Available online at http://scik.org
J. Math. Comput. Sci. 8 (2018), No. 2, 290-296
https://doi.org/10.28919/jmcs/3584
ISSN: 1927-5307

NONHOLONOMIC FRAMES FOR FINSLER SPACES WITH SPECIAL (α,β)-METRIC

BRIJESH KUMAR TRIPATHI[1] AND V. K. CHAUBEY[2,*]

[1]Department of Mathematics, L.E. College, Morbi, Samakathe Morbi-2,(Gujarat), 363642, India

[2]Department of Applied Sciences, Buddha Institute of Technology, GIDA Gorakhpur, U.P., 273209, India

Abstract. The purpose of present paper to determine the Finsler spaces due to deformation of special Finsler (α,β) metric. Consequently, we obtain the non-holonomic frame with help of Riemannian metric α , one form metric β and Randers metric such as forms

$I.$ $L(\alpha,\beta)=\alpha\beta+\beta^2$ i.e. product of Randers metric and 1-form metric and

$II.$ $L(\alpha,\beta)=\alpha^2+\alpha\beta$ i.e. product of Randers metric and Riemannian metric.

Keywords: Riemannian metric; one form metric; Randers metric; GL-metric; nonholonomic Finsler frame.

2010 AMS Subject Classification: 53C60.

1. Introduction

The concept of non-holonomic frame is a sort of plastic deformation which was arising from the consideration of a charged particle moving in an external electromagnetic field in the background of space time was developed by Holland in 1982 ([8], [9]). Further a gauge transformation was viewed in a non holonomic frame on the tangent bundle of a four dimensional base

*Corresponding author

E-mail address: vkchaubey@outlook.com

Received November 21, 2017

manifold ([3], [4]). These geometries shows that there is unified approach to gravitation and gauge symmetries.

In the present paper, the fundamental tensor field might be considered as the deformation of two different special Finsler spaces from the (α,β)- metrics. First deformation is on a nonholonomic frame for a Finsler space with (α,β)- metrics such as first product of Randers metric [11] and 1-form metric and second is the product of Randers metric and Riemannian metric. This is an extension work of Bucataru and Miron [6] by considering $a_{ij}(x)$ the components of a Riemannian metric on the base manifold M, $a(x,y) > 0$ and $b(x,y) \geq 0$. Two functions on TM and $B(x,y) = B_i(x,y)dx^i$ a vertical 1-form on TM. Then

$$g_{ij}(x,y) = a(x,y)a_{ij}(x) + b(x,y)B_i(x,y)B_j(x,y) \tag{1}$$

is a generalized Lagrange metric, called the Beil metric. The case $a(x,y) = 1$ with various choices of b and B_i was introduced and studied by Beil [4] for constructing a new unified field theory.

2. Preliminaries

The first Finsler spaces with (α,β)-metrics were introduced by the physicist Randers ([10], [13]) in 1940, are called Randers spaces. Beil considering a more general case i.e. Lagrange spaces with (α,β)-metric, which was discussed in detail by Bucataru [5]. The detail of Lagrange space with (α,β)-metric was studied by the Miron.

Definition 2.1. A Finsler space $F^n = \{M, F(x,y)\}$ is called with (α,β)-metric if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F : TM \to R$ is given by

$$F^2(x,y) = L\{\alpha(x,y), \beta(x,y)\} \tag{2}$$

where $\alpha^2(x,y) = a_{ij}(x)y^iy^j, \alpha$ is a Riemannian metric on the manifold M, and $\beta(x,y) = b_i(x)y^i$ is a 1-form on M.

Taking into account the homogeneity of α and F we have the following formulae [12]:

$$p^i = \frac{1}{a}y^i = a^{ij}\frac{\partial \alpha}{\partial y^j}; \quad p_i = a_{ij}p^j = \frac{\partial \alpha}{\partial y^i};$$

(3) $$l^i = \frac{1}{L}y^i = g^{ij}\frac{\partial l}{\partial y^i}; l_i = g_{ij}l^j = \frac{\partial L}{\partial y^i} = P_i + b_i$$

$$l^i = \frac{1}{L}p^i; l^i l_i = p^i p_i = 1; l^i p_i = \frac{\alpha}{L}; p^i l_i = \frac{L}{\alpha};$$

$$b_i P^i = \frac{\beta}{\alpha}; b_i l^i = \frac{\beta}{L}$$

with respect to these notations, the fundamental metric tensors g_{ij} of the Randers space (M,F) is given by [12],

(4) $$g_{ij}(x,y) = \frac{L}{\alpha}a_{ij} + b_i P_j + P_i b_j - \frac{\beta}{\alpha}p_i p_j = \frac{L}{\alpha}(a_{ij} - p_i p_j) + l_i l_j$$

Theorem 2.1. [6]: For a Finsler space (M,F) consider the metric with the entries:

(5) $$Y_j^i = \sqrt{\frac{\alpha}{L}}(\delta_j^i - l^i l_j + \sqrt{\frac{\alpha}{L}}p^i p_j)$$

defined on TM. Then $Y_j = Y_j^i(\frac{\partial}{\partial y^i}), \quad j \in 1,2,3,\ldots,n$ is a non holonomic frame.

Theorem 2.2. [1]: With respect to frame the holonomic components of the Finsler metric tensor $a_{\alpha\beta}$ is the Randers metric g_{ij}, i.e,

(6) $$g_{ij} = Y_i^\alpha Y_j^\beta a_{\alpha\beta}.$$

Now for a Finsler space with (α,β)-metric $F^2(x,y) = L\{\alpha(x,y), \beta(x,y)\}$ we have the Finsler invariants [12].

(7) $$\rho = \frac{1}{2\alpha}\frac{\partial L}{\partial \alpha}; \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial \beta^2}; \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial \alpha \partial \beta}; \rho_{-2} = \frac{1}{2\alpha^2}(\frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial \alpha})$$

where subscripts 1, 0, -1, -2 gives us the degree of homogeneity of these invariants.
For a Finsler space with (α, β)-metric we have,

(8) $$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0$$

with respect to the notations we have that the metric tensor g_{ij} of a Finsler space with (α,β)-metric is given by [12].

(9) $$g_{ij}(x,y) = \rho a_{ij}(x) + \rho_0 b_i(x) + \rho_{-1}\{b_i(x)y_j + b_j(x)y_i\} + \rho_{-2}y_i y_j$$

From (9) we can see that $g_{ij}$is the result of two Finsler deformations:

$$I. \qquad a_{ij} \to h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j)$$

$$II. \quad h_{ij} \to g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-1} - \rho_{-1}^2)b_ib_j \tag{10}$$

The nonholonomic Finsler frame that corresponding to the I^{st} deformation (10) is according to the theorem (7.9.1) in [6], given by,

$$X_j^i = \sqrt{\rho}\delta_j^i - \frac{1}{\beta^2}\{\sqrt{\rho} + \sqrt{\rho + \frac{\beta^2}{\rho_{-2}}}\}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j) \tag{11}$$

where $B^2 = a_{ij}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j) = \rho_{-1}^2 b^2 + \beta\rho_{-1}\rho_{-2}$.

This metric tensor a_{ij} and h_{ij} are related by,

$$h_{ij} = X_i^k X_j^l a_{kl} \tag{12}$$

Again the frame that corresponds to the II_{nd} deformation (10) given by,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\{1 \pm \sqrt{1 + \left(\frac{\rho_{-2}C^2}{\rho_0\rho_{-2} - \rho_{-1}^2}\right)}\}b^ib_j \tag{13}$$

where $C^2 = h_{ij}b^ib^j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-1}b^2 + \rho_{-2}\beta)^2$.

The metric tensor h_{ij} and g_{ij} are related by the formula;

$$g_{mn} = Y_m^i Y_n^j h_{ij} \tag{14}$$

Theorem 2.3. [6]: Let $F^2(x,y) = L\{\alpha(x,y), \beta(x,y)\}$ be the metric function of a Finsler space with (α,β) metric for which the condition (8) is true. Then

$$V_j^i = X_k^i Y_j^k$$

is a nonholonomic Finsler frame with X_k^i and Y_j^k are given by (11) and (13) respectively.

3. Nonholonomic frames for Finsler space with special (α,β)-metric

In this section we consider two cases of nonholonomic Finlser frames with special (α,β)-metrics, such as the first Finsler frame is the product of Randers metric and 1- form metric and second Finsler frame is the product of Randers metric and Riemannian metric.

3.1. Nonholonomic frame for $L(\alpha,\beta)=\alpha\beta+\beta^2$

Now for a Finsler space with the fundamental function $L(\alpha,\beta)=\alpha\beta+\beta^2$, the Finsler invariants (7) are given by

$$\rho=\frac{\beta}{2\alpha},\ \rho_0=1,\ \rho_{-1}=\frac{1}{2\alpha},\ \rho_{-2}=\frac{-\beta}{2\alpha^3},\ B^2=\frac{a^2b^2-\beta^2}{4\alpha^4} \tag{15}$$

Using (15) in (11) we have,

$$X^i_j=\sqrt{\frac{\beta}{2\alpha}}\delta^i_j-\frac{1}{4\alpha^2\beta^2}[\sqrt{\frac{\beta}{2\alpha}}+\sqrt{\frac{(\beta-4\alpha^4\beta)}{2\alpha}}](b^i-\frac{\beta}{\alpha^2}y^i)(b_j-\frac{\beta}{\alpha^2}y_j) \tag{16}$$

Again using (15) in (11) we have,

$$Y^i_j=\delta^i_j-\frac{1}{C^2}\{1\pm\sqrt{1+\frac{2\beta C^2}{\alpha+2\beta}}\}b^ib_j \tag{17}$$

where $C^2=\frac{\beta}{2\alpha}b^2-\frac{1}{2\alpha^3\beta}(\alpha^2b^2-\beta^2)^2$.

Theorem 3.1. Consider a Finsler space with special (α,β)- metric of type $L(\alpha,\beta)=\alpha\beta+\beta^2$ for which the condition (8) is satisfied then

$$V^i_j=X^i_kY^k_j$$

is nonholonomic Finsler Frame with X^i_k and Y^k_j are given by (16) and (17) respectively.

3.2. Nonholonomic frame for $L(\alpha,\beta)=\alpha^2+\alpha\beta$

Again for a Finsler space with the fundamental function $L(\alpha,\beta)=\alpha^2+\alpha\beta$ the Finsler invariants (7) are given by

$$\rho=1+\frac{\beta}{2\alpha},\ \rho_0=0,\ \rho_{-1}=\frac{1}{2\alpha},\ \rho_{-2}=\frac{-\beta}{2\alpha^3},\ B^2=\frac{a^2b^2-\beta^2}{4\alpha^4} \tag{18}$$

Using (18) in (13) we have,

$$X_j^i = \sqrt{1+\frac{\beta}{2\alpha}}\delta_j^i - \frac{1}{4\alpha^2\beta^2}[\sqrt{1+\frac{\beta}{2\alpha}} + \sqrt{\frac{(2\alpha+\beta-4\alpha^4\beta)}{2\alpha}}](b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j) \tag{19}$$

Again using (18) in (13) we have,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\{1 \pm \sqrt{1+\frac{2\beta C^2}{\alpha}}\}b^i b_j \tag{20}$$

where $C^2 = (1+\frac{\beta}{2\alpha})b^2 - \frac{1}{2\alpha^3\beta}(\alpha^2 b^2 - \beta^2)^2$.

Theorem 3.2. Consider a Finsler space with special (α,β)- metric of type $L(\alpha,\beta) = \alpha^2 + \alpha\beta$ for which the condition (8) is satisfied then

$$V_j^i = X_k^i Y_j^k$$

is nonholonomic Finsler Frame with X_k^i and Y_j^k are given by (19) and (20) respectively.

Conclusions

Non-holonomic frame relates a semi-Riemannian metric or Minkowski or Lorentz metric with an induced Finsler metric. Antonelli and Bucataru ([1], [2]), has been determined such a non-holonomic frame for two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces [7]. It appears a natural question: Does how many Finsler space with (α,β)-metrics have such a nonholonomic frame? The answer is yes, there are many Finsler space with (α,β)-metrics.

In the present work, we consider the Randers Finsler metrics, Riemannian metric and 1-form metric and determine the nonholonomic Finsler frames. But, in Finsler geometry, there are many (α,β)-metrics, in future work we can determine the frames for them also.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] Antonelli, P. L., Bucataru, I.: On Holland's frame for Randers space and its applications in physics. In: Kozma, L. (ed), Steps in Differential Geometry. Proceedings of the Colloquium on Differential Geometry, Debrecen, Hungary, July 25-30, 2000. Debrecen: Univ. Debrecen, Institute of Mathematics and Informatics, 39-54 (2001).

[2] Antonelli, P. L., Bucataru, I.: Finsler connections in an holonomic geometry of a Kropina space, Nonlinear Stud., 8 (1) (2001), 171-184.

[3] Beil, R. G.: Comparison of unified field theories. Tensor N.S., 56 (1995), 175-183.

[4] Beil, R. G.: Equations of Motion from Finsler Geometric Methods. In: Antonelli, P.L. (ed), Finslerian Geometries. A meeting of minds. Kluwer Academic Publisher, FTPH, no. 109(2000), 95-111.

[5] Bucataru, I.: Nonholonomic frames on Finsler geometry, Balkan J. Geom. Appl., 7 (1)(2002), 13-27.

[6] Bucataru, I. and Miron R.: Finsler-Lagrange Geometry. Applications to dynamical systems, Bucuresti: Editura Academiei Romane, (2007).

[7] Hrimiuc, D., Shimada, H.: On the L-duality between Lagrange and Hamilton Manifolds, Nonlinear World, 3 (1996), 613-641.

[8] Holland, P. R.: Electromagnetism, Particles and Anholonomy. Physics Lett. A, 91(6)(1982), 275-278.

[9] Holland, P. R.: An holonomic deformations in the ether: a significance for the electrodynamic potentials. In: Hiley, B.J. Peat, F.D. (eds.), Quantum Implications. Routledge and Kegan Paul, London and New York, 295-311 (1987).

[10] Ingarden, R. S.: On Physical interpretations of Finsler and Kawaguchi spaces. Tensor N. S., 46(1987), 354-360.

[11] Matsumoto, M.: Theory of Finsler spaces with (α,β)-metric, Rep. Math. Phys., 31 (1992), 43-83.

[12] Matsumoto. M: Foundations of Finsler geometry and special Finsler spaces, Kaishesha Press, Otsu, Japan, 1986.

[13] Randers, G.: On asymmetric metric in the four space of general relativity, Phys. Rev., 59(1941), 195-199.