Available online at http://scik.org
J. Math. Comput. Sci. 8 (2018), No. 6, 705-713
https://doi.org/10.28919/jmcs/3571
ISSN: 1927-5307

ON THE GREEN FUNCTION OF HIGHER ORDER DIFFERENTIAL EQUATION WITH NORMAL OPERATOR COEFFICIENTS ON THE SEMI-AXIS

HAMIDULLA I. ASLANOV[1,*], NIGAR A. GADIRLI[2]

[1]Institute of Mathematics and Mechanics of NASA, Azerbaijan

[2]Sumgayit State University, Azerbaijan

Abstract. In the paper we study the Green function for $2n$-th order differential equation with a normal operator coefficient on the semi-axis. To this end, at first we construct the Green function of the equation with "frozen" coefficients. Then we use the Levi method and obtain an integral equation for the Green function of the given equation. We prove the solvability of the integral equation. Using the integral equation we establish the main properties of the Green function.

Keywords: Hilbert space; operator-differential equation; spectrum, Green function; integral equation; eigenvalue; resolvent; operator.

2010 AMS Subject Classification: 34L10, 30E25, 34B24.

1. Introduction. Problem Statement.

Let H be a separable Hilbert space. Denote by $H_1 = L_2[H;[0,\infty)]$ a Hilbert space of strongly measurable functions $f(x)$ $(0 \le x < \infty)$ with the values from H_1 for which

*Corresponding author

E-mail address: aslanov.50@mail.ru

Received November 12, 2017

$$\int_0^\infty \|f(x)\|_H^2 dx < \infty.$$

The scalar product of the element $f(x), g(x) \in H_1$ is determined by the equality

$$[f,g]_{H_1} = \int_0^\infty (f(x), g(x))_H dx.$$

Let us consider in space H_1 an operator generated by the differential expression

$$l(y) = (-1)^n y^{(2n)} + Q(x)y \tag{1.1}$$

and the boundary conditions

$$y^{(l_1)}(0) = y^{(l_2)}(0) = ... = y^{(l_n)}(0) = 0,\ 0 \le l_1 < l_2 < ... < l_n \le 2n-1. \tag{1.2}$$

Here $y \in H_1$ and the derivatives are understood in the strong sense.

Let D' be a union of all functions of the form $\sum\limits_{k=1}^{p} \varphi_k(x) f_k$, where $\varphi_k(x)$ are finite, $2n$-times continuously differentiable scalar functions, and $f_k \in D(Q)$.

Denote by L an operator generated by differential expression (1.1) and boundary conditions (1.2) with domain of definition D'.

We will assume that the operator coefficient $Q(x)$ satisfies the following conditions:

1) The operator $Q(x)$ for almost all $x \in [0,\infty)$ is a normal operator in H and for almost all $x \ge 0$ has a common domain of definition $D(Q)$ in H.

2) For almost all $x \ge 0$ $Q(x)$ is inverse to completely continuous operator, and its eigen values lie in a complex plane out of the sector $\Omega = \{\lambda : |arg\lambda - \pi| < \varepsilon_0,\ 0 < \varepsilon_0 < \pi$ is a constant number$\}$. Let $\alpha_1(x), \alpha_2(x), ..., \alpha_n(x), ...$ be eigen values of the operator $Q(x)$ and assume that they are located in ascending order of the modulus, i.e.

$$|\alpha_1(x)| \le |\alpha_2(x)| \le ... \le |\alpha_n(x)| \le ...$$

and are measurable functions. It is supposed that the series $\sum\limits_{j=1}^{\infty} |\alpha_j(x)|^{\frac{1-4n}{2n}}$ converges for almost all $x \in [0,\infty)$, and its sum $f(x) \in L_1(0,\infty)$. 3) For all $x \in [0,\infty)$ and for $|x-\xi| \le 1$ $\|[Q(\xi) -$

$Q(x)]Q^{-a}(x)\|_H \leq A|x-\xi|$, where $0<a<\frac{2n+1}{2n}$, $\|Q^{-\frac{1}{2n}}(x)Q^{\frac{1}{2n}}(x)\|_H < C_1$, $\|Q^{-\frac{1}{2n}}(\xi)Q^{\frac{1}{2n}}(x)\|_H < C_2, A, C_1, C_2$ are constants. 4) For all $x \geq 0$ for $|x-\xi| > 1$ the following inequality is valid:

$$\left\| Q(\xi) exp\left(-\frac{Im\omega_1}{2}|x-\xi|Q^{\frac{1}{2n}}(x)\right)\right\|_H < B$$

where $Im\omega_1 = \min_i\{Im\omega_i > 0, \omega_2^{2n} = -1\}, B > 0 - const.$

The main goal of the paper is to study the Green function of the operator L.

We have the following theorem.

Theorem 1. If conditions 1) -4) are fulfilled, then for sufficiently large $\mu > 0$ there exists the inverse operator $R_\mu = (L+\mu E)^{-1}$ that is an integral operator with operator kernel $G(x,\eta;\mu)$ that will be called Green function of the operator L. $G(x,\eta;\mu)$ is an operator function in H and depends on two variables $x,\eta(0 \leq x,\eta < \infty)$, the parameter μ and satisfies the following conditions: a) $\frac{\partial^k G(x,\eta;\mu)}{\partial \eta^k}$ $k = 0,1,...,2n-2$ is strongly continuous with respect to the variables (x,η); b) there exists the strong derivative $\frac{\partial^{2n-1}G(x,\eta;\mu)}{\partial\eta^{2n-1}}$, and

$$\frac{\partial^{2n-1}G(x,x+0;\mu)}{\partial\eta^{2n-1}} - \frac{\partial^{2n-1}G(x,x-0;\mu)}{\partial\eta^{2n-1}} = (-1)^n E;$$

c)

$$(-1)^n \frac{\partial^{2n}G(x,\eta;\mu)}{\partial\eta^{2n}} + G(x,\eta;\mu)[Q(\eta)+E] = 0$$

d)

$$\left.\frac{\partial^{l_1}G}{\partial\eta^{l_1}}\right|_{\eta=0} = \left.\frac{\partial^{l_2}G}{\partial\eta^{l_2}}\right|_{\eta=0} = ... = \left.\frac{\partial^{l_n}G}{\partial\eta^{l_n}}\right|_{\eta=0} = 0.$$

The proof of the theorem is carried out in two stages. At first we construct the Green function $G_1(x,\eta;\mu)$ of the operator L_1, generated by the expression

$$l_1(y) = (-1)^n y^{(2n)} + Q(\xi)y + \mu y \tag{1.3}$$

and the boundary conditions

$$y^{(l_1)}(0) = y^{(l_2)}(0) = ... = y^{(l_n)}(0) = 0 \tag{1.4}$$

where ξ is a fixed number.

2. Constructing the Green function of the operator L_1.

We will look for the Green function $G_1(x,\eta,\xi;\mu)$ of the operator L_1 in the form

$$G_1(x,\eta,\xi;\mu) = g(x,\eta;\xi,\mu) + V(x,\eta,\xi;\mu) \tag{2.1}$$

where $g(x,\eta,\xi,\mu)$ is the Green function of the equation $l_1(y) = 0$ on the whole axis.

As is known [4], it has the form:

$$g(x,\eta,\xi;\mu) = \frac{K_\xi^{1-2n}}{2ni}\sum_{\alpha=1}^{n}\omega_\alpha e^{i\omega_\alpha K_\xi|x-\eta|} \tag{2.2}$$

Here ω_α denotes the roots from the (-1) degree of $2n$, lying in the upper halfplane, and $K_\xi = [Q(\xi)+\mu E]^{\frac{1}{2n}}$.

The function $V(x,\eta,\xi;\mu)$ is a bounded solution as $x\to\infty$ of the following problem:

$$l_1(y) = 0 \tag{2.3}$$

$$V^{(l_j)}(x,\eta,\xi;\mu)\Big|_{x=0} = -\, g^{(l_j)}(x,\eta,\xi;\mu)\Big|_{x=0},\ \ j = 1,2,...,n. \tag{2.4}$$

Then for $V(x,\eta,\xi;\mu)$ we have

$$V(x,\eta,\xi;\mu) = \frac{K_\xi^{1-2n}}{2ni}\sum_{p=1}^{n}A_p(\eta,\xi;\mu)e^{i\omega_p K_\xi x} \tag{2.5}$$

The coefficients $A_k(\eta,\xi;\mu)$ are determined from boundary conditions (2.4).

We will have:

$$\frac{K_\xi^{1-2n}}{2ni}\sum_{p=1}^{n}A_p(\eta,\xi;\mu)\omega_p^{l_j}(iK_\xi)^{l_j}e^{i\omega_p K_\xi x}\Bigg|_{x=0} =$$

$$-\frac{K_\xi^{1-2n}}{2ni}\sum_{\alpha=1}^{n}\omega_\alpha^{l_j+1}(iK_\xi)^{l_j}e^{i\omega_p K_\xi|x-\eta|}\Bigg|_{x=0},\ \ j = 1,2,...,n$$

or

$$-\frac{K^{1-2n+l_j}}{2ni}\sum_{p=1}^{n}A_p(\eta,\xi;\mu)\omega_p^{l_j} = -\frac{K^{1-2n+l_j}}{2ni}\sum_{\alpha=1}^{n}\omega_\alpha^{1+l_j}e^{i\omega_\alpha K_\xi\eta}.$$

Finally

$$\sum_{p=1}^{n}A_p(\eta,\xi;\mu)\omega_p^{l_j} = -\sum_{\alpha=1}^{n}\omega_\alpha^{1+l_j}e^{i\omega_\alpha K_\xi\eta},\ \ j = 1,2,...,n \tag{2.6}$$

Denote

$$\Delta_0 = \begin{vmatrix} \omega_1^{l_1} & \omega_2^{l_1} & \dots & \omega_n^{l_1} \\ \omega_1^{l_2} & \omega_2^{l_2} & \dots & \omega_n^{l_2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \omega_1^{l_n} & \omega_2^{l_n} & \dots & \omega_n^{l_n} \end{vmatrix}$$

$$\Delta_k = \begin{vmatrix} \omega_1^{l_1} & \dots & \omega_{k-1}^{l_1} & -\sum\limits_{\alpha=1}^{n} \omega_\alpha^{1+l_1} e^{i\omega_\alpha K_\xi \eta} & \omega_{k+1}^{l_1} & \dots & \omega_n^{l_1} \\ \omega_1^{l_2} & \dots & \omega_{k-1}^{l_2} & -\sum\limits_{\alpha=1}^{n} \omega_\alpha^{1+l_2} e^{i\omega_\alpha K_\xi \eta} & \omega_{k+1}^{l_2} & \dots & \omega_n^{l_2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \omega_1^{l_n} & \dots & \omega_{k-1}^{l_n} & -\sum\limits_{\alpha=1}^{n} \omega_\alpha^{1+l_n} e^{i\omega_\alpha K_\xi \eta} & \omega_{k+1}^{l_n} & \dots & \omega_n^{l_n} \end{vmatrix}$$

It is easy to show that $\Delta_k = -\omega_k e^{i\omega_k K_\xi \eta} \Delta_0$.

Then by the Cramer's formulas we get:

$$A_k(\eta,\xi;\mu) = \frac{\Delta_k}{\Delta_0} = -\omega_k e^{i\omega_k K_\xi \eta}.$$

Substituting the expression $A_k(\eta,\xi;\mu)$ in equality (2.5), we get

$$V(x,\eta,\xi;\mu) = -\frac{K_\xi^{1-2n}}{2ni} \sum_{p=1}^{n} \omega_p e^{i\omega_p K_\xi (x+\eta)} \tag{2.7}$$

Then the Green function of the problems (1.3)-(1.4) will have the form:

$$G_1(x,\eta,\xi;\mu) = \frac{K_\xi^{2n}}{2ni} \sum_{\alpha=1}^{n} \omega_\alpha e^{i\omega_\alpha K_\xi |x-\eta|} - \frac{K_\xi^{2n}}{2ni} \sum_{p=1}^{n} \omega_p e^{i\omega_p K_\xi (x-\eta)} \tag{2.8}$$

We can write the obtained formula in the form:

$$G_1(x,\eta,\xi;\mu) = \begin{cases} \frac{K_\xi^{1-2n}}{2ni} \sum\limits_{k=1}^{n} \omega_k e^{i\omega_k K_\xi (x-\eta)} \{E - e^{2i\omega_k K_\xi \eta}\}, x \geq \eta \\ \\ \frac{K_\xi^{1-2n}}{2ni} \sum\limits_{k=1}^{n} \omega_k e^{i\omega_k K_\xi (\eta-x)} \{E - e^{2i\omega_k K_\xi x}\}, x \leq \eta \end{cases} \tag{2.9}$$

As $Q(\xi)$ is an unbounded normal operator, and $Re(2i\omega_k K_\xi \eta) < 0$, $Re(2i\omega_k K_\xi x) < 0$, therefore $\|e^{2i\omega_k K_\xi \eta}\| \to 0$, $\|e^{2i\omega_k K_\xi x}\| \to 0$, as $\mu \to \infty$.

Thus, from (2.9) it follows

$$G_1(x,\eta,\xi;\mu)=\frac{K_\xi^{1-2n}}{2ni}\sum_{k=1}^{n}\omega_k e^{i\omega_k K_\xi|x-\eta|}\{E-r(x,\eta,\xi;\mu)\},$$

and as $\mu\to\infty$ we have $\|r(x,\eta,\xi;\mu)\|=o(1)$ uniformly with respect to (x,η).

Now let us construct and study some properties of the Green function of the operator L generated by the differential expression (1.1) and boundary conditions (1.2). The Green function $G(x,\eta;\mu)$ of the problem (1.1), (1.2) is the solution of the following integral equation:

$$G(x,\eta;\mu)=G_1(x,\eta;\mu)-\int_0^\infty G_1(x,\xi;\mu)[Q(\xi)-Q(x)]G(\xi,\eta;\mu)d\xi \tag{2.10}$$

It is proved that under sufficiently large μ, equation (2.10) is solvable and its solution is the Green function of the operator L.

Equation (2.10) is studied in Banach spaces X_1, X_2, $X_3^{(p)}$, $X_2^{(s)}$, $X_4^{(s)}$ and $X_5(p\geq 1, s\gtreqless 0)$, whose elements are the operator functions $A(x,\eta)$ in H, $0\leq x,\eta<\infty$. Definition and proof of their completeness was given by B.M. Levitan in [6].

Let us estimate the norm of the operator function $G_1(x,\eta;\mu)$. In the complex plane λ lying out of the sector Ω, the inequality $|\lambda+\mu|\geq\mu\sin\varepsilon_0$ is valid. By condition 1) $Q(x)$ for each $x\in[0,\infty)$ is a normal operator in H, therefore we can write spectral expansion for $G_1(x,\eta;\mu)$ as a function of normal operator $Q(x)$:

$$\|G_1(x,\eta;\mu)\|_H\leq$$

$$\leq\frac{1+o(1)}{2n}\int_0^\infty|\lambda+\mu|^{\frac{1-2n}{2n}}\left|\sum_{k=1}^{n}\max|\omega_k|e^{i\omega_k(\lambda+\mu)^{\frac{1}{2n}}|x-\eta|}dE(\lambda)\right|\leq$$

$$\leq c\cdot\mu^{\frac{1-2n}{2n}}\cdot r_0^{\frac{1-2n}{2n}}\cdot e^{-\rho_0 Im\omega_1\cdot\mu^{\frac{1}{2n}}\delta_0^{\frac{1}{2n}}|x-\eta|} \tag{2.11}$$

where $\delta_0=\sin\varepsilon_0$, $r_0=\cos\frac{arg(\lambda+\mu)+2\pi m}{2n}$, $m=0,1,2,...,2n-1$. Here we take that branch of the $2n$-th root $(\lambda+\mu)$ on which $Re(\lambda+\mu)^{\frac{1}{2n}}>0$, $Im\omega_1=\min\{Im\omega_k>0,\omega_k^{2n}=-1,k=1,2,...,n\}$.

From (2.11) we get

$$\|G_1(x,\eta;\mu)\|_H^2\leq c^2\mu^{\frac{1-2n}{n}}\cdot\delta_0^{\frac{1-2n}{n}}\cdot e^{-2Im\omega_1 r_0\mu^{\frac{1}{2n}}\delta_0^{\frac{1}{2n}}|x-\eta|}.$$

Hence

$$\int_0^\infty \|G_1(x,\eta;\mu)\|_H^2 d\eta \le c^2\mu^{\frac{1-2n}{n}}\delta_0^{\frac{1-2n}{n}}\int_0^\infty e^{-2Im\omega_1 r_0\mu^{\frac{1}{2n}}\delta_0^{\frac{1}{2n}}|x-\eta|}d\eta \le$$

$$\le \frac{c^2\mu^{\frac{1-2n}{n}}\cdot\delta_0^{\frac{1-2n}{n}}}{2Im\omega_1 r_0\cdot\mu^{\frac{1}{2n}}\cdot\delta_0^{\frac{1}{2n}}} = \frac{c^2}{2Im\omega_1 r_0}\cdot\frac{1}{\mu^{\frac{4n-1}{2n}}\cdot\delta_0^{\frac{4n-1}{2n}}} \tag{2.12}$$

From this estimation it follows that $G_1(x,\eta;\mu)\in X_3^{(1)}$.

Subject to condition 2) for the Hilbert-Schmidt norm of the operator function $G_1(x,\eta;\mu)$ we have:

$$\|G_1(x,\eta;\mu)\|_2^2 \le \frac{c^2}{4n^2}\sum_{i=1}^\infty\left|\sum_{k=1}^n[\alpha_k(x)+\mu]^{\frac{1-2n}{2n}}\omega_k e^{i\omega_k[\alpha_k(x)+\mu]^{\frac{1}{2n}}|x-\eta|}\right|^2 \le$$

$$\le \frac{c^2}{4n^2}\sum_{i=1}^\infty\left\{|\alpha_i(x)+\mu|^{\frac{1-2n}{n}}\left|\sum_{k=1}^n\omega_k e^{i\omega_k[\alpha_k(x)+\mu]^{\frac{1}{2n}}}|x-\eta|\right|^2\right\} \le$$

$$\le \frac{c^2}{4n^2}\sum_{i=1}^\infty\left\{|\alpha_i(x)+\mu|^{\frac{1-2n}{n}}\left|e^{-2Im\omega_1[\alpha_i(x)+\mu]^{\frac{1}{2n}}}|x-\xi|\right|^2\right\} \le$$

$$\le \frac{c^2}{4n^2}\sum_{i=1}^\infty|\alpha_i(x)+\mu|^{\frac{1-2n}{n}}e^{-2Im\omega_1|\alpha_i(x)+\mu|^{\frac{1}{2n}}}|x-\eta|.$$

Therefore

$$\int_0^\infty\|G_1(x,\eta;\mu)\|_2^2 d\eta \le$$

$$\le \frac{c^2}{4n}\sum_{i=1}^\infty|\alpha_i(x)+\mu|^{\frac{1-2n}{n}}\int_0^\infty e^{-2Im\omega_1|\alpha_i(x)+\mu|^{\frac{1}{2n}}|x-\eta|}d\eta \le$$

$$\le \frac{c^2}{8nIm\omega_1\cdot r_0}\sum_{i=1}^\infty|\alpha_i(x)+\mu|^{\frac{1-4n}{2n}} = \frac{c^2}{8nIm\omega_1\cdot r_0}\cdot F(x).$$

Hence

$$\int_0^\infty\left\{\int_0^\infty\|G_1(x,\eta;\mu)\|_2^2 d\eta\right\}dx \le \frac{c^2}{8n\cdot Im\omega_1\cdot r_0}\int_0^\infty F(x)dx < \infty. \tag{2.13}$$

Using conditions 3) 4) we can show that the operator function $G_1(x,\xi;\mu)[Q(x)-Q(x)]$ is a bounded operator in H with respect to $(x,\xi), 0<x,\xi<\infty$ for $\mu>0$. Therefore, it makes sense to consider the operator T generated by the kernel $G_1(x,\xi;\mu)[Q(\xi)-Q(x)]$:

$$TA(x,\eta) = \int_0^\infty G_1(x,\xi;\mu)[Q(\xi)-Q(x)]A(\xi,\eta)d\xi. \tag{2.14}$$

It is proved that the operator T is a compressive operator in Banach spaces introduced by B.M. Levitan in the paper [6].

Hence it follows that integral equation (2.10) has a unique solution that may be obtained by the iteration method if only the function $G_1(x,\eta;\mu)$ belongs to the corresponding space.

From estimation (2.12) it follows that $G_1(x,\eta;\mu) \in X_3^{(2)}$. Then for large $\mu > 0$ one van affirm that $G_0(x,\eta;\mu)$ also belongs to the space $X_3^{(1)}$. If we assume that conditions 1)-4) are fulfilled, then from estimation (2.13) it is seen that $G_1(x,\eta;\mu) \in X_2$, and therefore $G(x,\eta;\mu)$ for sufficiently large $\mu > 0$ also belongs to the space X_2.

Having imposed additional conditions on the function $Q(x)$ (see [4]) we show the operator function $G_1(x,\eta;\mu)$ belongs to the space $X_4^{(\pm\frac{1}{2})}$ and therefore $G(x,\eta;\mu)$ belongs to these spaces.

Using integral equation (2.10) we establish strong continuity of the derivatives $\frac{\partial^k G}{\partial \eta^k}$, $n = 0,1,2,...,2n-2$.

We also prove that the operator function $G(x,\eta;\mu)$ satisfies the equation

$$(-1)^n \frac{\partial^{2n} G(x,\eta;\mu)}{\partial \eta^{2n}} + G(x,\eta;\mu)[Q(\eta)+E] = 0$$

and the boundary conditions

$$\left.\frac{\partial^{l_1} G}{\partial \eta^{l_1}}\right|_{\eta=0} = \left.\frac{\partial^{l_2} G}{\partial \eta^{l_2}}\right|_{\eta=0} = ... = \left.\frac{\partial^{l_n} G}{\partial \eta^{l_n}}\right|_{\eta=0} = 0.$$

Note that the Green function of the Sturm-Liouville equation with normal operator coefficients was studied in the works [2], [5], for higher order equations the Green function was studied in [1], [3], [10].

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] Aslanov H.I. On the Green function of higher order equation with normal operator coefficients. Spectralnaya teoria operatorov. Baku "Elm". 1982, pp. 23-27. (in Russian)

[2] Aslanov H.I., Bayramoglu M. Studying the Green function of Sturm-Liouville operator equation in not self-adjoint case. Spektralnaya teoria operatorov. Baku, "Elm", 1984, pp. 69-77.(in Russian)

[3] Aliyev B.I., Bayramoglu M. The Green function of higher order ordinary differential equations with operator coefficients on the semi-axis. Izv. An Azerb. SSR. Ser. Fiz.tekhn. i matem. Nauk. 2 (4)(1981), 33-38.(in Russian)

[4] Bayramoglu M. Asymptotics of the number of eigen values of ordinary differential operators with operator coefficients. Coll: Funksionalniy analiz i ee priminenia. Baku Elm.1971, pp. 144-166.(in Russian)

[5] Kleiman E.G. On the Green function of Sturm-Liouville equation with normal operator coefficient. In: Vestnik Moskovskogo Universiteta, No 5, 1974, pp. 47-53.(in Russian)

[6] Levitan B.M. Studying the Green function of the Sturm-Liouville equation with operator coefficients. Matem. Sb., 76 (2) (1968), 239-270. (in Russian)

[7] Plesner A.I. Spectral theory of linear operators, M. Nauka, 1965. (in Russian)

[8] Gunay I. Kasymova. Investigation of the Green function of second order equations with normal operator coefficients on the axis. Trans. NAS Azerbaijan. Iss. math. Mech. Ser. XXVIII (4) (2008), 59-64.

[9] Gamidulla I.Aslanov, Gunay I. Kasumova. On the resolvent of Sturm-Liouville operator-differential equations on the finite segment. Math. Anal., Diff. Equ. Appl., Bulgaria, 2011, pp. 101-108.

[10] Gamidulla I. Aslanov, Konul H. Badalova. Investigation of Green function of higher order operator-differential equation in finite segment. Trans. NAS Azerbaijan, XXXI (4) (2011), 35-44.