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NORMAL AND RECTIFYING CURVES IN PSEUDO-GALILEAN SPACE G_3^1 AND THEIR CHARACTERIZATIONS

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Abstract: We defined normal and rectifying curves in Pseudo-Galilean Space G_3^1. Also we obtained some characterizations of this curves in G_3^1.

Keywords: Pseudo-Galilean Space, Rectifying Curve, Frenet Equations

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1. Introduction

In the Euclidean space E^3, the notion of rectifying curves was introduced by B.Y. Chen in [4]. By definition, a regular unit speed space curve $\alpha(s)$ is called a rectifying curve, if its position vector always lies its rectifying plane $\{\mathbf{t},\mathbf{b}\}$, spanned by the tangent and the binormal vector field. This subject have been studied by many researcher. The curves are studied from different way in [4,5,6,7].

A Galilean space may be considered as the limit case of a pseudo-Euclidean space in which the isotropic cone degenerates to a plane. This limit transition corresponds to the limit transition from the special theory of relativity to classical mechanics [10].

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The Pseudo-Galilean geometry is one of the real Cayley-Klein geometries (of projective signature $(0,0,+,-)$. The absolute of the Pseudo-Galilean geometry is an ordered triple $\{w, f, I\}$ where w is the ideal (absolute) plane, f is line in w and I is the fixed hyperbolic involution of points of f [2]. Differential geometry of the Pseudo - Galilean space G_3^1 has been largely developed in [1,2,3,8,9].

In the Pseudo-Galilean Space G_3^1, to each regular unit speed curve $r: I \to G_3^1$, $I \subset \mathbb{R}$, it is possible to associate three mutually ortogonal unit vector fields. The vectors $\mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ are called the tangent, the principal normal and the binormal vector field, respectively. The planes spanned by the vector fields $\{\mathbf{t}, \mathbf{n}\}, \{\mathbf{t}, \mathbf{b}\}$ and $\{\mathbf{n}, \mathbf{b}\}$ are defined as the osculating plane, the rectifying plane and the normal plane, respectively.

In this paper, we study the normal and rectifying curves in the Pseudo-Galilean Space G_3^1. By using similar method as in [4] we show that there is some characterizations of normal and rectifying curves.

2. Preliminaries

Let r be a spatial curve given first by

$$r(t) = (x(t), y(t), z(t)), \tag{2.1}$$

where $x(t), y(t), z(t) \in C^3$ (the set of three-times continuously differentiable functions) and t run through a real interval [2].

Definition 2.1. A curve r given by (2.1) is called admissible if

$$\dot{x}(t) \neq 0. \tag{2.2}$$

Then the curve r can be given by

$$r(x) = (x, y(x), z(x)) \tag{2.3}$$

and we assume in addition that, in [2]

$$y''^2(x) - z''^2(x) \neq 0. \tag{2.4}$$

Definition 2.2. For an admissible curve given by (2.1) the parameter of arc length is defined by

$$ds = |\dot{x}(t)dt| = |dx|. \tag{2.5}$$

For simplicity we assume $dx = ds$ and $x = s$ as the arc length of the curve r. From now on, we will denote the derivation by s by upper prime $'$ [2].

The vector $\mathbf{t}(s) = r'(s)$ is called the tangential unit vector of an admissible curve r in a point $\mathbf{P}(s)$. Further, we define the so called osculating plane of r spanned by the vectors $r'(s)$ and $r''(s)$ in the same point. The absolute point of the osculating plane is

$$H(0:0:y''(s):z''(s)). \tag{2.6}$$

We have assumed in (2.4) that H is not lightlike. H is a point at infinity of a line which direction vector is $r''(s)$. Then the unit vector

$$\mathbf{n}(s) = \frac{r''(s)}{\sqrt{|y''^2(s) - z''^2(s)|}} \tag{2.7}$$

is called the principal normal vector of the curve r in the point $\mathbf{P}$.

Now the vector

$$\mathbf{b}(s) = \frac{(0, \varepsilon z''(s), \varepsilon y''(s))}{\sqrt{|y''^2(s) - z''^2(s)|}} \tag{2.8}$$

is orthogonal in pseudo-Galilean sense to the osculating plane and we call it the binormal vector of the given curve in the point $\mathbf{P}$. Here $\varepsilon = +1$ or -1 is chosen by the criterion $\det(\mathbf{t}, \mathbf{n}, \mathbf{b}) = 1$. That means

$$|y''^2(s) - z''^2(s)| = \varepsilon(y''^2(s) - z''^2(s)). \tag{2.9}$$

By the above construction the following can be summarized [2].

Definition 2.3. In each point of an admissible curve in G_3^1 the associated orthonormal (in pseudo-Galilean sense) trihedron $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ can be defined. This trihedron is called pseudo-Galilean Frenet trihedron [2].

If a curve is parametrized by the arc length, i.e. given by (2.3), then the tangent vector is non-isotropic and has the form of

$$\mathbf{t}(s) = r'(s) = (1, y'(s), z'(s)). \tag{2.10}$$

Now we have

$$\mathbf{t}'(s) = r''(s) = (0, y''(s), z''(s)). \tag{2.11}$$

According to the clasical analogy we write (2.7) in the form

$$r''(s) = \kappa(s)\mathbf{n}(s), \tag{2.12}$$

and so the curvature of an admissible curve r can be defined as follows

$$\kappa(s) = \sqrt{\left| y''^2(s) - z''^2(s) \right|}. \tag{2.13}$$

Remark 2.1. In [2] for the pseudo-Galilean Frenet trihedron of an admissible curve r given by (2.3) the following derivative Frenet formulas are true.

$$\begin{aligned} \mathbf{t}'(s) &= \kappa(s)\mathbf{n}(s) \\ \mathbf{n}'(s) &= \tau(s)\mathbf{b}(s) \\ \mathbf{b}'(s) &= \tau(s)\mathbf{n}(s) \end{aligned} \tag{2.14}$$

where $\mathbf{t}(s)$ is a spacelike, $\mathbf{n}(s)$ is a spacelike and $\mathbf{b}(s)$ is a timelike vector, $\kappa(s)$ is the pseudo-Galilean curvature given by (2.13) and $\tau(s)$ is the pseudo-Galilean torsion of r defined by

$$\tau(s) = \frac{y''(s)z'''(s) - y'''(s)z''(s)}{\kappa^2(s)}. \tag{2.15}$$

The formula (2.15) can be written as

$$\tau(s) = \frac{\det(r'(s), r''(s), r'''(s))}{\kappa^2(s)}. \tag{2.16}$$

3. Normal and Rectifying Curves in Pseudo-Galilean Space G_3^1.

Definition 3.1. Let r be an admissible curve in 3-dimensional Pseudo-Galilean Space G_3^1. If the position vector of r always lies in its normal plane, then it is called normal curve in G_3^1.

By this definition, for a curve in G_3^1, the position vector of r satisfies

$$r(s)=\xi(s)\mathbf{n}(s)+\eta(s)\mathbf{b}(s), \tag{3.1}$$

where $\xi(s)$ and $\eta(s)$ are differentiable functions.

Theorem 3.1. Let r be an admissible curve in G_3^1, with $\kappa,\ \tau\in\mathbb{R}$. Then r is a normal curve if and only if the principal normal and binormal components of the position vector are respectively given by

$$<r,\mathbf{n}>=(c_1+c_2s)e^{-\tau s}+(c_3+c_4s)e^{\tau s}+\frac{\kappa}{\tau^2} \tag{3.2}$$

and

$$<r,\mathbf{b}>=(c_1+c_2s)e^{-\tau s}-(c_3+c_4s)e^{\tau s}, \tag{3.3}$$

where $c_1,c_2,c_3,c_4\in\mathbb{R}$.

Proof. Let us assume that r is a normal curve in G_3^1, then from Definition 2.1 we have

$$r(s)=\xi(s)\mathbf{n}(s)+\eta(s)\mathbf{b}(s). \tag{3.4}$$

Differentiating this with respect to s, we have

$$r'(s)=\xi'(s)\mathbf{n}(s)+\eta'(s)\mathbf{b}(s)+\xi(s)\mathbf{n}'(s)+\eta(s)\mathbf{b}'(s). \tag{3.5}$$

By using the Frenet equation (2.14), we write

$$\mathbf{t}=\xi'\mathbf{n}+\eta'\mathbf{b}+\xi\tau\mathbf{b}+\eta\tau\mathbf{n}. \tag{3.6}$$

Again differentiating this with respect to s and by using the Frenet equation (2.14), we get

$$\kappa\mathbf{n}=[(\xi'+\eta\tau)'+\tau(\xi\tau+\eta')]\mathbf{n}+[\tau(\xi'+\eta\tau)+(\xi\tau+\eta')']\mathbf{b} \tag{3.7}$$

From equation (3.7), we obtain the differential equation system

$$\begin{aligned} \xi'' + 2\tau\eta' + \tau^2\xi &= \kappa \\ \eta'' + 2\tau\xi' + \tau^2\eta &= 0. \end{aligned} \tag{3.8}$$

By solving this system, we obtain

$$\xi(s) = (c_1 + c_2 s)e^{-\tau s} + (c_3 + c_4 s)e^{\tau s} + \frac{\kappa}{\tau^2}, \quad c_1, c_2, c_3, c_4 \in \mathbb{R} \tag{3.9}$$

and

$$\eta(s) = (c_1 + c_2 s)e^{-\tau s} - (c_3 + c_4 s)e^{\tau s}, \quad c_1, c_2, c_3, c_4 \in \mathbb{R} \tag{3.10}$$

which completes the proof.

Definition 3.2. Let r be an admissible curve in 3-dimensional Pseudo-Galilean Space G_3^1. If the position vector of r always lies in its rectifying plane, then it is called rectifying curve in G_3^1.

By this definition, for a curve in G_3^1, the position vector of r satisfies

$$r(s) = \lambda(s)\mathbf{t}(s) + \mu(s)\mathbf{b}(s), \tag{3.11}$$

where $\lambda(s)$ and $\mu(s)$ are some differentiable functions.

Theorem 3.2. Let r be a rectifying curve in G_3^1, with curvature $\kappa > 0$, $<\mathbf{t},\mathbf{t}>=1$, $<\mathbf{n},\mathbf{n}>=1$, $<\mathbf{b},\mathbf{b}>=\varepsilon$, $\varepsilon = \mp 1$. Then the following statements hold:

(i)The distance function $\rho = \|r\|$ satisfies

$$\rho^2 = |<r,r>| = |s^2 + 2m_1 s + m_1^2 + \varepsilon n_1^2|$$

for some $m_1 \in \mathbb{R}$, $n_1 \in \mathbb{R} - \{0\}$.

(ii) The tangential component of the position vector of r is given by $<r,\mathbf{t}>= s + m_1$, where $m_1 \in \mathbb{R}$.

(iii) The normal component r^N of the position vector of the curve has a constant length and the distance function ρ is non-constant.

(iv) The torsion $\tau(s) \neq 0$ and binormal component of the position vector of the curveis constant, i.e. $<r, \mathbf{b}>$ is constant.

Proof. Let us assume that r is a rectifying curve in G_3^1. Then from Definition 2.3, we can write the position vector of r by

$$r(s) = \lambda(s)\mathbf{t}(s) + \mu(s)\mathbf{b}(s), \tag{3.12}$$

where $\lambda(s)$ and $\mu(s)$ are some differentiable functions of the ivariant parameters.

(i) Differentiating the equation (3.12) with respect to s and considering the Frenet equations (2.14), we get

$$\begin{gathered} \lambda'(s) = 1 \\ \lambda(s)\kappa(s) + \mu(s)\tau(s) = 0 \\ \mu'(s) = 0. \end{gathered} \tag{3.13}$$

Thus, we obtain

$$\begin{gathered} \lambda(s) = s + m_1, \quad m_1 \in \mathbb{R} \\ \mu(s) = n_1, \quad n_1 \in \mathbb{R} \\ \mu(s)\tau(s) = -\lambda(s)\kappa(s) \neq 0, \end{gathered} \tag{3.14}$$

and hence $\mu(s) = n \neq 0, \quad \tau(s) \neq 0.$ From the equation (3.12), we easily find that

$$\rho^2 = |<r, r>| = \left|s^2 + 2m_1 s + m_1^2 + \varepsilon n_1^2\right|, \quad \varepsilon = \mp 1 \tag{3.15}$$

(ii) If we consider equation (3.12), we get

$$<r, \mathbf{t}> = \lambda(s) \tag{3.16}$$

which means that the tangential component of the position vector of r is given by

$$<r, \mathbf{t}> = s + m_1, \quad m_1 \in \mathbb{R}. \tag{3.17}$$

(iii) From the equation (3.12), it follows that the normal component r^N of the position vector r is given by

$$r^N = \mu \mathbf{b}. \tag{3.18}$$

Therefore,

$$\|r^N\| = |\mu| = |n_1| \neq 0. \quad (3.19)$$

Thus we proved statement (iii).

(iv) If we consider equation (3.12), we easily get

$$< r, \mathbf{b} >= \varepsilon\mu = const., \quad \varepsilon = \mp 1 \quad (3.20)$$

and since $\tau(s) \neq 0$, the statement (iv) is proved.

Conversely, suppose that statement (i) or statement (ii) holds. Then we have

$$< r, \mathbf{t} >= s + m_1, \quad m_1 \in \mathbb{R}. \quad (3.21)$$

Differentiating equation (3.21) with respect to s, we obtain

$$\kappa < r, \mathbf{n} >= 0. \quad (3.22)$$

Since $\kappa > 0$, it follows that

$$< r, \mathbf{n} >= 0 \quad (3.23)$$

which means that r is a rectifying curve.

Next, suppose that statement (iii) holds. Let us can write

$$r(s) = l(s)\mathbf{t}(s) + r^N, \quad l(s) \in \mathbb{R}. \quad (3.24)$$

Then we easily obtain that

$$< r^N, r^N >= C = const. =< r, r > - < r, \mathbf{t} >^2 . \quad (3.25)$$

If we differentiate equation (3.25) with respect to s, we get

$$< r, \mathbf{t} >=< r, \mathbf{t} > [1 + \kappa < r, \mathbf{n} >]. \quad (3.26)$$

Since $\rho \neq const.$, we have

$$< r, \mathbf{t} >\neq 0. \quad (3.27)$$

Moreover, since $\kappa > 0$ and from (3.26) we obtain

$$< r, \mathbf{n} >= 0, \quad (3.28)$$

that is r is rectifying curve.

Finally, if the statement (iv) holds, then from the Frenet equations (2.14), we get

$$< r, \mathbf{n} >= 0, \quad (3.29)$$

which means that r is rectifying curve.

Theorem 3.3. Let r be a curve in G_3^1. Then the curve r is a rectifying curve if and only if there holds

$$\frac{\tau(s)}{\kappa(s)} = as + b \tag{3.30}$$

where $a \in \mathbb{R} - \{0\}, \quad b \in \mathbb{R}.$

Proof. Let us first suppose that the curve $r(s)$ is rectifying. From the equations (3.13) and (3.14) we easily find that

$$\frac{\tau(s)}{\kappa(s)} = as + b \tag{3.31}$$

where $a \in \mathbb{R} - \{0\}, \quad b \in \mathbb{R}.$

Conversely, let us suppose that $\frac{\tau(s)}{\kappa(s)} = as + b, \quad a \in \mathbb{R} - \{0\}, \quad b \in \mathbb{R}.$ Then we may choose

$$a = \frac{1}{n_1}, \qquad b = \frac{m_1}{n_1} \tag{3.32}$$

where $n_1 \in \mathbb{R} - \{0\}, \quad m_1 \in \mathbb{R}.$

Thus we have

$$\frac{\tau(s)}{\kappa(s)} = \frac{s + m_1}{n_1}. \tag{3.33}$$

If we consider the Frenet equations (2.14), we easily find that

$$\frac{d}{ds}[r(s) - (s + m_1)\mathbf{t}(s) - n_1\mathbf{b}(s)] = 0 \tag{3.34}$$

which means that r is a rectifying curve.

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