(\ker f_*)^{\perp} \perp (S(\ker f_*)^{\perp})^{\perp}.$$

Thus, from [2] , a quasi-orthonormal basis of $M_1$ along $(\ker f_*)$ can be constructed. Therefore, we obtain

$$\begin{aligned} g(\xi_i,\xi_j) &= g(N_i,N_j) = 0, \quad g(\xi_i,N_j) = \delta_{ij}, \\ g(W_\alpha,\xi_j) &= g(W_\alpha,N_j) = 0, \quad g(W_\alpha,W_\beta) = \varepsilon_\alpha \delta_{\alpha\beta}, \end{aligned} \tag{1}$$

$\{\xi_i\}$ is a basis of $\Delta$, $\{N_i\}$ are smooth lightlike vector fields of $(S(\ker f_*)^{\perp})^{\perp}$ and $\{W_\alpha\}$ is a basis of $S(\ker f_*)^{\perp}$. Let $ltr(\ker f_*)$ is the set of vector fields $\{N_i\}$ and consider

$$tr(\ker f_*) = ltr(\ker f_*) \perp S(\ker f_*)^{\perp}.$$

Using equation (2.1) , it is clear that $ltr(\ker f_*)$ and $Kerf_*$ are not orthogonal to each other. Denote by $\mathscr{V} = \ker f_*$ the vertical space of $T_p(M_1)$ and $H = tr(\ker f_*)$ the horizontal space of $T_p(M_1)$. Thus we have

$$T_p(M_1) = \mathscr{V}_p \oplus \mathscr{H}_p.$$

**Definition 1.1.** Let $(M_1, g_1)$ be a semi-Riemannian manifold and let $(M_2, g_2)$ be an $r$-lightlike manifold. Let $f : M_1 \to M_2$ be a submersion such that:

(a) $\dim \Delta = \dim\{(\ker f_*) \cap (\ker f_*)^{\perp}\} = r, 0 < r < \min\dim\{\dim(\ker f_*), \dim(\ker f_*)^{\perp}\}$

(b) $f_*$ preserves the length of horizontal vectors,i.e.

$$g_1(X,Y) = g_2(f_*X, f_*Y), \text{ for } X,Y \in \Gamma(\mathscr{H}). \tag{2}$$

*. Then f is called an r-lightlike submersion.*

**Case** 2: $\dim \Delta = \dim(\ker f_*) < \dim(\ker f_*)^{\perp}$. Then $\mathscr{V} = \Delta$, $\mathscr{H} = S(\ker f_*)^{\perp} \perp ltr(\ker f_*)$, and $f$ is called an isotropic submersion.

**Case** 3: $\dim \Delta = \dim(\ker f_*)^{\perp} < \dim(\ker f_*)$. Then $\mathscr{V} = S(\ker f_*) \perp \Delta$, $\mathscr{H} = ltr(\ker f_*)$, and $f$ is called a co-isotropic submersion.

**Case** 4: $\dim \Delta = \dim(\ker f_*) = \dim(\ker f_*)^{\perp}$. Then $\mathscr{V} = \Delta$, $\mathscr{H} = ltr(\ker f_*)$ and $f$ is called a totally lightlike submersion.

**Definition 1.2.** Let $(M, g, J)$ be an indefinite almost Hermitian manifold and $\bigtriangledown$ be the Levi-Civitia connection on $M$ with respect to $g$ such that

$$J^2 = -I, g(JX, JY) = g(X,Y), \tag{3}$$

for $X,Y$ on $M$. Then $M$ is called an indefinite nearly Kähler manifold if

$$(\bigtriangledown_X J)Y + (\bigtriangledown_Y J)X = 0, \text{ for all } X,Y \in \Gamma(TM). \tag{4}$$

It is well known that every Kähler manifold is a nearly Kähler manifold but converse is not true.

**Note:** Whatever it is need we have suppose the horizontal vector field to be basic. For any arbitrary tangent vector fields $V$ and $W$ on $M$, we have

$$(\bigtriangledown_V J)W = P_V W + Q_V W, \tag{5}$$

where $P_V W$ and $Q_V W$ denote the horizontal and vertical component of $(\bigtriangledown_V J)W$ respectively.

For a Kähler manifold $M$ we have

$$P = Q = 0.$$

If $M$ is a nearly Kähler manifold, then it can be easily seen that both $P$ and $Q$ are anti-symmetric in $V$ and $W$, hence

$$P_V W = -P_W V \text{ and } Q_V W = -Q_W V. \tag{6}$$

We need the statement of following theorem to define a slant lightlike submersion from an indefinite nearly Kähler manifold into a lightlike manifold.

**Theorem 2.1.** Let $f : M_1 \to M_2$ be an r-lightlike submersion from an indefinite almost Hermitian manifold $(M_1, g_1, J)$ where $g_1$is a semi-Riemannian metric of index $2r$, to an $r$-lightlike manifold $(M_2, g_2)$. Let $J\Delta$ be a distribution on M such that $\Delta \cap J\Delta = 0$. Then any distribution complementary to $J\Delta \oplus Jltr(\ker f_*)$ in $S(\ker f_*)$ is Riemannian.

## 3. SLANT LIGHTLIKE SUBMERSION

Let $M$ be an $r$-lightlike submanifold of an indefinite Hermitian manifold $M^{'}$of index $2r$. Then M is a slant lightlike submanifold of $M^{'}$if the following conditions are satisfied:

$(a)$ $Rad(TM)$ is a distribution on $M$ such that

$$JRadTM \cap RadTM = \{0\}.$$

$(b)$ For any non zero vector field tangent to $D$ for $p \in U \subset M$, the angle $\theta(X)$ between $JX$ and the vector space $D_p$ is constant i.e., it is independent of the choice of $p \in U \subset M$ and $X \in D_p$, where $D$ is the distribution complementary to $JRadTM \oplus Jltr(TM)$ in the screen distribution $S(TM)$.

This constant angle $\theta(X)$ is called the slant angle of the distribution $D$. If $D \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$ then a slant lightlike submanifold is said to be proper slant lightlike submanifold.

**Definition 3.1.** Let $(M_1, g_1, J)$ be a real $2m$-dimensional indefinite nearly Kähler manifold, where $g_1$ is a semi-Riemannian metric of index $2r, 0 < r < m$ and any other manifold $(M_2, g_2)$

which is an r-lightlike manifold. Let f be an r-lightlike submersion, $f: M_1 \to M_2$. Then f is said to be slant lightlike submersion if following conditions are satisfied:

(c) $J\Delta$ is a distribution in $\ker f_*$ such that $\Delta \cap J\Delta = \{0\}$.

(d) The angle $\theta(X)$ between $JX$ and $D$ is constant for each non zero vector field X tangent to $D$ , where D is the distribution complementary to $J\Delta \oplus Jltr(\ker f_*)$ in $S(\ker f_*)$.

Hence, we get

$$T_p(M_1) = \mathscr{V}_p \oplus \mathscr{H}_p,$$

$$T_p(M_1) = \{\Delta \perp (J\Delta \oplus J\, ltr(\ker f_*)) \perp D\} \oplus \{f(D) \perp \mu \perp ltr(\ker f_*)\},$$

where $\mu$ is the orthogonal subbundle complementary to $f(D)$ in $S(\ker f_*)$. Let $f$ be a slant lightlike submersion from an indefinite nearly Kähler manifold $(M_1, g_1, J)$ into an $r-$lightlike manifold $(M_2, g_2)$. Then any $X \in v_p$ can be written as

$$JX = \phi X + \omega X, \tag{7}$$

where $\phi X$ and $\omega X$ are the tangential and transverse components of JX, respectively. Similarly, for any $Z \in \mathscr{H}_p$, we get

$$JZ = \mathscr{B}Z + \mathscr{C}Z, \tag{8}$$

where $\mathscr{B}Z$ and $\mathscr{C}Z$ are the tangential and transversal component of JZ, respectively. Denote $P_1, P_2, Q_1$ and $Q_2$ the projections onto the distributions $\Delta$, $J\Delta$, $J\ ltr(kerf_*)$ and $D$ respectively. Thus, we can express X as

$$X = P_1X + P_2X + Q_1X + Q_2X, \tag{9}$$

for any $X \in \mathscr{V}_p$. Applying J to (9), we get

$$JX = JP_1X + JP_2X + \omega Q_1X + \phi Q_2X + \omega Q_2X, \tag{10}$$

for any $X \in \mathscr{V}_p$. Then, clearly,

$$\begin{aligned}
JP_1X &= \phi P_1X \in \Gamma(J\Delta)\\
JP_2X &= \phi P_2X \in \Gamma(\Delta)\\
\omega P_1X &= 0\\
\omega P_2X &= 0\\
\phi Q_1X &= 0\\
\omega Q_1X &\in \Gamma(ltr(kerf_*))\\
\phi Q_2X &\in \Gamma(D)\\
\omega Q_2X &\in \Gamma(f(D))
\end{aligned}$$

Therefore, we can write

$$\phi X = \phi P_1X + \phi P_2X + \phi Q_2X. \tag{11}$$

Since the geometry of Riemannian submersions is characterized by O'Neill's tensors T and A, Sahin [9] defined these tensors for lightlike submersions as follows:

$$\mathscr{A}_E F = \mathscr{H}\nabla_{\mathscr{H}E}\mathscr{V}F + \mathscr{V}\nabla_{\mathscr{H}E}\mathscr{H}F, \tag{12}$$

$$\mathscr{T}_E F = \mathscr{H}\nabla_{\mathscr{V}E}\mathscr{V}F + \mathscr{V}\nabla_{\mathscr{V}E}\mathscr{H}F, \tag{13}$$

for vector fields E and F on $M_1$, where $\nabla$ is the Levi-Civitia connection of $g_1$. It should be noted that T and A are skew-symmetric tensors in Riemannian submersions but not in lightlike submersions because the horizontal and vertical subspaces are not orthogonal to each other. The tensors T and A both reverse the horizontal and vertical subspaces and, moreover, T has the symmetric property ie.

$$\mathcal{T}_U W = \mathcal{T}_W U, \ \ \forall U, W \in \Gamma(\ker f_*). \tag{14}$$

**Lemma 3.1.** Let $(M_1, g_1, J)$ be an indefinite nearly Kähler manifold, where $g_1$ is a semi-Riemannian metric of index $2r$ and let $(M_2, g_2)$ be an r-lightlike manifold. If f be a slant lightlike submersion such as $f : M_1 \to M_2$ , then

$$\nabla_X Y = \mathcal{V}\nabla_X Y + \mathcal{T}_X Y \ , \tag{15}$$

$$\nabla_X V = \mathcal{H}\nabla_X V + \mathcal{T}_X V, \tag{16}$$

$$\nabla_V X = \mathcal{A}_V X + \mathcal{V}\nabla_V X, \tag{17}$$

$$\nabla_V U = \mathcal{H}\nabla_V U + \mathcal{A}_V U, \tag{18}$$

for any $X, Y \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^{\perp}$.

**Lemma 3.2.** Let $(M_1, g_1, J)$ be an indefinite nearly Kähler manifold, where $g_1$ is a semi-Riemannian metric of index $2r$ and let $(M_2, g_2)$ be an $r$-lightlike manifold. If f be a slant lightlike submersion such as $f : M_1 \to M_2$ ,then

$$(\nabla_X \omega)Y = \mathscr{C}\mathscr{T}_X Y - \mathcal{T}_X \phi Y - P_X Y, \tag{19}$$

$$(\nabla_X \phi)Y = \mathscr{B}\mathscr{T}_X Y - \mathcal{T}_X \omega Y - Q_X Y, \tag{20}$$

where

$$\begin{aligned}
(\nabla_X \omega)Y &= \mathcal{H}\nabla_X \omega Y - \omega\mathcal{V}\nabla_X Y, \\
(\nabla_X \phi)Y &= \mathcal{V}\nabla_X \phi Y - \phi\mathcal{V}\nabla_X Y,
\end{aligned}$$

for any $X, Y \in \Gamma(\ker f_*)$.

**Proof.** For any $X, Y \in \Gamma(\ker f_*)$, using equations $(5), (7), (8), (15)$ and $(16)$, we get

$$\mathcal{T}_X \phi Y + \mathcal{V}\nabla_X \phi Y + \mathcal{H}\nabla_X \omega Y + \mathcal{T}_X \omega Y + P_X Y + Q_X Y = \mathscr{B}\mathscr{T}_X Y + \mathscr{C}\mathscr{T}_X Y + \phi\mathcal{V}\nabla_X Y + \omega\mathcal{V}\nabla_X Y.$$

From above equation, we have

$$(\nabla_X\phi)Y + (\nabla_X\omega)Y = \mathscr{B}\mathscr{T}_XY + \mathscr{C}\mathscr{T}_XY - \mathscr{T}_X\phi Y - \mathscr{T}_X\omega Y - P_XY - Q_XY,$$

Comparing vertical and horizontal parts, we obtain

$$\begin{aligned}(\nabla_X\omega)Y &= \mathscr{H}\nabla_X\omega Y - \omega\mathscr{V}\nabla_XY,\\ (\nabla_X\phi)Y &= \mathscr{V}\nabla_X\phi Y - \phi\mathscr{V}\nabla_XY.\end{aligned}$$

**Theorem 3.1.** Let $(M_1, g_1, J)$ be an indefinite nearly Kähler manifold, where $g_1$ is a semi-Riemannian metric of index $2r$ and let $(M_2, g_2)$ be an $r$-lightlike manifold. Let $f$ be a slant lightlike submersion such as $f : M_1 \to M_2$ ,then f is a proper slant lightlike submersion if and only if

$(a)$ $J$ $(ltr(\ker f_*))$ is a distribution on $M_1$;

$(b)$ for any $X \in \Gamma(\ker f_*)$, there exists a constant $\lambda \in [-1, 0]$ such that

$$\phi^2 Q_2X = \lambda Q_2X, \tag{21}$$

moreover, in this case $\lambda = -cos^2\theta$.

**Proof.** Let f be a slant lightlike submersion. Then $J\Delta$ is a distribution on S(TM). Hence by virtue of Theorem (1), $J$ $(ltr(\ker f_*))$ is a distribution on $M_1$. Further the slant angle between J$Q_2X$ and $D_p$ is constant and given by

$$\cos\theta(Q_2X) = \frac{g(JQ_2X, \phi Q_2X)}{|JQ_2X|\,|\phi Q_2X|},$$

$$\cos\theta(Q_2X) = -\frac{g(Q_2X, \phi^2 Q_2X)}{|JQ_2X|\,|\phi Q_2X|}. \tag{22}$$

and also the $\cos\theta(Q_2X)$ is also given by

$$\cos\theta(Q_2X) = \frac{|\phi Q_2X|}{|JQ_2X|}. \tag{23}$$

Hence using (22) and (23), we obtain

$$\cos^2\theta(Q_2X) = -\frac{g(Q_2X, \phi^2 Q_2X)}{|Q_2X|^2}.$$

Since the angle $\theta(Q_2X)$ is constant on $D$, we have

$$\phi^2 Q_2 X = \lambda Q_2 X,$$

where $\lambda = -cos^2\theta$. (a) implies that J$\Delta$ is a distribution on $S(\ker f_*)$. Hence, in view of theorem (2.1), any distribution complementary to $J\Delta \oplus Jltr(\ker f_*)$ in $S(\ker f_*)$ is Riemannian.

**Corollary 3.1.** Let $(M_1, g_1, J)$ be an indefinite nearly Kähler manifold, where $g_1$ is a semi-Riemannian metric of index $2r$ and let $(M_2, g_2)$ be an r-lightlike manifold. Let $f$ be a proper slant lightlike submersion such as $f : M_1 \to M_2$ with slant angle $\theta$, then for any $X, Y \in \Gamma(\ker f_*)$

$$g_1(\phi X, \phi Y) = \cos^2\theta g_1(X, Y), \tag{24}$$

and

$$g_1(\omega X, \omega Y) = \sin^2\theta g_1(X, Y). \tag{25}$$

**Theorem 3.2.** Let $(M_1, g_1, J)$ be an indefinite nearly Kähler manifold, where $g_1$ is a semi-Riemannian metric of index $2r$ and let $(M_2, g_2)$ be an $r-$lightlike manifold. Let $f$ be a slant lightlike submersion such as $f : M_1 \to M_2$ , then $f$ is a proper slant lightlike submersion if and only if

$(a)$ $J(ltr(kerf_*))$ is a distribution on $M_1$;

$(b)$ for any vector field tangent to $M_1$, there exists a constant $v \in [-1, 0]$ such that

$$\mathscr{B}\omega Q_2 X = v Q_2 X, \tag{26}$$

where $v = -sin^2\theta$.

**Proof.** Let $f$ be a slant lightlike submersion, then $J$ $(ltr(\ker f_*))$ is a distribution on $M_1$. Using equations $(3), (7), (8)$ to $(10)$, we obtain

$$-X = -P_1 X - P_2 X + \phi^2 Q_2 X + \omega\phi Q_2 X + \mathscr{B}\omega Q_2 X + \mathscr{C}\omega Q_2 X + \mathscr{B}\omega Q_1 X + \mathscr{C}\omega Q_1 X.$$

Comparing the components of the distribution D on both sides of last equation, we get

$$-Q_2 X = \phi^2 Q_2 X + \mathscr{B}\omega Q_2 X. \tag{27}$$

Hence by using $(21)$, we get

$$\mathscr{B}\omega Q_2 X = -\sin^2\theta Q_2 X.$$

Conversely, by virtue of equations (3.20) and (3.21), we obtain

$$\phi^2 Q_2 X = -\cos^2\theta Q_2 X.$$

Further, we prove that the orthogonal complement subbundle $\mu$ of $f(D)$ in $S(\ker f_*)^\perp$ is holomorphic with respect to $J$ and determine the dimension.

**Theorem 3.3.** Let $(M_1, g_1, J)$be an indefinite nearly Kähler manifold, where $g_1$ is a semi-Riemannian metric of index $2r$ and let $(M_2, g_2)$ be an $r-$lightlike manifold. Let $f$ be a slant lightlike submersion such as $f : M_1 \to M_2$ ,then $\mu$ is invariant under $J$.

**Proof.** In view of equation (7), for any $U \in \Gamma(\mu)$ and $\omega X \in \Gamma(f(D))$, we get

$$\begin{aligned} g_1(JU, \omega X) &= g_1(JU, JX - \phi X). \\ g_1(JU, \omega X) &= -g_1(JU, \phi X). \end{aligned}$$

By virtue of Theorem (3.1), we have

$$\begin{aligned} g_1(JU, \omega X) &= g_1(U, J\phi X) = g_1(U, \phi^2 X) + g_1(U, \omega\phi X), \\ g_1(JU, \omega X) &= -\cos^2\theta g_1(U, X) + g_1(U, \omega\phi X), \\ g_1(U, \omega\phi X) &= 0. \end{aligned}$$

Similarly,

$$g_1(JU, Y) = -g_1(U, JY) = 0,$$

for any $Y \in \Gamma(\ker f_*)$. Moreover, for any N$\in \Gamma(ltr(ker f_*))$, we obtain

$$g_1(JU, N) = -g_1(U, JN) = 0.$$

Hence, the proof follows.

**Theorem 3.4.** Let $(M_1^m, g_1, J)$ be an indefinite nearly Kähler manifold, where $g_1$ is a semi-Riemannian metric of index $2r$ and let $(M_2^n, g_2)$ be an $r-$lightlike manifold. Let $f$ be a proper slant lightlike submersion such as $f : M_1^m \to M_2^n$ , then

$$\dim(\mu) = 2n - m + 2r.$$

If $\mu = \{0\}$, then $n = \frac{m-2r}{2}$.

**Proof.** Since $\dim S(\ker f_*)^{\perp} = n - r$, and $\dim S(\ker f_*) = m - n - 3r$, and we know that

$$\dim(\mu) = \dim S(\ker f_*)^{\perp} - \dim S(\ker f_*).$$

We conclude that $\dim(\mu) = 2n - m + 2r$.

Moreover, $M_1$ is an indefinite nearly Kähler manifold and its dimension m is even. Hence, the dimension of $\mu$ is also even.

**Lemma 3.3.** Let $(M_1^m, g_1, J)$ be an indefinite nearly Kähler manifold, where $g_1$ is a semi-Riemannian metric of index $2r$ and let $(M_2^n, g_2)$ be an $r$-lightlike manifold. Let $f$ be a proper slant lightlike submersion such as $f : M_1^m \to M_2^n$ . If $\{e_1, \ldots\ldots\ldots, e_{m-n-3r}\}$ be a local orthonormal basis of $D$, then

$\{\cos\theta\omega e_1, \ldots\ldots\ldots\ldots, \cos\theta\omega e_{m-n-3r}\}$ is a local orthonormal basis of $f(D)$.

**Proof.** Since $\{e_1, \ldots\ldots\ldots, e_{m-n-3r}\}$ be a local orthonormal basis of $D$ and $D$ is Riemannian, then in view of equation (25)

$$g_1(\cos\theta\omega e_i, \cos\theta\omega e_j) = \cos^2\theta\sin^2\theta g_1(e_i, e_j) = \delta_{ij},$$

for any $i, j \in \{1, \ldots\ldots, \frac{m-n}{2}\}$.

This proves the lemma.

We note that the above Theorem (3.3) tells that the distributions $\mu$ and $D \oplus f(D)$ are even dimensional. It implies that the distribution $D$ is even dimensional. More precisely, we have the following result whose proof is similar to the above lemma.

**Lemma 3.4.** Let $(M_1^m, g_1, J)$ be an indefinite nearly Kähler manifold, where $g_1$ is a semi-Riemannian metric of index $2r$ and let $(M_2^n, g_2)$ be an $r$-lightlike manifold. Let $f$ be a proper slant lightlike submersion such as $f : M_1^m \to M_2^n$ . If $\{e_1, \ldots\ldots\ldots, e_{\frac{m-n-3r}{2}}\}$ are unit vector fields in $D$, then

$\{e_1, \sec\theta\phi e_1, e_2, \sec\theta\phi e_2, \ldots\ldots\ldots, e_{\frac{m-n-3r}{2}}, \sec\theta\phi e_{\frac{m-n-3r}{2}},\}$ is a local orthonormal basis of $D$.

**Theorem 3.5.** Let $(M_1, g_1, J)$ be an indefinite nearly Kähler manifold, where $g_1$ is a semi-Riemannian metric of index $2r$ and let $(M_2, g_2)$ be an $r$-lightlike manifold. Let $f$ be a proper slant lightlike submersion such as $f : M_1 \to M_2$. If $\omega$ is parallel with respect to $\nabla$, then

$$\mathscr{T}_{\phi X}\phi X = -\cos^2\theta\mathscr{T}_X X + 2P_X\phi X,\ \mathscr{T}_{\phi X}\phi X = -\mathscr{T}_X X + 2P_X\phi X,\ \mathscr{T}_{\phi X}\phi X = 2P_X\phi X,$$

for any $X \in \Gamma(D)$, $X \in \Gamma(\Delta \perp J\Delta)$, and $X \in \Gamma(J(ltr(\ker f_*)))$, respectively.

**Proof.** Let $\omega$ be parallel, i.e. $(\nabla_X \omega)Y = 0$. From equation (19), we get

$$\mathscr{C}\mathscr{T}_X Y = \mathscr{T}_X \phi Y + P_X Y. \tag{28}$$

Using above equation (28), we have

$$\mathscr{C}\mathscr{T}_Y X = \mathscr{T}_Y \phi X + P_Y X. \tag{29}$$

From equations (6), (14), (28) and (29), we get

$$\mathscr{T}_{\phi X} \phi X = -\cos^2\theta\, \mathscr{T}_X X + 2P_X \phi X,$$

for any $X \in \Gamma(D)$.

Therefore, by virtue of Theorem (2) and the fact that $\phi^2 X = -X$, for any $X \in \Gamma(\Delta \perp J\Delta)$ and $\phi X = 0$ for any $X \in \Gamma(J(ltr(kerf_*)))$, we get $\mathscr{T}_{\phi X}\phi X = -\mathscr{T}_X X + 2P_X\phi X$ and $\mathscr{T}_{\phi X}\phi X = 2P_X \phi X$.

**Theorem 3.6.** Let $(M_1, g_1, J)$ be an indefinite nearly Kähler manifold, where $g_1$ is a semi-Riemannian metric of index $2r$ and let $(M_2, g_2)$ be an $r-$lightlike manifold. Let f be a proper slant lightlike submersion such as $f : M_1 \to M_2$. Then the distribution $\mathscr{V}$ defines a totally geodesic foliation on $M_1$ if and only if

$$\omega(\mathscr{V}\nabla_X \phi Y + \mathscr{T}_X \omega Y + Q_X Y) + \mathscr{C}(\mathscr{T}_X \phi Y + \mathscr{H}\nabla_X \omega Y + P_X Y) = 0,$$

for any $X, Y \in \Gamma(\mathscr{V})$.

**Proof.** For $X, Y \in \Gamma(\mathscr{V})$, using equations (3), (5), (7), (8), (15) and (16), we get

$$\begin{aligned}\nabla_X Y \quad = \quad & -(\mathscr{B}\mathscr{T}_X\phi Y + \mathscr{C}\mathscr{T}_X\phi Y + \phi\mathscr{V}\nabla_X\phi Y + \omega\mathscr{V}\nabla_X\phi Y + \mathscr{B}\mathscr{H}\nabla_X\omega Y + \mathscr{C}\mathscr{H}\nabla_X\omega Y \\ & +\phi\mathscr{T}_X\omega Y + \omega\mathscr{T}_X\omega Y) + \phi Q_X Y + \omega Q_X Y + \mathscr{B}P_X Y + \mathscr{C}P_X Y,\end{aligned}$$

Hence, $\nabla_X Y \in \Gamma(\mathscr{V})$, if and only if

$$\omega(\mathscr{V}\nabla_X \phi Y + \mathscr{T}_X \omega Y + Q_X Y) + \mathscr{C}(\mathscr{T}_X \phi Y + \mathscr{H}\nabla_X \omega Y + P_X Y) = 0.$$

**Theorem 3.7.** Let $(M_1, g_1, J)$ be an indefinite nearly Kähler manifold, where $g_1$ is a semi-Riemannian metric of index $2r$ and let $(M_2, g_2)$ be an $r$-lightlike manifold. Let $f$ be a proper slant lightlike submersion such as $f: M_1 \to M_2$. Then the distribution $\mathscr{H}$ defines a totally geodesic foliation on $M_1$ if and only if

$$\phi(\mathscr{V}\nabla_V \mathscr{B}W + \mathscr{A}_V \mathscr{C}W + Q_V W) + \mathscr{B}(\mathscr{A}_V \mathscr{B}W + \mathscr{H}\nabla_V \mathscr{C}W + P_V W) = 0,$$

for any $V, W \in \Gamma(\mathscr{H})$.

**Proof.** For V,W$\in \Gamma(\mathscr{H})$, using equations $(3), (5), (7), (8), (17)$ and $(18)$, we get

$$\begin{aligned} \nabla_V W &= -(\mathscr{B}\mathscr{A}_V \mathscr{B}W + \mathscr{C}\mathscr{A}_V \mathscr{B}W + \phi\mathscr{V}\nabla_V \mathscr{B}W + \omega\mathscr{V}\nabla_V \mathscr{B}W + \mathscr{B}\mathscr{H}\nabla_V \mathscr{C}W + \mathscr{C}\mathscr{H}\nabla_V \mathscr{C}W \\ &\quad + \phi\mathscr{A}_V \mathscr{C}W + \omega\mathscr{A}_V \mathscr{C}W) + \phi Q_V W + \omega Q_V W \\ &\quad + \mathscr{B}P_V W + \mathscr{C}P_V W = 0. \end{aligned}$$

Hence, $\nabla V, W \in \Gamma(\mathscr{H})$, if and only if

$$\phi(\mathscr{V}\nabla_V \mathscr{B}W + \mathscr{A}_V \mathscr{C}W + Q_V W) + \mathscr{B}(\mathscr{A}_V \mathscr{B}W + \mathscr{H}\nabla_V \mathscr{C}W + P_V W) = 0.$$

**Corollary 3.2.** Let $(M_1, g_1, J)$ be an indefinite nearly Kähler manifold, where $g_1$ is a semi-Riemannian metric of index $2r$ and let $(M_2, g_2)$ be an $r$-lightlike manifold. Let $f$ be a proper slant lightlike submersion such as $f: M_1 \to M_2$. Then $M_1$ is a locally product Riemannian manifold if and only if

$$\begin{aligned} \omega(\mathscr{V}\nabla_X \phi Y + \mathscr{T}_X \omega Y + Q_X Y) + \mathscr{C}(\mathscr{T}_X \phi Y + \mathscr{H}\nabla_X \omega Y + P_X Y) &= 0, \\ \phi(\mathscr{V}\nabla_V \mathscr{B}W + \mathscr{A}_V \mathscr{C}W + Q_V W) + \mathscr{B}(\mathscr{A}_V \mathscr{B}W + \mathscr{H}\nabla_V \mathscr{C}W + P_V W) &= 0, \end{aligned}$$

for any $X, Y \in \Gamma(\mathscr{V})$ and $V, W \in \Gamma(\mathscr{H})$.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

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