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SINGULAR VALUES OF THREE-PARAMETER FAMILIES $\lambda(\frac{e^{az}-1}{z})^{\mu}$ AND $\lambda(\frac{z}{e^{az}-1})^{\eta}$

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Abstract. The main goal of the present paper is to investigate the singular values of three-parameter families of transcendental (i) entire functions $f_{\lambda,a,\mu}(z) = \lambda\left(\frac{e^{az}-1}{z}\right)^{\mu}$ and $f_{\lambda,a,\mu}(0) = \lambda a^{\mu}$; $\mu > 0$, λ, $a \in \mathbb{R}\backslash\{0\}$, $z \in \mathbb{C}$ (ii) meromorphic functions $g_{\lambda,a,\eta}(z) = \lambda\left(\frac{z}{e^{az}-1}\right)^{\eta}$ and $g_{\lambda,a,\eta}(0) = \frac{\lambda}{a^{\eta}}$, $\eta > 0$; $\lambda, a \in \mathbb{R}\backslash\{0\}$, $z \in \hat{\mathbb{C}}$. It is obtained that all the critical values of $f_{\lambda,a,\mu}(z)$ and $g_{\lambda,a,\eta}(z)$ lie in the right half plane for $a < 0$ and in the left half plane for $a > 0$. It is also shown that all these critical values of $f_{\lambda,a,\mu}(z)$ and $g_{\lambda,a,\eta}(z)$ are interior and exterior of the open disk centered at origin and having radii $|\lambda a^{\mu}|$ and $|\frac{\lambda}{a^{\eta}}|$ respectively. Further, it is described that the functions $f_{\lambda,a,\mu}(z)$ and $g_{\lambda,a,\eta}(z)$ both have infinitely many singular values.

Keywords: Critical values; singular values; entire function; meromorphic function.

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1. Introduction

The importance of singular values in the dynamics of transcendental functions can be seen in [1, 4, 7, 17]. The dynamics of one-parameter family λe^{z}, that has only one singular value, is studied in detail, for instance see [2, 3]. This exponential family is simpler than other families

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which have more than one or infinitely many singular values. Often it is crucial to investigate the dynamical properties of such functions. Some dynamical properties of families of such kinds of functions associated with exponential map are found in [5, 6, 8, 9, 10, 15]. These investigations are enormously applicable for describing Julia sets and Fatou sets.

This work investigates the singular values of three-parameter families of transcendental entire and transcendental meromorphic functions. For this purpose, let us consider three-parameter families of transcendental entire and meromorphic functions respectively which are neither even nor odd and not periodic:

$$\mathscr{E} = \left\{ f_{\lambda,a,\mu}(z) = \lambda\left(\frac{e^{az}-1}{z}\right)^{\mu} \text{and } f_{\lambda,a,\mu}(0) = \lambda a^{\mu} : \mu > 0, \lambda, a \in \mathbb{R}\backslash\{0\}, z \in \mathbb{C} \right\}$$

$$\mathscr{M} = \left\{ g_{\lambda,a,\eta}(z) = \lambda\left(\frac{z}{e^{az}-1}\right)^{\eta} \text{and } g_{\lambda,a,\eta}(0) = \frac{\lambda}{a^{\eta}} : \eta > 0, \lambda, a \in \mathbb{R}\backslash\{0\}, z \in \hat{\mathbb{C}} \right\}$$

The family $\mathscr{E}$ is a generalization of families of entire functions $\lambda \dfrac{e^z-1}{z}$ [5], $\lambda \frac{e^{az}-1}{z}$ [14] and $\lambda\left(\dfrac{e^z-1}{z}\right)^m$ [12]. The family $\mathscr{M}$ is a generalization of families of meromorphic functions $\lambda \dfrac{z}{e^z-1}$ [11], $\lambda \frac{z}{e^{az}-1}$ [14] and $\lambda\left(\dfrac{z}{e^z-1}\right)^m$ [13]. In spite of these, assuming $a = 1$, our families of functions are associated to the well known generalized *(i)* Bernoulli's generating function $\left(\frac{z}{e^z-1}\right)^{\alpha} e^{tz} = \sum_{k=0}^{\infty} B_k^{(\alpha)}(t)\frac{z^k}{k!}$ by choosing $\alpha > 0$ and $t = 0$ *(ii)* Apostol-Bernoulli's generating function $\left(\frac{z}{\lambda e^z-1}\right)^{\alpha} e^{tz} = \sum_{k=0}^{\infty} B_k^{(\alpha)}(t;\lambda)\frac{z^k}{k!}$ by choosing $\alpha > 0$, $\lambda = 1$ and $t = 0$ [16].

A point z^* is said to be a critical point of $f(z)$ if $f'(z^*) = 0$. The value $f(z^*)$ corresponding to a critical point z^* is called a critical value of $f(z)$. A point $w \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is said to be an asymptotic value for $f(z)$, if there exists a continuous curve $\gamma : [0,\infty) \to \hat{\mathbb{C}}$ satisfying $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = w$. A singular value of f is defined to be either a critical value or an asymptotic value of f.

In Theorem 1, it is seen that the functions $f'_{\lambda,a,\mu}(z)$ and $g'_{\lambda,a,\eta}(z)$ have no zeros in the left half plane and the right half plane for $a < 0$ and $a > 0$ respectively. It is found that, in Theorem 2, the functions $f_{\lambda,a,\mu} \in \mathscr{E}$ and $g_{\lambda,a,\eta} \in \mathscr{M}$ map (i) the right half plane interior and exterior of the open disk for $a < 0$ respectively (ii) the left half plane interior and exterior of the open disk for $a > 0$ respectively. In Theorem 3, it is shown that all the critical values of $f_{\lambda,a,\mu} \in \mathscr{E}$ and $g_{\lambda,a,\eta} \in \mathscr{M}$ lie interior and exterior of the open disk centered at origin and having radii $|\lambda a^{\mu}|$

and $|\frac{\lambda}{a^{\eta}}|$ respectively. Further, it is described that the functions $f_{\lambda,a,\mu} \in \mathscr{E}$ and $g_{\lambda,a,\eta} \in \mathscr{M}$ have infinitely many singular values in Theorem 4.

2. Singular Values of $f_{\lambda,a,\mu} \in \mathscr{E}$ and $g_{\lambda,a,\eta} \in \mathscr{M}$

Let us denote the right half and left half planes by $H^+ = \{z \in \hat{\mathbb{C}} : Re(z) > 0\}$ and $H^- = \{z \in \hat{\mathbb{C}} : Re(z) < 0\}$ respectively.

Lemma 1. *The equation $e^{-w} = 1 - w$ has no solutions in the right half plane H^+.*

The rigorous proof of this lemma can be seen in [5] and a short proof is given in [11] by using graphs.

Let $D_r(0)$ be the open disk centered at origin and radius r. In the following theorem, it is found that the functions $f_{\lambda,a,\mu} \in \mathscr{E}$ and $g_{\lambda,a,\eta} \in \mathscr{M}$ have no critical points in the left half plane and the right half plane for $a < 0$ and $a > 0$ respectively:

Theorem 1. *Let $f_{\lambda,a,\mu} \in \mathscr{E}$ and $g_{\lambda,a,\eta} \in \mathscr{M}$. Then,*

(a) for $a < 0$, $f'_{\lambda,a,\mu}(z)$ and $g'_{\lambda,a,\eta}(z)$ have no zeros in the left half plane H^-.

(b) for $a > 0$, $f'_{\lambda,a,\mu}(z)$ and $g'_{\lambda,a,\eta}(z)$ have no zeros in the right half plane H^+.

Proof. We have $f'_{\lambda,a,\mu}(z) = \lambda\mu(\frac{e^{az}-1}{z})^{\mu-1}\frac{(az-1)e^{az}+1}{z^2}$ for $z \neq 0$ and $f'_{\lambda,a,\mu}(0) = \lambda\mu\frac{a^{\mu+1}}{2}$. Then, for critical points of $f_{\lambda,a,\mu}(z)$, we get $e^{az} = 1$ and $e^{-az} = 1 - az$. Hence, the zeros of $f'_{\lambda,a,\mu}(z)$ are $z = \frac{2p\pi i}{a}$ on imaginary axis, where p is nonzero integer; and the solutions of the equation $e^{-az} = 1 - az$.

Similarly as above, we have $g'_{\lambda,a,\eta}(z) = \lambda\eta(\frac{z}{e^{az}-1})^{\eta-1}\frac{(1-az)e^{az}-1}{(e^{az}-1)^2}$ for $z \neq 0$ and $g'_{\lambda,a,\eta}(0) = -\lambda\eta\frac{1}{2a^{\eta-1}}$, so that for zeros of $g'_{\lambda,a,\eta}(z)$, we get $e^{-az} = 1 - az$.

Let $w = az$. Then, $e^{-w} = 1 - w$ for above both cases. By Lemma 1, this equation has no any solution in H^+.

Therefore, it shows that both the functions $f'_{\lambda,a,\mu}(z)$ and $g'_{\lambda,a,\eta}(z)$ have no zeros in H^- for $a < 0$ and in H^+ for $a > 0$.

Lemma 2. *Let $h(z) = e^{az}$ for an arbitrary fixed $z \in \mathbb{C}$. Then, for (i) $a < 0$ and $z \in H^+$, and (ii) $a > 0$ and $z \in H^-$, the following inequality holds*

$$|e^{az} - 1| < |z||a| \tag{1}$$

Proof. Suppose that the line segment γ is defined by $\gamma(t) = tz$, $t \in [0,1]$.

Since $M \equiv \max_{t\in[0,1]} |h(\gamma(t))| = \max_{t\in[0,1]} |(e^a)^{tz}| < 1$ for $a < 0$ and $z \in H^+$, then

$$\int_\gamma h(z)dz = \int_0^1 h(\gamma(t))\gamma'(t)dt = z\int_0^1 e^{atz}dt = \frac{1}{a}(e^{az} - 1)$$

$$|e^{az} - 1| = \left|a\int_\gamma h(z)dz\right| \le M|z||a| < |z||a|$$

Using similar arguments as above, we can easily deduce the inequality for $a > 0$.

It is shown in the following theorem that the functions $f_{\lambda,a,\mu} \in \mathscr{E}$ and $g_{\lambda,a,\eta} \in \mathscr{M}$ map the right half plane and the left half plane interior and exterior of the open disk respectively:

Theorem 2. *Let $f_{\lambda,a,\mu} \in \mathscr{E}$ and $g_{\lambda,a,\eta} \in \mathscr{M}$. Then,*

(i) for $a < 0$, $f_{\lambda,a,\mu}(z)$ and $g_{\lambda,a,\eta}(z)$ map the right half plane H^+ interior $D_{|\lambda a^\mu|}(0)$ and exterior of $D_{|\frac{\lambda}{a^\eta}|}(0)$ respectively.

(ii) for $a > 0$, $f_{\lambda,a,\mu}(z)$ and $g_{\lambda,a,\eta}(z)$ map the left half plane H^- interior $D_{|\lambda|a^\mu}(0)$ and exterior of $D_{\frac{|\lambda|}{a^\eta}}(0)$ respectively.

Proof. *(i)* For $a < 0$, using Inequality (1) of Lemma 2, we have

$$\left|\frac{e^{az}-1}{z}\right| < |a| \text{ for all } z \in H^+$$

$$\left|\left(\frac{e^{az}-1}{z}\right)^\mu\right| = \left|\frac{e^{az}-1}{z}\right|^\mu < |a|^\mu \text{ since } \mu > 0$$

$$|f_{\lambda,a,\mu}(z)| = \left|\lambda\left(\frac{e^{az}-1}{z}\right)^\mu\right| < |\lambda a^\mu| \text{ for all } z \in H^+.$$

It proves that the function $f_{\lambda,a,\mu} \in \mathscr{E}$ maps H^+ interior $D_{|\lambda a^\mu|}(0)$.

Again, from Inequality (1), we get

$$\left|\frac{z}{e^{az}-1}\right| > \frac{1}{|a|} \text{ for all } z \in H^+$$
$$\left|\left(\frac{z}{e^{az}-1}\right)^{\eta}\right| = \left|\frac{z}{e^{az}-1}\right|^{\eta} > \frac{1}{|a|^{\eta}} \text{ since } \eta > 0$$
$$|g_{\lambda,a,\eta}(z)| = \left|\lambda\left(\frac{z}{e^{az}-1}\right)^{\eta}\right| > \left|\frac{\lambda}{a^{\eta}}\right| \text{ for all } z \in H^+.$$

This shows that the function $g_{\lambda,a,\eta} \in \mathscr{M}$ maps H^+ exterior of $D_{|\frac{\lambda}{a^{\eta}}|}(0)$.

(ii) The proof of this part can be obtained similar as part *(i)* by using Inequality (1) of Lemma 2(ii), hence it is omitted. This completes the proof of theorem.

In the following theorem, it is proved that the functions $f_{\lambda,a,\mu} \in \mathscr{E}$ and $g_{\lambda,a,\eta} \in \mathscr{M}$ have critical values interior and exterior of the open disk respectively:

Theorem 3. *Let $f_{\lambda,a,\mu} \in \mathscr{E}$ and $g_{\lambda,a,\eta} \in \mathscr{M}$. Then, all the critical values of $f_{\lambda,a,\mu}(z)$ and $g_{\lambda,a,\eta}(z)$ lie interior of $D_{|\lambda a^{\mu}|}(0)$ and exterior of $D_{|\frac{\lambda}{a^{\eta}}|}(0)$ respectively.*

Proof. For $a < 0$, by Theorem 1(a), the function $f'_{\lambda,a,\mu}(z)$ has no zeros in H^-. Therefore, all the critical points of $f_{\lambda,a,\mu}(z)$ lie in H^+. By Theorem 2(i), the function $f_{\lambda,a,\mu}(z)$ maps H^+ interior of $D_{|\lambda a^{\mu}|}(0)$. It gives that all the critical values of $f_{\lambda,a,\mu} \in \mathscr{E}$ are lying interior of $D_{|\lambda a^{\mu}|}(0)$.

For $a > 0$, by Theorem 1(b), the function $f'_{\lambda,a,\mu}(z)$ has no zeros in H^+. Hence, all the critical points of $f_{\lambda,a,\mu}(z)$ lie in H^-. By Theorem 2(ii), the function $f_{\lambda,a,\mu}(z)$ maps H^- interior of $D_{|\lambda|a^{\mu}}(0)$. It follows that all the critical values of $f_{\lambda,a,\mu} \in \mathscr{E}$ are lying interior of $D_{|\lambda|a^{\mu}}(0)$.

Similarly, using analogous arguments as above, we can obtain the proof that all the critical values of $g_{\lambda,a,\eta} \in \mathscr{M}$ lie exterior of $D_{|\frac{\lambda}{a^{\eta}}|}(0)$ for both $a < 0$ and $a > 0$. This completes the proof.

Lemma 3. *The equation $\frac{v}{\sin(v)} - e^{v\cot(v)-1} = 0$ has infinitely many solutions.*

The proof of this lemma can be seen in [5, 11].

The following theorem shows that the functions $f_{\lambda,a,\mu} \in \mathscr{E}$ and $g_{\lambda,a,\eta} \in \mathscr{M}$ possess infinitely many singular values:

Theorem 4. *Let $f_{\lambda,a,\mu} \in \mathscr{E}$ and $g_{\lambda,a,\eta} \in \mathscr{M}$. Then, the functions $f_{\lambda,a,\mu}(z)$ and $g_{\lambda,a,\eta}(z)$ have infinitely many singular values.*

Proof. For critical points of $f_{\lambda,a,\mu} \in \mathscr{E}$, we have $f'_{\lambda,a,\mu}(z) = 0$. Then, we get $e^{-az} = 1 - az$ and $z = \frac{2p\pi i}{a}$, where p is nonzero integer. Note that $f_{\lambda,a,\mu}(\frac{2p\pi i}{a}) = 0$. Similarly, for critical points of $g_{\lambda,a,\eta} \in \mathscr{M}$, $g'_{\lambda,a,\eta}(z) = 0$, we get $e^{-az} = 1 - az$.

Now, separating the real and imaginary parts of $e^{-az} = 1 - az$, then

$$\frac{ay}{\sin(ay)} - e^{ay\cot(ay)-1} = 0 \tag{2}$$

$$x = \frac{1}{a} - y\cot(ay) \tag{3}$$

Substituting $v = ay$ in Equation (2),

$$\frac{v}{\sin v} - e^{v\cot v - 1} = 0 \tag{4}$$

By Lemma 3, Equation (4) has infinitely many roots. Therefore, it follows that Equation (2) has infinitely many solutions for both $a < 0$ and $a > 0$. Let $\{y_k\}_{k=-\infty, k\neq 0}^{k=\infty}$ be the infinitely solutions of Equation (2). Then, from Equation (3), $x_k = \frac{1}{a} - y_k \cot(ay_k)$ for $k = \pm 1, \pm 2, \pm 3, \ldots$.

For critical points $z_k = x_k + iy_k$ of $f_{\lambda,a,\mu}(z)$ and $g_{\lambda,a,\eta}(z)$, the critical values $f_{\lambda,a,\mu}(z_k) = \lambda\left(\dfrac{e^{az_k}-1}{z_k}\right)^{\mu}$ and $g_{\lambda,a,\eta}(z_k) = \lambda\left(\dfrac{z_k}{e^{az_k}-1}\right)^{\eta}$ are distinct for distinct k. It shows that $f_{\lambda,a,\mu} \in \mathscr{E}$ and $g_{\lambda,a,\eta} \in \mathscr{M}$ have infinitely many critical values for $a < 0$ and $a > 0$.

The point 0 is the finite asymptotic value of $f_{\lambda,a,\mu} \in \mathscr{E}$ and $g_{\lambda,a,\eta} \in \mathscr{M}$ since $f_{\lambda,a,\mu}(z) \to 0$ as $z \to \infty$ along both positive and negative real axes for $a < 0$ and $a > 0$ respectively; and $g_{\lambda,a,\mu}(z) \to 0$ as $z \to \infty$ along both negative and positive real axes for $a < 0$ and $a > 0$ respectively.

Thus, the functions $f_{\lambda,a,\mu} \in \mathscr{E}$ and $g_{\lambda,a,\eta} \in \mathscr{M}$ have infinitely many singular values. This completes the proof.

Conclusion

In the present paper, the singular values of three-parameter families of transcendental entire and transcendental meromorphic functions have investigated. We have found that all the critical values of both families lie interior and exterior of the disk centered at origin and having finite radii respectively. We have also shown that both the families of transcendental functions possess infinitely many singular values.

Conflict of Interests

The author declares that there is no conflict of interests.

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