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HYBRID PROJECTIVE SYNCHRONIZATION BETWEEN THE FRACTIONAL ORDER SYSTEMS

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Abstract. In this paper we have investigated Hybrid Projective Synchronization, between the fractional order chaotic systems of different dimensions. We have synchronized the fractional order chaotic Lü system, as a master system, with the fractional order hyperchaotic Rössler system as the slave system. Further, a fractional order hyperchaotic system is controlled by the fractional order chaotic financial system as the slave system. Numerical simulations are carried out using Matlab to show the effectiveness of the method.

Keywords: synchronization; Rössler system; Lü system; financial system.

2010 AMS Subject Classification: 34A08, 34D06, 34H10.

1. Introduction

Chaos is an interesting phenomenon of non linear systems. Since Pecora and Carroll [1] established a chaos synchronization scheme for two identical systems, with different initial conditions, chaos synchronization has attracted much attention of the researchers. Various effective

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methods have been proposed to synchronize chaotic systems such as the sliding mode control method [2], active control method [3-6], linear and non linear feedback control method [7-8], adaptive control method [9-10], backstepping control [11-12] and impulse control method [13-14]. Using these methods, numerous synchronization problems of well- known chaotic systems such as Lü, Rössler, Lorenz, Chen, Genesio have been studied.

Fractional calculus is a classical mathematical notion with a history as long as calculus itself. Fractional calculus deals with derivatives and integration of arbitrary order and has deep connections with many fields of applied mathematics, physics and engineering. Many systems display fractional order dynamics such as dielectric polarization [15], electromagnetic waves [16], viscoelastic systems [17-18], electrode - electrolyte polarization [19], quantitative finance [20], bio-engineering [21] etc.

The first attempt to study the synchronization in fractional order systems was by Deng and Li [22] and they have summarized the theory and technique of synchronization in [23]. Recently, synchronization in fractional order chaotic systems is discussed by various researchers [24-28] due to its potential applications in secure communication and control processing.

A hyperchaotic system is characterized as a chaotic system with at least two positive Lyapunov exponents together with a zero exponent and a negative exponent to ensure the boundedness of the solution. It is believed that chaotic systems with higher dimensions have much wider applications. Synchronization in hyperchaotic fractional order systems is discussed by various researchers [29-31]. The activation feedback control technique is used by Wang and Song [32] to synchronize hyperchaotic Lorenz system, the Laplace tranformation theory and variational iteration method is used by Yu and Li to study Rössler system [33], feedback control technique is used in [34-35] to synchronize various hyperchaotic systems, Wang, Yu and Diao in [36] discussed the synchronization between the fractional order chaotic systems of different dimensions. Zhang and Lü in [37] introduced a new type of synchronization called full state hybrid log projective synchronization and applied it to Rössler system and hyperchaotic Lorenz system to numerically verify their results.

The co-existance of Complete synchronization and Anti synchronization, known as Hybrid synchronization, has good application prospects in digital communication. It may enhance security in communication and chaotic encryption schemes.

This paper is organized as follows: In section 2, the fractional derivative and its applications are studied, section 3 gives a brief introduction of the methodology for synchronizing the fractional order chaotic systems. In section 4 we have studied the increased order synchronization by synchronizing a fractional order chaotic system by a fractional order hyperchaotic system together with the numerical simulations, section 5 is devoted to the the study of decreased order synchronization, where a hyperchaotic system is being synchronized by a fractional order chaotic system, with numerical simulations. In section 6 we provide the conclusion of the paper.

2. Fractional Derivative and its Approximation

Fractional calculus is a generalization of integration and differentiation to a non-integer-order integro-differential operator ${}_aD_t^q$ defined by

$$ {}_aD_t^q = \begin{cases} \frac{d^q}{dt^q} & \text{if } \mathbb{R}(q) > 0 \\ 1 & \text{if } \mathbb{R}(q) = 0 \\ \int_a^t (d\tau)^{-q} & \text{if } \mathbb{R}(q) < 0 \end{cases} $$

where q is the fractional order which can be a complex number, $\mathbb{R}(q)$ denotes the real part of q and $a < t$, where a is the fixed lower terminal and t is the moving upper terminal.

There are two commonly used definitions for fractional derivatives [38], they are Grunward -Letnikov definition and Riemann- Liouville definition. The Riemann- Liouville definition is given by

$$ D^q x(t) = \frac{d^\eta}{dt^\eta} J^{\eta - q} x(t), \qquad q > 0 $$

where η is the first integer that is not less than q , J^β is the β - order Riemann- Liouville integral operator defined as follows:

$$J^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t f(t)(t-\tau)^{\beta-1}\, d\tau$$

where $\Gamma(.)$ is the Gamma function, $0 < \beta \leq 1$.

3. Methodology

Consider the fractional order chaotic system

(1) $$D^q x = f(x)$$

as a master system ,where $x \in \mathbb{R}^n$ is the state vector of the master system and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector function, and the fractional order chaotic system

(2) $$D^q y = g(y) + u(x,y)$$

as a slave system , where $y \in \mathbb{R}^m$ is the state vector of the slave system, $g : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous vector function and $u : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is the control function to be determined.

Writing the fractional order chaotic systems (1) and (2) as

(3) $$D^q x = Ax + F(x)$$

and

(4) $$D^q y = By + G(y) + u(x,y)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$ are linear parts and $F : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^m \to \mathbb{R}^m$ are non linear parts of the master and slave systems respectively.
Choosing a real matrix $C \in \mathbb{R}^{m\times n}$, if we define the error function $e = (e_1, e_2, ..., e_m)^T$ as

(5) $$e = y - Cx$$

then the systems (3) and (4) are synchronized when $\lim_{t\to\infty} \|e\| = 0$.

Using equations (3) and (4), the error system (5) reduces to

$$
\begin{aligned}
D^q e &= D^q y - CD^q x \\
&= By + G(y) + u(x,y) - CAx - CF(x) \\
&= By - BCx + G(y) + u(x,y) + BCx - CAx - CF(x) \\
&= Be + G(y) + BCx - CAx - CF(x) + u(x,y)
\end{aligned} \tag{6}
$$

Choosing the control function $u(x,y)$ as

$$u(x,y) = (A-B)Cx + CF(x) - G(y) + Ke \tag{7}$$

where $K \in \mathbb{R}^{m\times m}$ is the control Gain matrix(to be determined), the error system (6)reduces to

$$D^q e = (B+K)e \tag{8}$$

The system (8) is asymptotically stable if and only if all the eigenvalues λ_i of $B+K$ satisfy $|arg(\lambda_i)| > \frac{q\pi}{2}$, $i = 1,2,...,m$ i.e., $\lim_{t\to\infty} \|e\| = 0$ or the system (3) and(4) are hybrid projective synchronized.

4. Increased order synchronization

In this section, the method discussed in the above section is being applied on the chaotic systems. Here we will consider the case of the increased order synchronization when $n < m$

Consider the fractional order chaotic Lü system (master system) given by

$$
\left.
\begin{aligned}
\frac{d^q x_1}{dt^q} &= a_1(x_2 - x_1), \\
\frac{d^q x_2}{dt^q} &= -x_1 x_3 + c_1 x_2, \\
\frac{d^q x_3}{dt^q} &= x_1 x_2 - b_1 x_3
\end{aligned}
\right\} \tag{9}
$$

which exhibits the chaotic nature when $q = 0.9$ and $(a_1, b_1, c_1) = (35, 3, 28)$. The chaotic attractor of the Lü system are shown in figure 1.

The fractional order hyper chaotic Rössler system(slave system) given by

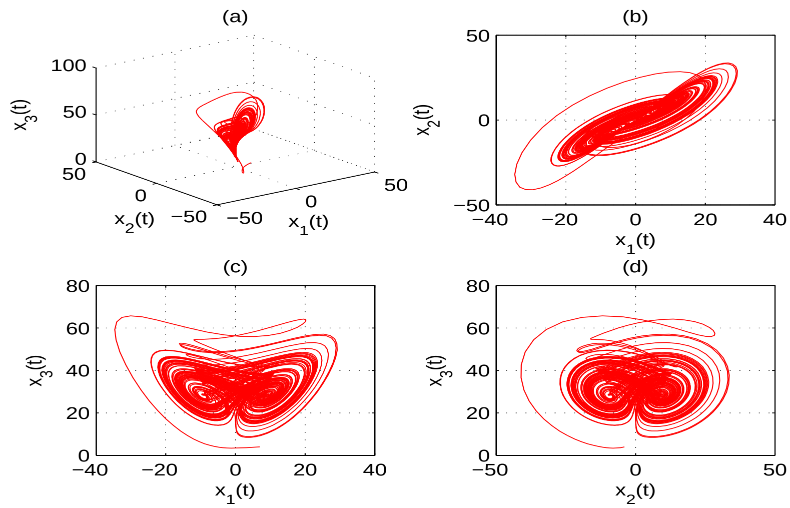

FIGURE 1. Phase Potraits of Lü fractional order chaotic Dynamical system in (a) the $x_1 - x_2 - x_3$ space and the projections on (b) the $x_1 - x_2$ plane , (c) the $x_1 - x_3$ plane and (d) the $x_2 - x_3$ plane.

$$
\left.\begin{aligned}
\frac{d^q y_1}{dt^q} &= -(y_2 + y_3) + u_1, \\
\frac{d^q y_2}{dt^q} &= y_1 + a_2 y_2 + y_4 + u_2, \\
\frac{d^q y_3}{dt^q} &= b_2 + y_1 y_3 + u_3, \\
\frac{d^q y_4}{dt^q} &= -c_2 y_3 + d_2 y_4 + u_4
\end{aligned}\right\} \tag{10}
$$

which exhibits the hyper chaotic behaviour when $q = 0.9$ and $(a_2, b_2, c_2, d_2) = (0.32, 3, 0.5, 0.05)$ as the slave system , and u_1, u_2, u_3, u_4 are control functions to be determined.The chaotic attractors of the Rössler system are shown in figure 2.

Comparing systems (9) and (10) with the systems (3) and (4) respectively, we get

$$
A = \begin{bmatrix} -a_1 & a_1 & 0 \\ 0 & c_1 & 0 \\ 0 & 0 & -b_1 \end{bmatrix}, F(x) = \begin{bmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{bmatrix}
$$

and

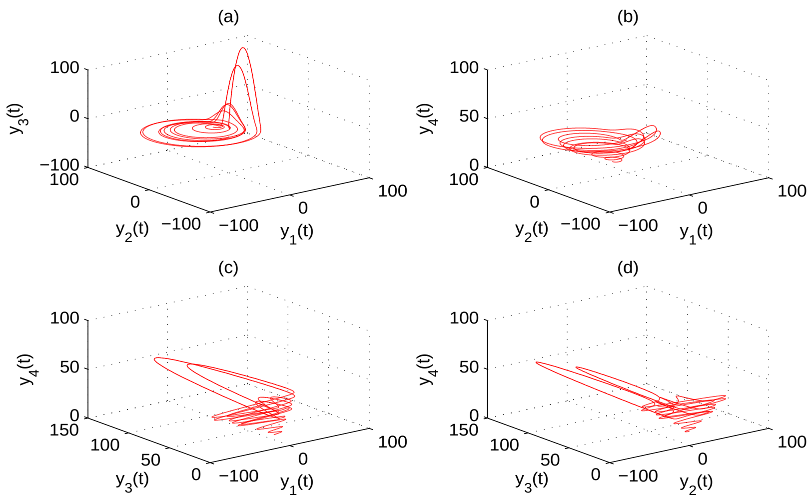

FIGURE 2. Phase Potraits of Rössler fractional order chaotic Dynamical system in (a) the $y_1 - y_2 - y_3$ space, (b) the $y_1 - y_2 - y_4$ space , (c) the $y_1 - y_3 - y_4$ space and (d) the $y_2 - y_3 - y_4$ space.

$$B = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & a_2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c_2 & d_2 \end{bmatrix}, G(y) = \begin{bmatrix} 0 \\ 0 \\ b_2 + y_1 y_3 \\ 0 \end{bmatrix}, u(x,y) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

where $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3, y_4)^T$ are the state vectors of the master and slave systems respectively. Choosing the real matrix

$$C = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

the error system $e = y - Cx$ between the systems (9) and (10) becomes

$$\left.\begin{aligned} e_1 &= y_1 - (x_1 - x_2), \\ e_2 &= y_2 - (x_1 + x_2 + x_3), \\ e_3 &= y_3 - (x_1 - x_3), \\ e_4 &= y_4 - (x_2 + x_3) \end{aligned}\right\} \tag{11}$$

Then as discussed in section 3, choosing the controller

$$u = C(A-B)x + CF(x) - G(y) + Ke, K \in \mathbb{R}^{m\times m} \tag{12}$$

we have

$$\left.\begin{aligned} u_1 &= a_1(x_2 - x_1) - c_1x_2 + 2x_1 + x_2 + x_1x_3 + Ke, \\ u_2 &= a_1(x_2 - x_1) + c_1x_2 - b_1x_3 - (x_1 + x_3) - a_2(x_1 + x_2 + x_3) - x_1x_3 + x_1x_2 + Ke, \\ u_3 &= a_1(x_2 - x_1) + b_1x_3 - x_1x_2 - b_2 - y_1y_3 + Ke, \\ u_4 &= c_1x_2 - b_1x_3 + c_2(x_1 - x_3) - d_2(x_2 + x_3) - x_1x_3 + x_1x_2 + Ke \end{aligned}\right\}$$

and the 4×4 matrix K as

$$K = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & -1.32 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0.5 & -1.05 \end{bmatrix}$$

the error system (11) reduces to

$$
\left.\begin{aligned}
\frac{d^q e_1}{dt^q} &= -(e_1+e_2+e_3),\\
\frac{d^q e_2}{dt^q} &= (a_2-1.32)e_2,\\
\frac{d^q e_3}{dt^q} &= -e_3,\\
\frac{d^q e_4}{dt^q} &= (d_2-1.05)e_4
\end{aligned}\right\} \tag{13}
$$

which implies that $D^q e = (B+K)e$, such that all the eigenvalues λ_i of $B+K$ satisfies the condition $|arg(\lambda_i)| > \frac{q\pi}{2}$, $i = 1,2,3,4$. The trajectories of synchronization error are shown in figure 4 which shows that hybrid synchronization between the systems (9) and(10) is achieved.

4.1. Numerical Simulation

In numerical simulations, the parameters of the Lü system are taken as $[a_1,b_1,c_1] = [35,3,28]$ and that of the Rössler system are taken as $[a_2,b_2,c_2,d_2] = [0.32,3,0.5,0.05]$. Time step size is taken as 0.005 The initial values of the master and slave systems are taken as $[x_1(0),x_2(0),x_3(0)] = [7,-4,4]$ and $[y_1(0),y_2(0),y_3(0),y_4(0)] = [-20,0,0,15]$ respectively. Thus the initial condition for error system becomes $[e_1(0),e_2(0),e_3(0),e_4(0)] = [-23,-7,-3,15]$. It is observed from figure (4) that it takes higher time for synchronization of the two fractional order chaotic systems considered for the fractional order $q_i = 0.99$ for $i = 1,2,3,4$.

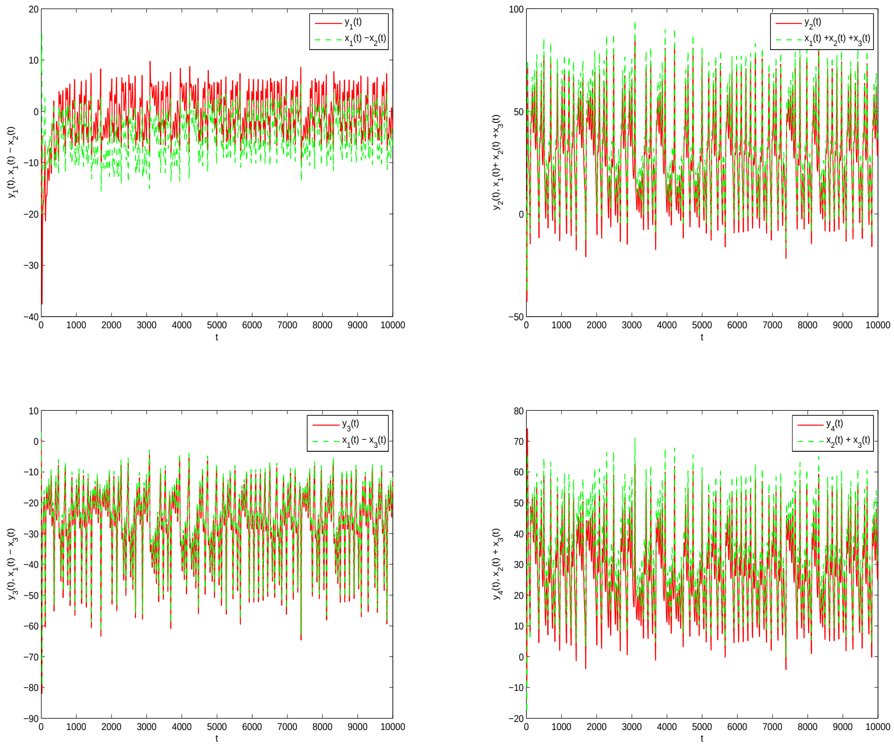

FIGURE 3. *The chaotic tracjectories of the drive and response system after the controllers are applied.*

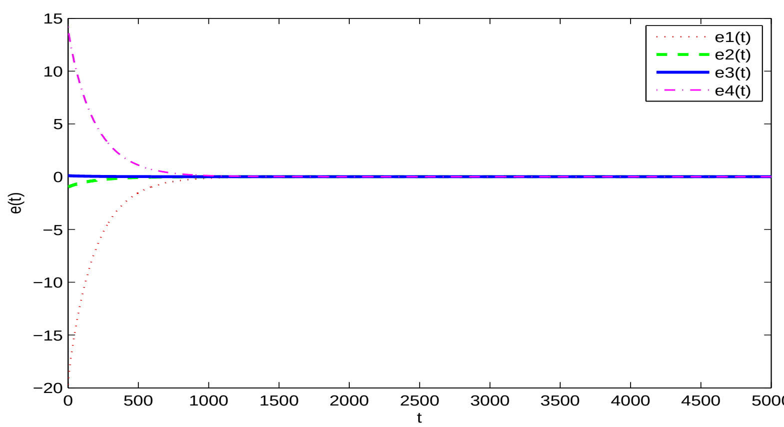

FIGURE 4. *Error Dynamics versus time t.*

5. Reduced order synchronization

Here we discuss the method studied in section 3 and apply it to synchronize a fractional order hyperchaotic system by a fractional order chaotic system i.e.,the case of decreased order synchronization where $m < n$. Consider the fractional order hyper chaotic system proposed by Xin and Ling [39]

$$\left.\begin{aligned} \frac{d^q x_1}{dt^q} &= a_1(x_2 - x_1) + x_4, \\ \frac{d^q x_2}{dt^q} &= b_1 x_1 + x_1 x_3 - x_4, \\ \frac{d^q x_3}{dt^q} &= -c_1 x_3 - d_1 x_1^2, \\ \frac{d^q x_4}{dt^q} &= c_1 x_1 \end{aligned}\right\} \tag{14}$$

which has hyper chaotic nature when $a_1 = 10, b_1 = 40, c_1 = 2.5, d_1 = 4$ and $q = 0.97$, as the master system. The phase potraits are shown in figure 5

and the slave system, the fractional order chaotic Financial system [40] given by

$$\left.\begin{aligned} \frac{d^q y_1}{dt^q} &= y_3 + (y_2 - a_2)y_1 + u_1, \\ \frac{d^q y_2}{dt^q} &= 1 - b_2 y_2 - y_1^2 + u_2, \\ \frac{d^q y_3}{dt^q} &= -y_1 - c_2 y_3 + u_3 \end{aligned}\right\} \tag{15}$$

where u_1, u_2, u_3 are the controllers to be determined. This system exhibits a chaotic nature when $q = 0.97$ and $(a_2, b_2, c_2) = (3, 0.1, 1)$. See figure 6 for phase potraits.

Comparing the systems (14) and (15) with the systems (3) and (4), we get

$$A = \begin{bmatrix} -a_1 & a_1 & 0 & 1 \\ b_1 & 0 & 0 & -1 \\ 0 & 0 & -c_1 & 0 \\ c_1 & 0 & 0 & 0 \end{bmatrix}, F(x) = \begin{bmatrix} 0 \\ x_1 x_3 \\ -d_1 x_1^2 \\ 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -a_2 & 0 & 1 \\ 0 & -b_2 & 0 \\ -1 & 0 & -c_2 \end{bmatrix}, G(y) = \begin{bmatrix} y_1 y_2 \\ 1 - y_1^2 \\ 0 \end{bmatrix}, u(x,y) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

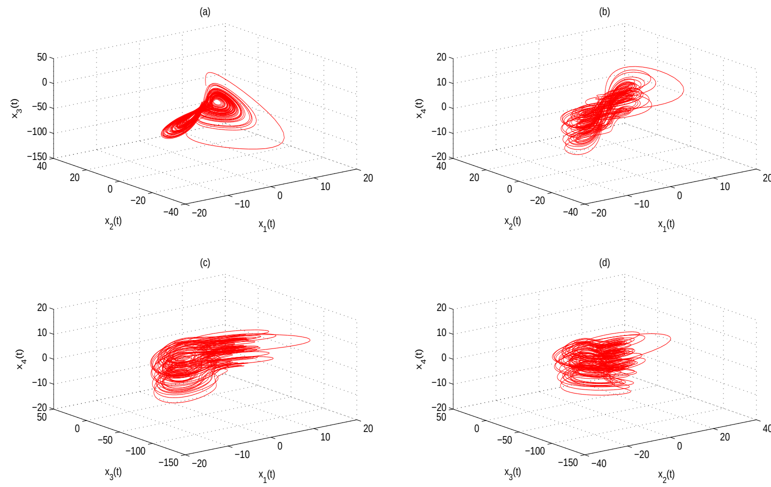

FIGURE 5. Phase Potraits of New fractional order chaotic Dynmical system in (a) the $x_1 - x_2 - x_3$ space , (b) the $x_1 - x_2 - x_4$ space , (c) the $x_1 - x_3 - x_4$ space and (d) the $x_2 - x_3 - x_4$ space.

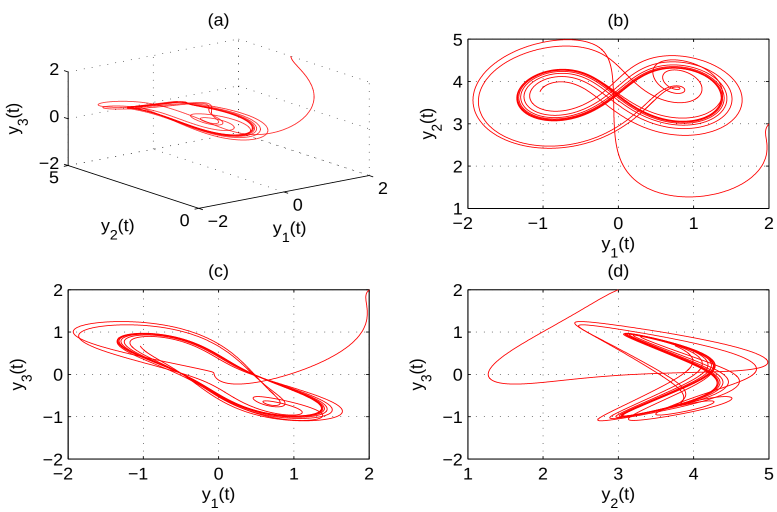

FIGURE 6. Phase Potraits of Financial system in (a) the $y_1 - y_2 - y_3$ space and the projections on (b) the $y_1 - y_2$ plane , (c) the $y_1 - y_3$ plane and (d) the $y_2 - y_3$ plane.

where $x = (x_1, x_2, x_3, x_4)^T$ and $y = (y_1, y_2, y_3)^T$ are the state vectors of the master and slave systems respectively.

Choosing the real matrix

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

the error system $e = y - Cx$ reduces to

$$\left.\begin{aligned} e_1 &= y_1 - (x_1 + x_2), \\ e_2 &= y_2 - x_2, \\ e_3 &= y_3 - (x_3 - x_4) \end{aligned}\right\} \tag{16}$$

Then as discussed in section 3, choosing the controllers

$$\left.\begin{aligned} u_1 &= a_1(x_2 - x_1) + b_1x_1 + a_2(x_1 + x_2) - x_3 + x_4 + x_1x_3 - y_1y_2 + Ke, \\ u_2 &= b_1x_1 - x_4 + b_2x_2 + x_1x_3 + y_1^2 - 1 + Ke, \\ u_3 &= -c_1(x_3 + x_1) + x_1 + x_2 + c_2(x_3 - x_4) - d_1x_1^2 + Ke \end{aligned}\right\}$$

and the 3×3 matrix K as

$$K = \begin{bmatrix} 2 & 0 & -1 \\ 0 & -0.9 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

the error system (16) reduces to

$$\left.\begin{aligned} \frac{d^qe_1}{dt^q} &= (2 - a_2)e_1, \\ \frac{d^qe_2}{dt^q} &= -(b_2 + 0.9)e_2, \\ \frac{d^qe_3}{dt^q} &= -c_2e_3 \end{aligned}\right\} \tag{17}$$

i.e., $D^qe = (B + K)e$ such that all the eigenvalues λ_i of $B + K$ are equal to -1 which satisfy the condition $|arg(\lambda_i)| > \frac{q\pi}{2}$, $i = 1, 2, 3$. The curves of synchronization error are shown in figure (8) which shows that the systems (14) and(15) are hybrid synchronized.

5.1. Numerical Simulation

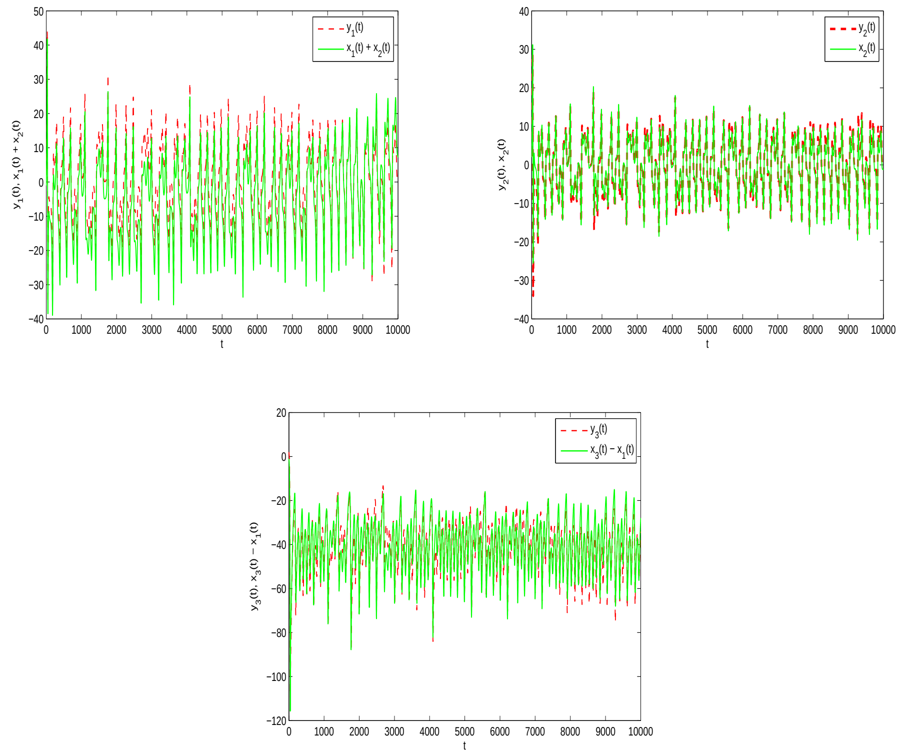

FIGURE 7. *State trajectories of the master and slave systems after the controllers are applied.*

In numerical simulations, the parameters of the hyperchaotic system are taken as $(a_1,b_1,c_1,d_1) = (10,40,2.5,4)$ and that of the Financial system are taken as $(a_2,b_2,c_2) = (3,0.1,1)$. Time step size is taken as 0.005. The initial values of the master and slave system are taken as $(x_1(0),x_2(0),x_3(0),x_4(0)) = (1,2,3,4)$ and $(y_1(0),y_2(0),y_3(0)) = (2,3,2)$ respectively. Thus the initial conditions for error system becomes $(e_1(0),e_2(0),e_3(0)) = (-1,1,3)$. It can be seen from figure 7 that it takes higher time for synchronization of the two fractional order chaotic systems considered for the fractional order $q_i = 0.99$ for $i = 1,2,3$.

6. Conclusion

In this paper, we have investigated hybrid projective synchronization between two fractional order chaotic systems of different dimensions.The numerical simulation result shows that that

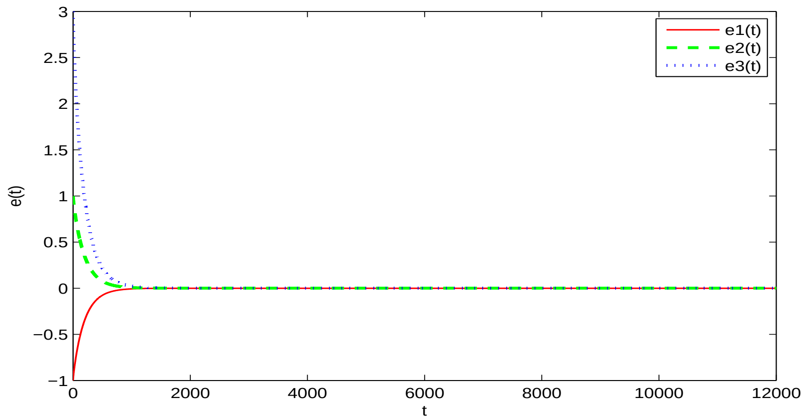

FIGURE 8. *Error Dynamics versus time t.*

the states of the fractional order chaotic Lü system and hyperchaotic Rössler system are synchronized and of the hyperchaotic system proposed by Xin and Ling and chaotic Financial system are also asymptotically synchronized. Hybrid Projective synchronization is more general than projective synchronization, in which the master and the slave systems can be synchronized upto a vector function factor. In hybrid projective synchronization the vector function factor has more unpredictibility than the same scaling factor in projective synchronization, which gives more secure communication.

Conflict of Interests

The authors declare that there is no conflict of interests.

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